On Some Properties of Riemannian Manifolds with Locally Conformal Almost Cosymplectic Structures

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Abstract. Let $M$ be a $2m + 1$-dimensional Riemannian manifold and let $\nabla$ be the Levi-Civita connection and $\xi$ be the Reeb vector field, $\eta$ the Reeb covector field and $X$ be the structure vector field satisfying a certain property on $M$. In this paper the following properties are proved:

1. $\xi$ and $X$ define a 3-covariant vanishing structure;
2. the Jacobi bracket corresponding to $\xi$ vanishes;
3. the harmonic operator acting on $X$ gives
$$\Delta X^i = f \|X\|^2 X^i,$$
which proves that $X^i$ is an eigenfunction of $\Delta$, having $f \|X\|^2$ as eigenvalue;
4. the 2-form $\Omega$ and the Reeb covector $\eta$ define a Pfaffian transformation, i.e.
$$\mathcal{L}_X \Omega = 0,$$
$$\mathcal{L}_X \eta = 0;$$
5. $\nabla X$ defines the Ricci tensor;
6. $\nabla X$ one has
$$\nabla_X X = f X, \quad f = \text{scalar},$$
which show that $X$ is an affine geodesic vector field;
7. the triple $(X, \xi, \phi X)$ is an involutive 3-distribution on $M$, in the sense of Cartan.

1. Introduction

Let $M(g, \Omega, \phi, \eta, \xi)$ be an $2m + 1$-dimensional Riemannian manifold with metric tensor $g$ and associated Levi-Civita connection $\nabla$. The quadruple $(\Omega, \phi, \eta, \xi)$ consists of a structure 2-form $\Omega$ of rank $2m$, and endomorphism $\phi$ of the tangent bundle, the Reeb vector field $\xi$, and its corresponding Reeb covector field $\eta$, respectively. We assume that the 2-form $\Omega$ satisfies the relation
$$\phi \Omega = \lambda \eta \wedge \Omega,$$  \hspace{1cm} (1)
where $\lambda$ is constant, and that the 1-form $\eta$ is given by
\[ \eta = \lambda \text{d}f, \]
for some scalar function $f$ on $M$. We may therefore notice that a locally conformal almost cosymplectic structure \cite{1,3,4} is defined on the manifold $M$. In addition, we assume that the field $\phi$ of endomorphism of the tangent spaces defines a quasi-Sasakian structure, thus realizing in particular the identity
\[ \phi^2 = -Id + \eta \otimes \xi. \]
Moreover, we will assume the presence on $M$ of a structure vector field $X$ satisfying the property
\[ \nabla X = f \text{d}p + \lambda \nabla \xi, \]
where $dp$ is a canonical vector valued 1-form on $M$.

In the present paper various properties involving the above mentioned objects are studied. In particular, for the Lie differential of $\Omega$ and $\pi$ with respect to $X$, one has
\[ L_X \eta = 0, \]
\[ L_X \Omega = 0, \]
which shows that $\eta$ and $\Omega$ define Pfaffian transformations \cite{1}.

2. Preliminaries

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $\nabla$ be the covariant differential operator defined by the metric tensor. We assume in the sequel that $M$ is oriented and that the connection $\nabla$ is symmetric. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle $TM$, and
\[ \iota: TM \ni T \mapsto \iota(T) \in T^* M \quad \text{and} \quad \sharp: TM \ni T \mapsto \sharp(T) \in T^* M \]
the classical isomorphism defined by the metric tensor $g$ (i.e. $\iota$ is the index-lowering operator, and $\sharp$ is the index-raising operator). Following \cite{1}, we denote by
\[ A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM), \]
the set of vector valued $q$-forms ($q < \dim M$), and we write for
\[ d^\nabla: A^q(M, TM) \to A^{q+1}(M, TM). \]
the covariant derivative operator with respect to $\nabla$. It should be noticed that in general $d^{\nabla} = d^{\nabla} \circ d^{\nabla} \neq 0$, unlike $d^3 = d \circ d = 0$. Furthermore, we denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of $M$, which is also called the soldering form of $M$ \cite{3}; since $\nabla$ is assumed to be symmetric, we recall that the identity $d^{\nabla}(dp) = 0$ is valid. The operator
\[ d^* = d + c(\omega), \]
acting on $\Lambda M$ is called the cohomology operator \cite{3}. Here, $c(\omega)$ means the exterior product by the closed 1-form $\omega$, i.e.
\[ d^* u = du + \omega \wedge u, \]
with \( u \in \Lambda M \). A form \( u \in \Lambda M \) such that
\[
d^\pi u = 0,
\]
is said to be \( d^\pi \)-closed, and \( \omega \) is called the cohomology form. A vector field \( X \in \Xi(M) \) which satisfies
\[
d^\pi (\nabla X) = \nabla^2 X = \pi \wedge dp \in \Lambda^2(M, TM), \quad \pi \in \Lambda^1 M, \tag{5}
\]
and where \( \pi \) is conformal to \( X^\dagger \), is defined to be an exterior concurrent vector field [1]. In this case, if \( \mathcal{R} \) denotes the Ricci tensor field of \( \nabla \), one has
\[
\mathcal{R}(X, Z) = -2m \lambda^3 (\kappa + \eta) \wedge dp, \quad Z \in \Xi(M).
\]

3. Results

In terms of a local field of adapted vectorial frames \( \mathcal{O} = \text{vect}[e_A | A = 0, \cdots, 2m] \) and its associated coframe \( \mathcal{O}^* = \text{covect}[\omega^A | A = 0, \cdots, 2m] \), the soldering form \( dp \) can be expressed as
\[
dp = \sum_{A=0}^{2m} \omega^A \otimes e_A.
\]
We recall that E.Cartan’s structure equations can be written as
\[
\nabla e_A = \sum_{B=0}^{2m} \theta^B_A \otimes e_B, \tag{6}
\]
\[
d\omega^A = - \sum_{B=0}^{2m} \theta^A_B \wedge \omega^B, \tag{7}
\]
\[
d\theta^A_B = - \sum_{C=0}^{2m} \theta^C_B \wedge \theta^A_C + \Theta^A_B, \tag{8}
\]
In the above equation \( \theta \) (respectively \( \Theta \)) are the local connection forms in the tangent bundle \( TM \) (respectively the curvature 2-forms on \( M \)). In terms of the frame fields \( \mathcal{O} \) and \( \mathcal{O}^* \) with \( e_0 = \xi \) and \( \omega^0 = \eta \), the structure vector field \( X \) and the 2-form \( \Omega \) can be expressed as
\[
X = \sum_{a=1}^{2m} X^a e_a, \tag{9}
\]
\[
\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^i, \quad i^* = i + m. \tag{10}
\]
Taking the Lie differential of \( \Omega \) and \( \eta \) with respect to \( X \), one calculates
\[
\mathcal{L}_X \eta = 0, \tag{11}
\]
\[
\mathcal{L}_X \Omega = 0. \tag{12}
\]
According to [2] the above equations prove that \( \eta \) and \( \Omega \) define a Pfaffian transformation [3]. Next, by (2) one gets that
\[
\theta_0^a = \lambda \omega^a. \tag{13}
\]
Since we also assume that
\[ \nabla X = f dp + \lambda \nabla \xi, \]  
we further also derive that
\[ \nabla \xi = \lambda(dp - \eta \otimes \xi). \]  
Since the \( q \)-th covariant differential \( \nabla^q Z \) of a vector field \( Z \in \mathfrak{X}(M) \) is defined inductively, i.e.
\[ \nabla^q Z = d^\nabla \left( \nabla^{q-1} Z \right), \] this yields
\[ \nabla^2 \xi = \lambda^2 \eta \otimes dp, \]  
\[ \nabla^3 \xi = 0. \]  
Hence, one may say that the 3-covariant Reeb vector field \( \xi \) is vanishing. Next, by (13), one derives that
\[ \nabla^2 X = \lambda^3(df + \eta) \wedge dp = \frac{1 + \lambda}{\lambda} \eta \wedge dp, \]  
and consecutively one gets that
\[ \nabla^3 X = 0. \]  
This shows that both vector fields \( \xi \) and \( X \) together define a 3-vanishing structure. Moreover, by reference to [3], it follows from (18) that one may write that
\[ \nabla^2 X = -\frac{1}{2m} \text{Ric}(X) - X^3 \wedge dp, \]  
where \( \text{Ric} \) is the Ricci tensor. Reminding that by the definition of the operator \( \phi \)
\[ \phi e_i = e_i, \quad i \in \{1, \cdots m\}, \]
\[ \phi e_i^* = -e_i, \quad i^* = i + m, \] one can check that indeed \( \phi^2 = -Id \). Acting with \( \phi \) on the vector field \( X \), one obtains in a first step that
\[ \phi X = \sum_{i=1}^{m} X^i e_i^* - X^{i^*} e_i \quad i^* = i + m. \]  
Calculating the Lie derivative of \( \phi \) w.r.t. \( \xi \), one gets
\[ (\mathcal{L}_\xi \phi)X = [\xi, \phi X] - \phi[\xi, X]. \]  
Since clearly
\[ [\xi, \phi X] = 0, \]  
there follows that
\[ (\mathcal{L}_\xi \phi)X = 0. \]  
Hence, the Jacobi bracket corresponding to the Reeb vector field \( \xi \) vanishes. By reference to the definition of the divergence
\[ \text{div} Z = \sum_{A=0}^{2m} \omega^A(\nabla_{e_A} Z), \]  
one obtains in the case under consideration that
\[ \text{div} X = 2m(\lambda + f^2). \]
and
\[ \text{div} \phi X = 0. \]  
(26)

Calculating the differential of the dual form \(X^b\) of \(X\), one gets
\[ dX^b = \sum_{a=1}^{2m} \left( dX^a + \sum_{b=1}^{2m} X^b \theta^a_b \right) \wedge \omega^a. \]  
(27)

Since
\[ dX^a + \sum_{b=1}^{2m} X^b \theta^a_b = \lambda \omega^a, \]  
(28)

one has that
\[ dX^b = 0, \]  
(29)

which means that the Pfaffian \(X^1\) is closed. This implies that \(X^b\) is an eigenfunction of the Laplacian \(\Delta\), and one can write that
\[ \Delta X^1 = f ||X||^2 X^1. \]

If we set
\[ 2l = ||X||^2, \]  
(30)

one also derives by (28) that
\[ dl = \lambda X^1. \]  
(31)

Returning to the operator \(\phi\), one calculates that
\[ \nabla (\phi X) = \lambda \phi dp - \sum_{i=1}^{m} \left( \sum_{a=1}^{2m} (X^a \theta^a_a \otimes c_i) + \sum_{a=1}^{2m} (X^a \theta^a_i) \otimes c_i \right). \]  
(32)

Hence there follows that
\[ [\xi, X] = \rho \xi - \phi C, \]  
(33)
\[ [\xi, \phi X] = (C^0)^2 + C^0(1 - \lambda)) \xi, \]  
(34)
\[ [X, \phi X] = \nabla_\xi \phi C = C^0 \xi - C \]  
(35)

which shows that the triple \(\{ X, \xi, \phi X \}\) defines a 3-distribution on \(M\). It is also interesting to draw the attention on the fact that \(X\) possesses the following property. From (14) and (15) one derives that
\[ \nabla_X X = f X, \]  
(36)

which means that \(X\) is an affine geodesic vector field. Finally, if we denote by \(\Sigma\) the exterior differential system which defines \(X\), it follows by Cartan’s test \([1]\) that the characteristic numbers are
\[ r = 3, \quad s_0 = 1, \quad s_1 = 2. \]

Since \(r = s_0 + s_1\), it follows that \(\Sigma\) is an involution and the existence of \(X\) depends on the arbitrary function of 1 argument. Summarizing, we can organize our results into the following

**Theorem 3.1.** Let \(M\) be a \(2m+1\)-dimensional Riemannian manifold and let \(\nabla\) be the Levi-Civita connection and \(\xi\) be the Reeb vector field, \(\eta\) the Reeb covector field and \(X\) be the structure vector field satisfying the property (3) on \(M\). One has the following properties:

- (i) \(\xi\) and \(X\) define a 3-covariant vanishing structure;
• (ii) the Jacobi bracket corresponding to $\xi$ vanishes;
• (iii) the harmonic operator acting on $X^i$ gives
  \[ \Delta X^i = \nabla \|X\|^2 X^i, \]
  which proves that $X^i$ is an eigenfunction of $\Delta$, having $\nabla \|X\|^2$ as eigenvalue;
• (iv) the 2-form $\Omega$ and the Reeb covector $\eta$ define a Pfaffian transformation, i.e.
  \[ \mathcal{L}_X \Omega = 0, \]
  \[ \mathcal{L}_X \eta = 0; \]
• (v) $\nabla^2 X$ defines the Ricci tensor;
• (vi) one has
  \[ \nabla_X X = f X, \quad f = \text{scalar}, \]
  which show that $X$ is an affine geodesic vector field;
• (vii) the triple $(X, \xi, \phi X)$ is an involutive 3-distribution on $M$, in the sense of Cartan.

References


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