CONSTRUCTION OF $B$–FOCAL CURVES OF SPACELIKE BIHARMONIC $B$–SLANT HELICES ACCORDING TO BISHOP FRAME IN $\mathbb{E}(1,1)$

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ABSTRACT. In this paper we find parametric equations of $B$–focal curves of spacelike biharmonic $B$–slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

1. Introduction

A smooth map $\phi : N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_n,$$

where $\mathcal{T}(\phi) := \text{tr} \nabla^0 d\phi$ is the tension field of $\phi$.

The Euler–Lagrange equation of the bienergy [1–5] is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \text{tr} R(\mathcal{T}(\phi), d\phi) \, d\phi,$$

and called the bitension field of $\phi$. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: First, we study $B$–focal curves of spacelike biharmonic $B$–slant helices. Finally, we find parametric equations of $B$–focal curves of spacelike biharmonic $B$–slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$
Topologically, \( E(1, 1) \) is diffeomorphic to \( \mathbb{R}^3 \) under the map

\[
\mathbb{E}(1, 1) \rightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \rightarrow (x, y, z).
\]

Its Lie algebra has a basis consisting of

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \\
X_3 &= \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},
\end{align*}
\]

for which

\[
\begin{align*}
[X_1, X_2] &= X_3, \\
[X_2, X_3] &= 0, \\
[X_1, X_3] &= X_2.
\end{align*}
\]

Put

\[
x^1 = x, \ x^2 = \frac{1}{2} (y + z), \ x^3 = \frac{1}{2} (y - z).
\]

Then, we get

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial x^1}, \\
X_2 &= \frac{1}{2} \left( e^{x^1} \frac{\partial}{\partial x^2} + e^{-x^1} \frac{\partial}{\partial x^3} \right), \\
X_3 &= \frac{1}{2} \left( e^{x^1} \frac{\partial}{\partial x^2} - e^{-x^1} \frac{\partial}{\partial x^3} \right).
\end{align*}
\]

The bracket relations are

\[
\begin{align*}
[X_1, X_2] &= X_3, \\
[X_2, X_3] &= 0, \\
[X_1, X_3] &= X_2.
\end{align*}
\]

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis \( \{X_1, X_2, X_3\} \). We consider left-invariant Lorentzian metric [6], given by

\[
g = -(dx^1)^2 + \left( e^{-x^1} dx^2 + e^{x^1} dx^3 \right)^2 + \left( e^{-x^1} dx^2 - e^{x^1} dx^3 \right)^2, \tag{2.2}
\]

where

\[
g(X_1, X_1) = -1, \ g(X_2, X_2) = g(X_3, X_3) = 1.
\]

Let coframe of our frame be defined by

\[
\begin{align*}
\theta^1 &= dx^1, \\
\theta^2 &= e^{-x^1} dx^2 + e^{x^1} dx^3, \\
\theta^3 &= e^{-x^1} dx^2 - e^{x^1} dx^3.
\end{align*}
\]

### 3. Spacelike biharmonic \( B \)-slant helices in the Lorentzian group of rigid motions \( \mathbb{E}(1, 1) \)

Let \( \gamma : I \rightarrow \mathbb{E}(1, 1) \) be a non geodesic spacelike curve on the \( \mathbb{E}(1, 1) \) parametrized by arc length. Let \( \{T, N, B\} \) be the Frenet frame fields tangent to the \( \mathbb{E}(1, 1) \) along \( \gamma \) defined as follows:

- \( T \) is the unit vector field \( \gamma' \) tangent to \( \gamma \), \( N \) is the unit vector field in the direction of \( \nabla_T T \) (normal to \( \gamma \)), and \( B \) is chosen so that \( \{T, N, B\} \) is a positively oriented orthonormal basis. Then, we have the following Frenet formulas [7]:

\[
\begin{align*}
\nabla_T T &= \kappa N, \\
\nabla_T N &= \kappa T + \tau B, \tag{3.1} \\
\nabla_T B &= \tau N,
\end{align*}
\]
where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion and
\[
\begin{align*}
g(T,T) &= 1, \ g(N,N) = -1, \ g(B,B) = 1, \\
g(T,N) &= g(T,B) = g(N,B) = 0.
\end{align*}
\]

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as [8–14]:
\[
\nabla_T T = k_1 M_1 - k_2 M_2, \\
\nabla_T M_1 = k_1 T, \\
\nabla_T M_2 = k_2 T,
\]
where
\[
\begin{align*}
g(T,T) &= 1, \ g(M_1,M_1) = -1, \ g(M_2,M_2) = 1, \\
g(T,M_1) &= g(T,M_2) = g(M_1,M_2) = 0.
\end{align*}
\]

Here, we shall call the set $\{T, M_1, M_2\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures and $\tau(s) = \psi'(s)$, $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by
\[
\begin{align*}
k_1 &= \kappa(s) \sinh \psi(s), \\
k_2 &= \kappa(s) \cosh \psi(s).
\end{align*}
\]

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write
\[
\begin{align*}
T &= T^1 e_1 + T^2 e_2 + T^3 e_3, \\
M_1 &= M_1^1 e_1 + M_1^2 e_2 + M_1^3 e_3, \\
M_2 &= M_2^1 e_1 + M_2^2 e_2 + M_2^3 e_3.
\end{align*}
\]

Definition 3.2. (see Ref. [3]) A regular spacelike curve $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is called a $B-$slant helix provided the timelike unit vector $M_1$ of the curve $\gamma$ has constant angle $\theta$ with some fixed timelike unit vector $u$, that is
\[
g(M_1(s), u) = \cosh \varphi \text{ for all } s \in I. \tag{3.4}
\]

Lemma 3.3. (see Ref. [3]) Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed spacelike curve with non-zero natural curvatures. Then $\gamma$ is a $B-$slant helix if and only if
\[
\frac{k_1}{k_2} = \tanh \varphi. \tag{3.5}
\]

4. $B-$focal curves of spacelike biharmonic $B-$slant helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$

Denoting the focal curve by $\text{focal},$ we can write
\[
\text{focal}(\gamma)(s) = (\gamma + f_1 B M_1 + f_2 B M_2)(s), \tag{4.1}
\]
where the coefficients $f_1 B , f_2 B$ are smooth functions of the parameter of the curve $\gamma$, called the first and second focal curvatures of $\gamma,$ respectively.
To separate a focal curve according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as \( B\)-focal curve.

**Theorem 4.1.** Let \( \gamma : I \rightarrow \mathbb{E}(1,1) \) is a non geodesic spacelike biharmonic \( B \)-slant helix with timelike \( M_1 \) and focal, its focal curve in the Lorentzian group of rigid motions \( \mathbb{E}(1,1) \). Then, the vector equation of focal \( B(s) \) is

\[
(- \sinh \varphi s + a_1 + p \cosh \varphi)X_1 + \left[ -Ae^{-\sinh \varphi s + a_1}[(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1}ight.

\[
- Ae^{\sinh \varphi s - a_1}[(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1}

\[
+ \frac{1 + pk_1}{k_2} \sin [A_1 s + A_2]]X_2 + \left[ -Ae^{-\sinh \varphi s + a_1}[(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1}ight.

\[
+ \frac{1 - pk_1}{k_2} \cos [A_1 s + A_2]]X_3,
\]

where \( a_1, p, A_1, A_2 \) are constants of integration and

\[
A = \frac{\cosh \varphi}{2 (A_1^2 + \sinh^2 \varphi)}.
\]

**Proof.** Assume that \( \gamma \) is a unit speed spacelike biharmonic \( B \)-slant helix with timelike \( M_1 \) and focal, its focal curve in the Lorentzian group of rigid motions \( \mathbb{E}(1,1) \). So, by differentiating of the formula (2.1), we get

\[
\text{focal}^B \gamma(s)' = (1 + t_1^B k_1 + t_2^B k_2)T + (t_1^B)' M_1 + (t_2^B)' M_2.
\]

On the other hand, from Definition 3.2, we obtain

\[
M_1 = \cosh \varphi X_1 + \sinh \varphi \cos [A_1 s + A_2] X_2 + \sinh \varphi \sin [A_1 s + A_2] X_3.
\]

Using (2.1) in (4.4), we may be written as

\[
M_2 = - \sin [D_1 s + D_2] X_2 + \cos [A_1 s + A_2] X_3.
\]

Furthermore, from above equations we get

\[
T = - \sinh \varphi X_1 - \cosh \varphi \cos [A_1 s + A_2] X_2 - \cosh \varphi \sin [A_1 s + A_2] X_3.
\]

On the other hand, the first 2 components of Eq.(4.6) vanish, we get

\[
t_1^B k_1 + t_2^B k_2 = -1,
\]

\[
(t_1^B)' = 0.
\]

Considering second equation above system, we chose

\[
t_1^B = p \text{ =constant } \neq 0.
\]
Then, it holds that

\[ f_{B_2} = -1 - \frac{p k_1}{k_2}. \]  

(4.8)

By means of obtained equations, we express

\[ \tilde{\xi}_B^B(s) = (\gamma + p M_1 + \frac{-1 - p k_1}{k_2} M_2)(s), \]  

(4.9)

where \( p \) is a constant.

Considering equations (4.5) and (4.6) by the (4.9), we get (4.2). This completes the proof.

Corollary 4.2. Let \( \gamma : I \rightarrow \mathbb{E}(1, 1) \) is a non geodesic spacelike biharmonic \( B-\)slant helix with timelike \( M_1 \) and \( \text{focal}_\gamma \) its focal curve in the Lorentzian group of rigid motions \( \mathbb{E}(1, 1) \). Then, the focal curvatures of \( \text{focal}_\gamma \) are

\[ f_{B_2} = -1 - f_{B_1} \frac{k_1}{k_2} \tanh \varphi = \text{constant} \neq 0. \]  

(4.10)

Proof. Suppose that \( \gamma \) is a non geodesic spacelike biharmonic \( B-\)slant helix with timelike \( M_1 \) and \( \text{focal}_\gamma \) its focal curve. From (3.5) and (4.8) the focal curvature of \( \text{focal}_\gamma \) takes the form (4.10). This completes the proof.

Then, we give the following theorem.

Theorem 4.3. Let \( \gamma : I \rightarrow \mathbb{E}(1, 1) \) is a non geodesic spacelike biharmonic \( B-\)slant helix with timelike \( M_1 \) and \( \text{focal}_\gamma \) its focal curve in the Lorentzian group of rigid motions \( \mathbb{E}(1, 1) \). Then, the vector equation of \( \text{focal}_B^B(s) \) is

\[ x_1^1(s) = -\sinh \varphi s + a_1 + p \cosh \varphi, \]

\[ x_1^2(s) = \frac{1}{2} \exp[-\sinh \varphi s + a_1 + p \cosh \varphi] \]

\[ \left[ -A e^{-\sinh \varphi s + a_1} [(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1} \right. \]

\[ \left. -A e^{\sinh \varphi s - a_1} [(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1} \right] \]

\[ + p \sinh \varphi \cos [A_1 s + A_2] + \frac{1 + p k_1}{k_2} \sin [A_1 s + A_2] \]

\[ + \frac{1}{2} \exp[\sinh \varphi s - a_1 - p \cosh \varphi] \left[ -A e^{-\sinh \varphi s + a_1} [(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1} \right. \]

\[ + A e^{\sinh \varphi s - a_1} [(\sinh \varphi - A_1) \cos [A_1 s + A_2] + (\sinh \varphi + A_1) \sin [A_1 s + A_2]] + a_2 e^{\sinh \varphi s - a_1} \]

\[ + p \sinh \varphi \sin [A_1 s + A_2] + \frac{-1 - p k_1}{k_2} \cos [A_1 s + A_2], \]
\[ x^3(s) = \frac{1}{2} \exp[- \sinh \varphi s + a_1 + p \cosh \varphi] \\
- \mathcal{A} e^{- \sinh \varphi s + a_1} \left[ (\sinh \varphi - A_1) \cos (A_1 s + A_2) + \sinh \varphi s + a_1 \right] \\
+ (\sinh \varphi + A_1) \sin (A_1 s + A_2) + a_2 \sinh \varphi s + a_1 \\
+ \mathcal{A} e^{\sinh \varphi s - a_1} \left[ (\sinh \varphi - A_1) \cos (A_1 s + A_2) + (\sinh \varphi + A_1) \sin (A_1 s + A_2) + a_3 \sinh \varphi s + a_1 \right] + p \sinh \varphi \cos (A_1 s + A_2) + \frac{1 + p k_1}{k_2} \sin (A_1 s + A_2) \\
- \frac{1}{2} \exp[\sinh \varphi s - a_1 - p \cosh \varphi] \left[ - \mathcal{A} e^{- \sinh \varphi s + a_1} \left[ (\sinh \varphi - A_1) \cos (A_1 s + A_2) + (\sinh \varphi + A_1) \sin (A_1 s + A_2) + a_2 \sinh \varphi s + a_1 \right] \\
+ \mathcal{A} e^{\sinh \varphi s - a_1} \left[ (\sinh \varphi - A_1) \cos (A_1 s + A_2) + (\sinh \varphi + A_1) \sin (A_1 s + A_2) + a_3 \sinh \varphi s + a_1 \right] + p \sinh \varphi \sin (A_1 s + A_2) + \frac{-1 - p k_1}{k_2} \cos (A_1 s + A_2)], \]

where \( p, A_1, A_2 \) are constants of integration and 

\[ \mathcal{A} = \frac{\cosh \varphi}{2 (A_1^2 + \sinh^2 \varphi)}. \]

**Proof.** Assume that \( \gamma \) is a non geodesic spacelike biharmonic \( B \)-slant helix and its focal curve is \( \text{focal}_\gamma \). Substituting (2.1) to (4.2), we have (4.11) as desired. This concludes the proof of theorem.

**5. Conclusions**

Consider a curve in a space and suppose that the curve is sufficiently smooth so that the Bishop Frame adapted to it is defined; the curvatures \( k_1 \) and \( k_2 \) then provide a complete characterization of the curve. In this paper we have found parametric equations of \( B \)-focal curves of spacelike biharmonic \( B \)-slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions.

**References**


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