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ON THE DYNAMICAL EQUIVALENCE OF LAGRANGIANS DIFFERING UP TO THE TOTAL TIME DERIVATIVE OF AN ARBITRARY FUNCTION OF COORDINATES AND TIME

A NOTE ON THE FOUNDATION OF ELECTROMAGNETIC THEORY AS A GAUGE THEORY

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ABSTRACT. It is a remarkable occurence that if two Lagrangians differ up to the total time derivative of an arbitrary function of coordinates and time they are dynamically equivalent, *i.e.*, they imply the same equations of motion despite these Euler-Lagrange equations are derived by a variational procedure or otherwise ("direct problem" versus "inverse problem"). In this note it is shown that by a proper identification of such redundance term the whole of the Maxwell equations can be readily derived, the gauge symmetry of the electromagnetic theory emerging from the outset as the fundamental symmetry.

1. Introduction

> *"Quantum mechanics was built upon a foundation of analogy with the Hamiltonian theory of classical mechanics ... Now there is an alternative foundation for classical dynamics, provided by the Lagrangian. This requires one to work in terms of coordinates and velocities instead of coordinates and momenta. The two formulations are, of course, closely related, but there are reasons for believing that the Lagrangian one is the more fundamental." (P. A. M. Dirac)*

To some extent this contribution aims to convey into that vigorous river that streamed after Dyson's paper [1] on the proposal by Feynman to derive the Lorentz force law

$$m\frac{d^2x^i}{dt^2} = E^i(t,x) + \varepsilon^{ijk}\frac{dx_j}{dt}H_k(t,x) \tag{1}$$

and the homogeneous Maxwell equations (M. E.)

$$\boldsymbol{\nabla}\wedge\boldsymbol{E} + \partial_t\boldsymbol{H} = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{H} = 0 \tag{2}$$

from a Newtonian force law

$$m\ddot{x}^i = F^i(t, x, \dot{x}), \qquad \dot{x}^i = \frac{dx^i}{dt} \tag{3}$$

assuming that Euclidean coordinates and velocities satisfy (quantum) canonical commutation relations:

$$\left[x^i, x^j\right] = 0, \qquad \left[\dot{x}, \dot{x}^j\right] = 0, \qquad m\left[x^i, \dot{x}^j\right] = i\hbar\,\delta^{ij} \tag{4}$$

Severe criticism was raised against this “derivation” primarily due to symmetry requirements and implied invariance properties [2]. Indeed it is quite wondering to claim for a Lorentz-invariant statement after starting from a dynamical equation that is not Lorentz-invariant and canonical commutation relations that are assigned at each given instant of time t. Moreover, it has to be noticed that both the Lorentz dynamical law and the first pair of the M. E. are Gauge invariant: indeed are these last equations to asses the Gauge Symmetry! On the other side, looking at the most autoritative and rightly widespread treatises and review papers on Electrodynamics [3] (just to limit to the most renewed and frequently consulted and quoted), emphasis is on the Lorentz invariance of the M. E., whereas their Gauge Symmetry properties appear to be quite a consequence of this last. That such a procedure to deduce the Gauge invariance of the M. E. after stating the Lorentz symmetry of the Lagrangian describing the interaction between charged particles and electromagnetic fields could be looked at as doubtful was previously raised up in [4] on the basis of various considerations that will be discussed in some details in what follows. Perhaps, in this context, it is not meaningless to remind the reader of the lively debate about Proca’s Lagrangian [5] where the term $\alpha^2 A_\mu A^\mu$, that should appear in a Lorentz covariant description from the outset, would be responsible of the breaking of the Gauge Symmetry and so omitted in the Lagrangian customarily invoked to derive Electrodynamics. So we feel strongly urged to better ascertain the relationship between the two classes of symmetries. To this end the best advice is to resort to a Lagrangian approach even if with some difference with respect to the traditionally followed path. Indeed, instead of following the commonly adopted procedure used to “derive” Lagrangians as the result of some “inverse problem solution” (*i.e.*, after the partial integration of a given set of so called “dynamical equations”), we would like to investigate what can be deduced from some *a priori* properties of the Lagrangian formalism as they can be stated by the simple fact that some basic given identities have to be satisfied. Such an attitude amounts to underline the eidetic, apodictic character of the Lagrangian method instead of its heuristic-pragmatic potentialities. Once more, as in [4] the purpose is to reach well-known fundamental results from another point of view that is considered best suited to avoid misunderstandings.

2. The Maxwell equations (“derived”)

Starting from the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0, \qquad i = 1, \dots, s \tag{5}$$

(for the sake of simplicity we write them in terms of Euclidean coordinates and velocities for a sistem with a finite number of degrees of freedom s, but our considerations can be directly extended to generalized coordinates or to field theories) it is well known that the same set of equations is obtained if the Lagrangian $\mathcal{L}$ which characterizes a given dynamical system Σ is changed into

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{d}{dt} f(\boldsymbol{x}(t), t) \equiv \mathcal{L} + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f + \partial_t f \tag{6}$$

$f(\boldsymbol{x}(t), t)$ being an arbitrary function of coordinates and time. As a matter of fact, we note that this occurence is intrinsic with the formal structure of the equations in themselves, *i.e.* it does not depend on the procedure followed to arrive to Eqs. (5), either they are obtained by a variational procedure ("direct problem") or after a partial integration starting from some given equations assumed as the evolutive equations ("inverse problem"). Such dynamical equivalence could appear quite wondering due to the very simple fact that under the most general circumstances both momentum and energy are altered:

$$\boldsymbol{p} \equiv \boldsymbol{\nabla}_{\dot{\boldsymbol{x}}} \mathcal{L} \to \boldsymbol{p'} = \boldsymbol{p} + \boldsymbol{\nabla}_{\boldsymbol{x}} f \tag{7}$$

$$\mathcal{E} \equiv (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\dot{\boldsymbol{x}}} \mathcal{L} - \mathcal{L}) \to \mathcal{E}' = -\partial_t \mathcal{L} + \mathcal{E} \tag{8}$$

We note that energy is unchanged only if the function f does not depend on time explicitly, whereas momentum is changed altogheter. If this redundance would be looked at as a dynamical coupling with some other systems, it is worth noticing that the interaction breaks up the homogeneity of the Lagrangian with respect to the velocity, unless $\mathcal{L}$ is homogenous of first degree in $\dot{\boldsymbol{x}}$. Moreover, in this restricted scenario if f is not explicitly time-dependent "zero-energy" coupling is attained ("minimal coupling"?) and time could not be looked at as absolute, insofar if the Lagrangian is homogenous of degree one with respect to the velocity time cannot be "absolute" [6]. Let us concentrate our attention on the r.h.s. of Eq. (6). Inasmuch as the function f enters the definition of $\mathcal{L}'$ via its space-time gradients $(\boldsymbol{\nabla}_{\boldsymbol{x}} f, \partial_t f)$, we are allowed to establish a correspondence between the function f and two vector fields $\boldsymbol{A}(\boldsymbol{x}(t), t)$ and $\boldsymbol{E}(\boldsymbol{\nabla} f, \boldsymbol{A})$ via the equations:

$$\boldsymbol{E} = -\boldsymbol{\nabla}_{\boldsymbol{x}} f - \frac{\partial \boldsymbol{A}}{\partial t} \tag{9}$$

$$-\partial_t f = \boldsymbol{\nabla} \cdot \boldsymbol{A} \tag{10}$$

in accordance with Helmholtz's decomposition rule [3]. We see that Eq. (10) defines quite "naturally" the Lorentz gauge as the most general gauge constraint. The first pair of the M. E.:

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0 \tag{11}$$

$$\nabla \cdot \boldsymbol{H} = 0 \tag{12}$$

with

$$\boldsymbol{H} = \nabla \wedge \boldsymbol{A} \tag{13}$$

can be readily obtained with the operator $\nabla\wedge$ acting on the l.h.s. of Eq. (9) and with the operator $\nabla\cdot$ acting on Eq. (13), respectively. Moreover, if we take the divergence of Eq. (9) and take into account Eq. (13), we readily obtain

$$\nabla \cdot \boldsymbol{E} = \rho \qquad\qquad (\rho = -\nabla^2 f + \frac{\partial^2 f}{\partial t^2}) \tag{14}$$

whereas, if we take the curl of Eq. (13), upon using the identity $\nabla\wedge(\nabla\wedge)\equiv-\nabla^2 + \nabla(\nabla\cdot)$ and according to Eq. (9), we obtain

$$\nabla \wedge \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t} \qquad (\boldsymbol{J} = -\nabla^2 \boldsymbol{A} + \frac{\partial^2 \boldsymbol{A}}{\partial t^2}) \tag{15}$$

3. Discussion and conclusions

Inasmuch as the main motivation underlying this note was the competitive role played by Gauge invariance requirements vs. Lorentz invariance when stating the M. E., a few comments seem to be in order. On this respect it is worth noticing that there are at least three points to motivate the criticism initially put forward in [4]:

a) firstly, the Lorentz group is a group of external symmetries whereas the Gauge Group can lead to both internal (local) and external (global) symmetries [7], as identically verified by the first and second couple of the M. E. respectively;

b) secondly, the transversality of the solution is a direct consequence of the Gauge Symmetry alone (it is worth mentioning that the causality requirement is completely fulfilled if the transversality of the solution is realized [8]);

c) thirdly, as previously noted, the Gauge Symmetry can be traced back to a Lagrangian which is a homogeneous function of degree one with respect to the velocity, which in turn implies that time cannot be considered absolute and, consequently, the theory depends on the ratios of the involved velocities with respect to some invariant velocity.

As a matter of fact one could be tempted, *sic stantibus rebus*, to look at Eq. (6) as to the feasibility to introduce the "interaction Lagrangian"

$$\mathcal{L}_{INT} = \frac{Q}{c} \boldsymbol{A} \cdot \dot{\boldsymbol{x}} - Q\Phi \tag{16}$$

provided the identifications

$$\boldsymbol{A} = \nabla_{\boldsymbol{x}} f, \qquad \partial_t f = \Phi \tag{17}$$

are performed and an "invariant charge" Q is presumed to exist: it is quite obvious that such a procedure could not lead to any theory of the electromagnetic field.

From the foregoing "derivation" of the M. E., the Gauge invariance of $\boldsymbol{E}$ and $\boldsymbol{H}$ under the "local" transformations:

$$\boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\chi \tag{18}$$

$$f' = f - \partial_t \chi \tag{19}$$

emerges quite neatly and this allows to look at Eqs. (11) and (12) as constraints. The criticism put forward by Rousseaoux [9] can be rejected and $\boldsymbol{A}$ and f can be safely looked at as Lagrangian parameters: indeed, they satisfy separate (i.e., decoupled) wave equations (as it has to be for fields) only in the Lorentz gauge which, in our approach, is introduced from the beginning in a straightforward way. It is worth noticing that in the approach followed in this note the primary field is the Maxwell field $(\boldsymbol{A}, f)$, *i.e.*, a gauge field and its wave equation is obtained without any recursion to a dynamical equation for particles. Moreover, it has to be stressed the analogy with a hydrodynamical theory, the magnetic field $\boldsymbol{H}$ being the vorticity of the field $\boldsymbol{A}$: in this context charges appear as topological invariants as it has to be, and a direct connection with vortex theory is established. Indeed, as for the "global" implications of a gauge theory, *i.e.* the existence of conserved currents and of the corresponding total charge, it is worth noticing that the usual continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{20}$$

is readily obtained from Eqs. (14) and (15) jointly with the topological charge conservation law:

$$\frac{d}{dt} Q \equiv \frac{d}{dt} \int_{universe} \rho d\Omega = 0 \tag{21}$$

It should also be remembered that such a hydrodynamical point of view can be more or less deeply connected with items like the superfluidity of the physical vacuum [10] or the superconductivity of condensed matter [11], both phenomena being viewed as spontaneously broken symmetry phenomena, the involved symmetry being the gauge symmetry.

Up to this point we have been essentially concerned with the feasibility of introducing a Gauge-invariant Electromagnetic Theory, *i.e.* the whole set of Maxwell's equations that, in turn, lead to the wave equations for the involved fields $\boldsymbol{E}$ and $\boldsymbol{H}$ (the so called "physical" fields):

$$\Box \boldsymbol{E} = -\boldsymbol{\nabla}\rho - \partial_t \boldsymbol{J} \tag{22}$$

$$\Box \boldsymbol{H} = -(\boldsymbol{\nabla} \wedge \boldsymbol{J}) \tag{23}$$

Now, for the sake of completeness, some words have to be spent about the construction of an Electrodynamic Theory where the motion of some properly chosen system (the so called "charged" system) has to be accounted for according to some dynamical equation which has to be consistent with the fields $\boldsymbol{E}$ and $\boldsymbol{H}$, as given by Maxwell's equations. In the language of a Lagrangian approach this amounts to find a correct criterion which may allow one to obtain the "interaction Lagrangian" under Eq. (16), after taking into account the observations under Eqs. (7) and (8). To those who could object that this could be looked at as an unessential nominalistic *distinguo*, we emphasize that the problem is not of a purely logic valence but rather of a truly semantic relevance, due to the almost exclusive formal treatment which has been followed up to now. To start with, upon looking at Eqs. (14) and (20), we notice that a point-like charge $|Q|$ is allotted to be defined, the relative density function being $\rho(\boldsymbol{x}) = |Q|\,\delta(\boldsymbol{x})$ (the absolute value is introduced so that the Lagrangian we are looking for can be used to construct an action functional whose spectrum is well defined, *i.e.*, it is non-negative almost everywhere). Then, accounting for the fact that the "physical" fields $\boldsymbol{E}$ and $\boldsymbol{H}$ jointly enter in Eqs. (11) and (12), the first being a pure vector, the other a pseudo-vector, the simplest pure vector (as should be the time derivative of a linear momentum) which can be builded up via the three vectors, namely $\boldsymbol{E}$, $\boldsymbol{H}$ and $\dot{\boldsymbol{x}}$, the velocity of the moving point-like charge $|Q|$, is the linear superposition

$$|Q|\,(\boldsymbol{E} + \dot{\boldsymbol{x}}_{(c=1)} \wedge \boldsymbol{H}). \tag{24}$$

With this in mind, we can look for the "interaction Lagrangian" through the equation:

$$\frac{d}{dt}(\boldsymbol{\nabla}_{\dot{\boldsymbol{x}}}\mathcal{L}) = \boldsymbol{\nabla}_{\boldsymbol{x}}\mathcal{L} \equiv |Q|\,(\boldsymbol{E} + \dot{\boldsymbol{x}} \wedge \boldsymbol{H}), \qquad \mathcal{L} = m\frac{\dot{x}^2}{2} - \mathcal{L}_{(INT)} \tag{25}$$

Inserting Eqs. (9) and (13) into the r.h.s. of Eq. (25), we have:

$$|Q|\,(\boldsymbol{E} + \dot{\boldsymbol{x}} \wedge \boldsymbol{H}) \rightarrow |Q|\,(-\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}_{\boldsymbol{x}} f + \dot{\boldsymbol{x}} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{A})) \tag{26}$$

We observe that, due to the two well-known vectorial identities, namely

$$\dot{\boldsymbol{x}} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{A}) = \boldsymbol{\nabla}(\dot{\boldsymbol{x}} \cdot \boldsymbol{A}) - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\dot{\boldsymbol{x}} - (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{A} - \dot{\boldsymbol{x}} \wedge (\boldsymbol{\nabla} \wedge \dot{\boldsymbol{x}}) = \boldsymbol{\nabla}(\dot{\boldsymbol{x}} \cdot \boldsymbol{A}) - (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{A}$$

and

$$\frac{d\boldsymbol{A}}{dt} = (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \partial_t \boldsymbol{A}$$

Eq. (25) can be written as

$$\frac{d}{dt}(m\dot{\boldsymbol{x}} + |Q|\,\boldsymbol{A}) = (\boldsymbol{\nabla}(\dot{\boldsymbol{x}} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}_{\boldsymbol{x}} f')\,|Q| \tag{27}$$

from which Eq. (16) is easily recovered, with $\Phi = \boldsymbol{\nabla}_{\boldsymbol{x}} f'$ and $f' = f - \beta \equiv f - \partial_t(\beta t)$, in accordance with Eq. (19).

To conclude this note we have to observe that, although many tracks to take advantage of the dynamical equivalence of Lagrangians differing up to the total time derivative of an arbitrary function of coordinates and time when affording the treatment of a theory of the electromagnetic field can be found in some highly qualified literature [12], we still claim the original character of the idea put forward here. Indeed, we don't start from an "interaction Lagrangian" and then operate a gauge transformation to discover that the initial form is changed up to the total time derivative of the gauge function as happens in any approach following the reduction theory initiated by Marsden [13] and primarily suggested by Dirac [6]. We stress once more that the dynamical equivalence in question is just our starting point. In any case, as for the spirit of the discussion, we feel that we can conclude quoting once more Henneaux and Teitelboim [14] when they assess that:

"Physical theories of fundamental significance tend to be gauge theories. These are theories in which the physical system being dealt with is described by more variables than there are physically independent degrees of freedom. The physically meaningful degrees of freedom then reemerge as being those invariant under a transformation connecting the variables (gauge transformation). Thus, one introduces extra variables to make the description more transparent and brings in at the same time a gauge symmetry to extract the physically relevant content. It is a remarkable occurrence that the road to progress has invariably been towards enlarging the number of variables and introducing a more powerful symmetry rather than conversely aiming at reducing the number of variables and eliminating the symmetry".

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