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GENERALIZED PLANE GRAVITATIONAL WAVES OF NON-SYMMETRIC UNIFIED FIELD THEORIES IN PLANE SYMMETRY

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ABSTRACT. In this paper we investigated the plane wave solutions of both the weak and strong non-symmetric unified field equations of Einstein and Bonner in a generalized plane symmetric space-time in the sense of Taub [*Ann. Math.* **53**, 472 (1951)] for plane gravitational waves. We show that the plane wave solutions of Einstein and Bonner field equations exist in plane symmetry.

1. Introduction

Takeno [1] has investigated the solutions of plane gravitational waves of the field equations in general relativity and those of the various field equations in non-symmetric unified field theories for his space-time. However, Takeno [1] has first mooted the idea of generalization of space-time metric. He proposed that all components of the metric tensor may be chosen to be arbitrary functions of z and t, but not those of $Z = z - t$ or (t/z)-type waves. He stated that the solutions of various field equations may be found out, though the calculations involved may be very complicated. This approach motivates that the results corresponding to plane gravitational waves can be deduced by suitable choice of the phase function Z.

Motivated by the work of Takeno we have deduced the generalized plane symmetric metric in the sense of Taub [2]:

$$ds^2 = -A\left(dx^2 + dy^2\right) - \phi^2 B dz^2 + B dt^2, \tag{1}$$

where A, B, and ϕ are functions of Z.

Accordingly the space-time metric (1) will become a generalized space-time if the function ϕ has not necessarily taken as a function of Z. Recently we have found the wave solutions of field equations of general relativity in the space-time (1). The theory of plane wave solutions or the generalized plane gravitational wave solutions of the field equations in general relativity and non-symmetric unified field theories have been studied by many investigators as Takeno [1, 3–5], Ghosh [6], Pandey [7], Lal and Shafiullah [8], Katore *et al.* [9], and Bhoyar *et al.* [10].

In this paper we have studied the generalized plane gravitational waves of the field equations in general relativity and those of the field equations in non-symmetric unified field theories of Einstein [11] and Bonnor [12] in the space-time metric (1). The paper is organized as follows: Section 2 contains the metric and field equations. The solutions of the non-symmetric metric tensor g_{ij} are given in Section 3. Section 4 deals with the solutions of $g_{ij;k}$, the covariant differentiation of the metric tensor, and the affine connection Γ^k_{ij}. The solutions of the unified field equations (E_1) and (E_2) are obtained in Sections 5 and 6, respectively. The solutions of Bonnor equations are considered in Section 7. Section 8 deals with particular cases in which some exact solutions of the field equations in these theories are obtained by using a definite choice of the functions ϕ and the electromagnetic field tensor components ρ and σ. The last section contains the conclusions.

2. Metric and field equations

We have introduced the generalized plane gravitational waves of the field equations in general relativity in the space-time given by

$$ds^2 = -A\left(dx^2 + dy^2\right) - \phi^2 B dz^2 + B dt^2, \tag{2}$$

where A and B are functions of Z and $\phi = \frac{Z_{,3}}{Z_{,4}} = \frac{u}{v}$ ($Z_{,i} = \frac{\partial Z}{\partial x^i}, \quad u = Z_{,3} \neq 0$ and $v = Z_{,4} \neq 0$). The quadratic form $A\left(dx^2 + dy^2\right)$ is positive definite, $A > 0$, and ϕ is real. The electromagnetic field is composed of plane waves which are transverse electromagnetic waves (or principal waves) of electric and magnetic fields propagating in one direction with unit velocity. Einstein's non-symmetric unified field theory predicts a non-linear interaction between electromagnetic waves.

The field equations of Einstein's [11] unified field theory are:

Set (E_1)

$$g_{ij;k} \equiv g_{ij,k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{kj} = 0 \tag{3}$$

$$\Gamma_k = \Gamma^s_{[ks]} = 0 \tag{4}$$

$$R_{(ij)} = 0 \tag{5a}$$

$$R_{[ij],k} + R_{[jk],i} + R_{[ki],j} = 0 \tag{5b}$$

Set (E_2)

$$\text{Eq. (3), Eq. (4), and } R_{ij} = 0 \tag{6}$$

where

$$R_{ij} = \Gamma^s_{ij,s} - \Gamma^s_{is,j} - \Gamma^s_{tj}\Gamma^t_{is} + \Gamma^s_{ts}\Gamma^t_{ij} \tag{7}$$

is the generalized Ricci tensor; round and square brackets enclosing a pair of indices denote their symmetric and skew-symmetric parts, respectively. (E_1) and (E_2) are known as weak and strong field equations.

The field equations of Bonnor [12] are given by the field equations (3), (4) and

$$R_{(ij)} + p^2 U_{(ij)} = 0 \tag{8a}$$

$$(R_{[ij],k} + R_{[jk],i} + R_{[ki],j}) + p^2(U_{[ij],k} + U_{[jk],i} + U_{[ki],j}) = 0 \tag{8b}$$

where p is an arbitrary real or imaginary constants and $U_{ij}$is given by

$$U_{ij} = g_{[ji]} - g^{[mn]} g_{im} g_{nj} + \frac{1}{2} g^{[mn]} g_{nm} g_{ij}, \tag{9}$$

where g^{ij} is the contravariant tensor defined from g_{ij} by $g_{ij} g^{kj} = \delta_i^k$.

3. Solution of non-symmetric g_{ij}

The unified field theories are the consistent physical theories describing geometrical structure of unification of the fields of gravitation and electromagnetism. In these theories, the fundamental tensor is a non-symmetric metric tensor

$$g_{ij} = g_{(ij)} + g_{[ij]} = h_{ij} + f_{ij}, \tag{10}$$

where $g_{(ij)} = h_{ij}$ is the symmetric part of g_{ij} representing a field of gravitation and $g_{[ij]} = f_{ij}$ is the skew-symmetric part of g_{ij} signifying the electromagnetic field. Therefore, following the method of Takeno [1, 3–5], we have:

I Ikeda [13] introduced a relation between $F_{ij}$and g_{ij}(=h_{ij}) by the equation

$$F_{ij} = \frac{1}{2} \varepsilon_{ijkh} \sqrt{-g} g^{kh}, \quad g = (\det g_{(ij)}), \tag{11}$$

where $\varepsilon_{ijkh} = \pm 1$ is the Levi-Civita symbol and F_{ij} is the electromagnetic field tensor whose components are

$$F_{12} = F_{34} = 0, \quad uF_{41} = -vF_{13} = uv\sigma_1, \quad uF_{42} = -vF_{23} = uv\rho_1, \tag{12}$$

where ρ_1 and σ_1 are the functions of Z.

II The space-time defined by $g_{(ij)} = h_{ij}$ is given by Eq. (2).

III $g = -mB^2\phi^2$, where $m = A^2$.

IV The skew-symmetric part of $f_{ij}$of g_{ij}, from (10), (I) and (II), is given by

$$f_{12} = f_{34} = 0, \quad f_{13} = \phi f_{14} = u\rho, \quad f_{23} = \phi f_{24} = u\sigma, \tag{13}$$

where $\rho = \frac{(A\rho_1)}{\sqrt{m}}$ and $\sigma = \frac{(-B\sigma_1)}{\sqrt{m}}$ are functions of Z.

Therefore from Eqs. (2) and (13), we obtain

$$g_{ij} = h_{ij} + f_{ij} = \begin{bmatrix} -A & 0 & u\rho & v\rho \\ 0 & -A & u\sigma & v\sigma \\ -u\rho & -u\sigma & -B\phi^2 & 0 \\ -v\rho & -v\sigma & 0 & B \end{bmatrix}, \tag{14}$$

satisfying $g = h = -mB^2\phi^2$ and

$$g^{ij} = \begin{bmatrix} -\frac{A}{m} & 0 & \frac{uU}{\phi^2} & -vU \\ 0 & -\frac{A}{m} & \frac{uV}{\phi^2} & -vV \\ -\frac{uU}{\phi^2} & -\frac{uV}{\phi^2} & -\frac{1}{B\phi^2} + \frac{v^2W}{\phi^2} & 0 \\ vU & vV & 0 & \frac{1}{B} + \frac{v^2W}{\phi^2} \end{bmatrix}, \tag{15}$$

where $U = \frac{(A\rho)}{mB}$, $V = \frac{(A\sigma)}{mB}$, and $W = \frac{A(\sigma^2+\rho^2)}{mB^2}$.

4. Solutions of $g_{ij;k} = 0$ and Γ^k_{ij}

Let us assume

$$\Gamma^k_{ij} = p^k_{ij} + q^k_{ij}, \tag{16}$$

where $p^k_{ij} = \Gamma^k_{(ij)}$ and $q^k_{ij} = \Gamma^k_{[ij]}$ are the symmetric and skew-symmetric parts of Γ^k_{ij}, respectively.

Assuming $E^k_{ij} = g_{ij,k}$ with $i = j$, $E^{k+}_{ij} = g_{ij;k} + g_{ji;k}$, $E^{k-}_{ij} = g_{ij;k} - g_{ji;k}$ with $i < j$ and Eqs. (3) and (14), we have 64 equations in p and q form. Following the concept of Takeno, Ikeda and Abe to solve the 64 equations, we first express all p's in terms of q's and $\left\{ \begin{array}{c} k \\ ij \end{array} \right\}$ by the formula

$$p^k_{ij} = \left\{ \begin{array}{c} k \\ ij \end{array} \right\} + h^{kl}(q^m_{li} f_{jm} + q^m_{lj} f_{im}) \quad (k = 1, 2, 3, 4), \tag{17}$$

where $\left\{ \begin{array}{c} k \\ ij \end{array} \right\}$ is the Christoffel symbol of second kind calculated from h_{ij}, and h^{ik} is defined by $h_{ij}h^{ik} = \delta^j_k$. The non-vanishing components of $\left\{ \begin{array}{c} k \\ ij \end{array} \right\}$ become as follows:

$$\left\{ \begin{array}{c} 3 \\ 11 \end{array} \right\} = -\frac{\bar{A}u}{2B\phi^2}, \left\{ \begin{array}{c} 4 \\ 11 \end{array} \right\} = \frac{\bar{A}v}{2B}, \left\{ \begin{array}{c} 3 \\ 22 \end{array} \right\} = -\frac{\overline{A}u}{2B\phi^2}, \left\{ \begin{array}{c} 4 \\ 22 \end{array} \right\} = \frac{\overline{A}v}{2B} \tag{18a}$$

$$\left\{ \begin{array}{c} 1 \\ 13 \end{array} \right\} = \frac{(\overline{A}A)u}{2m}, \left\{ \begin{array}{c} 1 \\ 14 \end{array} \right\} = \frac{(\overline{A}A)v}{2m}, \left\{ \begin{array}{c} 3 \\ 33 \end{array} \right\} = \frac{\overline{(\phi^2 B)}\, u}{2B\phi^2}, \left\{ \begin{array}{c} 4 \\ 34 \end{array} \right\} = \frac{\overline{B}\, u}{2B} \tag{18b}$$

$$\left\{ \begin{array}{c} 2 \\ 23 \end{array} \right\} = \frac{(A\overline{A})u}{2m}, \left\{ \begin{array}{c} 2 \\ 24 \end{array} \right\} = \frac{(A\overline{A})v}{2m}, \left\{ \begin{array}{c} 3 \\ 34 \end{array} \right\} = \frac{\overline{(\phi^2 B)}\, v}{2B\phi^2} \tag{18c}$$

$$\left\{ \begin{array}{c} 4 \\ 33 \end{array} \right\} = \frac{\overline{(\phi^2 B)}\, v}{2B}, \left\{ \begin{array}{c} 3 \\ 44 \end{array} \right\} = \frac{\overline{B}\, u}{2B\phi^2} \left\{ \begin{array}{c} 4 \\ 44 \end{array} \right\} = \frac{\overline{B}\, v}{2B} \tag{18d}$$

where a bar over a letter denotes the derivative with respect to Z. Using Eqs. (16), (17), and (18) in Eq. (3), we find the 24 components of q^k_{ij}:

$$uq^3_{13} = -vq^4_{13} = -\frac{uv^2\alpha}{B} \quad , \quad uq^3_{14} = -vq^4_{14} = -\frac{v^3\alpha}{B}, \tag{19a}$$

$$uq^3_{23} = -vq^4_{23} = -\frac{uv^2\beta}{B} \quad , \quad uq^3_{24} = -vq^4_{24} = -\frac{v^3\beta}{B}, \tag{19b}$$

where

$$\alpha = \overline{\rho} - \frac{\rho(\bar{A}A)}{2m} - \frac{\rho\,\overline{B}}{B} + \rho\left(\frac{v_{,4}}{v^2}\right) \tag{20}$$

and

$$\beta = \overline{\sigma} - \frac{\sigma(A\overline{A})}{2m} - \frac{\sigma\overline{B}}{B} + \sigma\left(\frac{v_{,4}}{v^2}\right). \tag{21}$$

Using Eqs. (19) we obtained the components of p^k_{ij} in terms of A, B, ρ and σ and their derivatives. This completes the solutions of Eq. (17). Substituting the relevant quantities of p^k_{ij} and q^k_{ij} into Eq. (16), we obtain the affine connection Γ^k_{ij} as

$$\Gamma^k_{11} = \left[0\ ,\ 0\ ,\ -\frac{\bar{A}u}{2B\phi^2}\ ,\ \frac{\bar{A}v}{2B}\right] \quad \Gamma^k_{\substack{12\\21}} \equiv [0\ ,0\ ,\ 0\ ,\ 0] \tag{22a}$$

$$\Gamma^k_{\substack{13\\31}} \equiv \left[\frac{(\bar{A}A)u}{2m}\ ,0,0,0\right] \quad \Gamma^k_{\substack{14\\41}} \equiv \left[\frac{(\bar{A}A)v}{2m}\ \ ,\ 0\ ,\ \mp\frac{v^3\alpha}{uB},\ \pm\ \frac{v^2\alpha}{B}\right] \tag{22b}$$

$$\Gamma^k_{22} = \left[0\ ,\ 0\ ,\ -\frac{\overline{A}u}{2B\phi^2}\ ,\ \frac{\overline{A}v}{2B}\right] \quad \Gamma^k_{\substack{23\\32}} \equiv \left[0\ ,\ \frac{(A\overline{A})u}{2m}\ ,\ \mp\frac{v^2\beta}{B}\ ,\pm\ \frac{uv\beta}{B}\right] \tag{22c}$$

$$\Gamma^k_{\substack{24\\42}} \equiv \left[0\ \ ,\ \frac{(A\overline{A})v}{2m}\ ,\ \mp\frac{v^3\beta}{uB},\pm\ \frac{v^2\beta}{B}\right] \quad \Gamma^k_{33} \equiv \left[0\ ,\ 0\ ,\ \frac{\overline{(\phi^2B)}\,u}{2\phi^2B}\ ,\ \frac{\overline{(\phi^2B)}\,v}{2B}\ \right] \tag{22d}$$

$$\Gamma^k_{34} = \Gamma^k_{43} \equiv \left[0\ ,\ 0\ ,\ \frac{\overline{(\phi^2B)}\,v}{2\phi^2B}\ ,\ \frac{\overline{B}\,u}{2B}\ \right] \quad \Gamma^k_{44} \equiv \left[0\ ,\ 0,\ \frac{\overline{B}\,u}{2\phi^2B}\ ,\ \frac{\overline{B}\,v}{2B}\ \right] \tag{22e}$$

5. Solution of the field equations $\Gamma_k = \Gamma^s_{[ks]} = 0$

Using Γ^k_{ij} from Eqs. (22) in Eq. (4), we find that Γ_1 and Γ_2 are satisfied identically for $t = 1$ and 2, while for $t = 3$ and 4, Eq. (4) gives

$$v\Gamma_3 = u\Gamma_4 = 0. \tag{23}$$

6. Solutions of Einstein's field equations ($\mathbf{E}_1$) and ($\mathbf{E}_2$)

Using Eq. (23) in Eq. (7), we find the components of the generalized Ricci tensor R_{ij}:

$$R_{33} = -u^2\left[\Theta - \frac{\phi_{44}}{uv}\right], R_{44} = -v^2\left[\Theta + \frac{\phi_{44}}{uv}\right], R_{34} = R_{43} = -uv\ \ [\Theta]\,, \tag{24}$$

where

$$\xi = \frac{\overline{\overline{m}}}{2m} - \frac{\bar{m}^2}{4m^2} - \frac{\bar{m}\overline{B}}{2mB} \tag{25}$$

and

$$\Theta = \xi + \frac{\bar{m}}{2m}\left(\frac{v_{,4}}{v^2}\right). \tag{26}$$

In view of Eq. (24), the strong field equation (2.5) yields:

$$\phi_{44} = 0 \tag{27a}$$

$$\xi + \frac{\bar{m}}{2m}\left(\frac{v_{,4}}{v^2}\right) = 0. \tag{27b}$$

In view of Eq. (24), the weak field equation (5a) yields Eqs. (27a) and (27b), while the field equation (5b) yields Eq. (27a).Thus g_{ij} given by Eq. (14) represent the generalized plane gravitational waves or wave solutions of Einstein's strong unified field equations (6) under the condition (27a) and (27b), while the same g_{ij} are the solutions of weak field equations (5a) and (5b) under the conditions (27a) in the space-time (2).

7. Solutions of Bonner field equations

The solutions of Bonner field equations (3) and (4), which are same as the first two Einstein's field equations, are given by Eqs. (14) and (23), respectively. The components of U_{ij}, obtained from Eqs. (9), (14) and (15), are the following:

$$\frac{U_{13}}{u} = -\frac{U_{31}}{u} = \frac{U_{14}}{v} = -\frac{U_{41}}{v} = -2\rho \tag{28a}$$

$$\frac{U_{23}}{u} = -\frac{U_{32}}{u} = \frac{U_{24}}{v} = -\frac{U_{42}}{v} = -2\sigma \tag{28b}$$

$$\frac{U_{33}}{u^2} = \frac{U_{34}}{uv} = \frac{U_{43}}{uv} = \frac{U_{44}}{v^2} = -\frac{2A(\sigma^2+\rho^2)}{m}. \tag{28c}$$

In view of Eqs. (24) and (28c), Bonner's unified field equation (8a) yields Eq. (27a) and

$$2m\xi + \bar{m}\left(\frac{v_{,4}}{v^2}\right) + 4p^2 A(\sigma^2+\rho^2) = 0, \tag{29}$$

Hence the g_{ij} given by Eq. (14) represent the wave solutions of Bonner's unified field equations (8a) under the conditions (27a) and (29) in the metric (2). These solutions give a relation between the gravitational and electromagnetic components. When p vanishes, the field equations of Bonnor reduce to (E_1). When the non-symmetric field vanishes, *i.e.*, the electromagnetic components ρ and σ vanish, the field equations (E_1) and (8b) reduce to Einstein's vacuum field equation $R_{(ij)} = 0$ if $\xi = 0$, *i.e.*, $\overline{\overline{m}} - \frac{\bar{m}^2}{2m} - \frac{\bar{m}\bar{B}}{B} = 0$.

8. Two special cases

It should be noted that the electromagnetic field tensor components ρ and σ are functions of Z. This is obvious because of the choice of the electromagnetic components ρ and σ. Therefore the relevant solutions (27a), (27b), and (29) obtained from the field equations of Einstein and Bonnor in the space-time (2) have a definite choice of the functions ϕ, ρ and σ. For this reason, we consider two particular cases.

Case (I): When $\rho = \rho(z-t)$, $\sigma = \sigma(z-t)$ and $\phi = \phi(z-t) = -1$, Eq. (2) reduces to

$$ds^2 = -A\left(dx^2+dy^2\right) - B\left(dz^2 - dt^2\right), \tag{30}$$

where A and B are arbitrary functions of Z (= $z-t$) when $\phi = -1$.

Case (II): When $\rho = \rho(t/z)$, $\sigma = \sigma(t/z)$ and $\phi = \phi(t/z) = -Z$, Eq. (2) reduces to

$$ds^2 = -A\left(dx^2+dy^2\right) - B\left(Z^2 dz^2 - dt^2\right), \tag{31}$$

where A and B are arbitrary functions of Z (= t/z) when $\phi = -Z$.

Hence we have the following theorems:

Theorem: A necessary and sufficient condition that the values of g_{ij} given by Eq. (14) for the metric (30) and (31) be a solution of Einstein's unified field equations (E_1) and (E_2) is $\xi = 0$.

Theorem: A necessary and sufficient condition that the values of g_{ij} from Eq. (14) for the metric (30) and (31) be a solution of Bonner's unified field equations (8a) and (8b) is

$$\xi + p^2 A(\rho^2+\sigma^2) = 0 \quad and \quad Z^2\xi + \frac{2p^2(\rho^2+\sigma^2)}{A} = 0.$$

9. Conclusions

In the present paper some exact solutions of the generalized plane gravitational waves of non-symmetric unified field theories given by Einstein and Bonner have been obtained in the space-time (2) introduced by us. The most remarkable feature is that, as far as the solutions dealt with in this paper are concerned with both the field equations of Einstein and Bonner, the role of electromagnetic field tensor components seem to be closely related with the geometry of space-time where the matter does not exist at all and the electromagnetic components are some functions of x, y and Z.

When p and the non-symmetric field f_{ij} vanish, we get pure gravitational field and the symmetric part becomes the gravitational potential in the same way as the metric tensor does in general relativity. In such cases, the weak field equations of both Einstein and Bonner become identical and, therefore, these field equations reduce to Einstein's vacuum field equation $R_{ij} = 0$ if $\xi = 0$.

It is observed that the gravitational field and the electromagnetic field which are propagating in the same z-direction and are quite independent of each other can coexist in the non-symmetric theory. Hence, the gravitational potential is affected by the existence of the electromagnetic field.

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