

SHEAR BANDS AS GROWING INSTABILITIES IN VISCOANELASTIC MEDIA WITH MEMORY

MARINA DOLFIN ^{a*} AND MAURO FRANCAVIGLIA ^{b†}

ABSTRACT. In this paper we investigate the critical conditions under which a small perturbation in an homogeneous continuum can possibly grows into a shear band instability. In particular, we analyze from a thermodynamical viewpoint the phenomenon of shear bands in viscoanelastic media with memory. It is emphasized, in the scientific literature, that the specific adopted rheology strongly affects the results so that a special attention has to be paid, also for engineering purposes, to the accuracy of the rheological model. Several well-known rheological model (for instance the so called Maxwell or Jeffreys media) are particular cases of the general model we adopt in the paper to analyze shear bands. Instability conditions, giving rise to shear bands formation, are obtained by introducing small perturbations around an homogeneous deformation into the system of differential equations governing the problem of homogeneous deformations in the considered continuous medium; as a result a non-homogeneous linear dynamical system is obtained whose stability is analyzed. A research perspective in view of a possible comparison with experimental results is proposed; in particular the simple methodology proposed in the paper should be applied in view of using the phenomenon of the initiation of shear bands to calculate the thermomechanical coefficients of real materials.

1. Introduction

The phenomenon of localization of plastic flow in small but finite deformation zones is known as “shear bands”. Experiments show that even if a body is initially and globally (i.e. macroscopically) homogeneous and also if it undergoes a homogeneous deformation, its substructure (at the microscopic level of observation) could cause an additional flow of plastic nature. From a physical viewpoint this means that a smoothly varying deformation pattern often gives way, rather abruptly, to a pattern exhibiting a loss of smoothness in the form of jump discontinuities in the displacement gradients across specific curved surfaces, while the displacements themselves remain continuous across these surfaces. It is experimentally seen that the deformation inside the zone of plastic localization is predominantly due to a shear parallel to the interface of the region of localization and the adjacent material; this is why the localized deformations are termed as “shear bands”.

[†]*In memoriam:* Prof. Mauro Francaviglia, a highly esteemed corresponding member of the Accademia Peloritana dei Pericolanti, passed away unexpectedly while this paper was being published.

In rheology, the macroscopic anelastic behavior is supposed to be due to local inhomogeneities in the material distributions and/or deformations. Such local inhomogeneities in the material distribution are at the microscale (where the microscale needs not to be, and it is not in our case, of atomic or molecular dimension). Once such bands occur they persist and the strain inside the material could eventually become very large, leading to fractures (Terzaghi and Peck 1984). Shear bands are considered an important precursor to the occurrence of a fracture (which is a jump in the displacements themselves). Experimentally observed phenomena are, for instance, the formation of shear bands in geotechnical structures due to ground motion and the onset of localization is often considered as the point of inception of rupture for many engineering purposes (Atkinson 1997; Lambe and Whitman 2005). It is therefore important, also for engineering purposes, to find the critical stress level at which they first appear together with their geometry, although a detailed analysis about shear bands geometry is beyond the scope of this paper. It is to be said that we focus on small disturbances which can go on for a long time resulting in shear band instability: for instance it is the case of a building near a road usually crossed by heavy vehicles, so that this model it is not suited for instance to analyze phenomena resulting from ground motion due to seismic waves.

It is often reported in the literature how strongly the rheology of the model affects the results in shear bands instability. In this paper we adopt a rheological model obtained with the procedures of the thermodynamic of irreversible processes with internal variables; the general framework, which applies also for finite deformations, has been extensively analyzed by the authors in previous papers (Dolfin, Francaviglia, *et al.* 2012; Ciancio *et al.* 2008). From a physical viewpoint we are considering a viscoanelastic media with memory. The general theory deals with several microscopic phenomena giving rise to anelastic strains (slip, dislocations, etc.) which are introduced as internal variables contributing to the total deformation gradient. The methods of non-equilibrium thermodynamics with internal variables allows one to obtain explicit expressions for the different involved stress tensors and a general stress-strain relation which covers many well-known particular cases. From the viewpoint of the mathematical model the starting point is that we have to introduce a length-scale parameter related to the material substructure into the macroscopic material constitutive description: in our case an internal variable (see, e.g., Maugin and Muschik (1994a) and Maugin and Muschik (1994b)) which we choose to be a second-order displacement gradient (accounting for non local effects). Moreover, the internal variables we introduce would help to describe the dissipation effects when one would like to examine the model exploiting the dissipation inequality (Maugin 1990; Dolfin, Francaviglia, *et al.* 2012; Dolfin *et al.* 2012, 2010). In Section 2 we introduce the thermodynamical model that we adopt to characterize the rheology of the considered medium: we introduce the basic physical assumptions, derive the explicit forms of the balance equations of internal energy and entropy and the general stress-strain relation by the procedure of irreversible thermodynamics with internal variables. In Section 3 we analyze the instability of small perturbations introduced in the system of differential equations characterizing a viscoanelastic medium of order 1 with memory. In Section 4 we study the particular case of planar deformation introducing also some experimental results taken from the literature in view of a possible comparison with the theory. Finally, in Section 5 we propose a possibility of

using the simple methodology illustrated in the paper to obtain values of the rheological coefficients in the case of specimen of real materials.

2. The rheological model

2.1. Basic assumptions. Following Kluitenberg (1968), we adopt a rheological equation for a viscoanelastic medium of order 1 with memory. In this case we consider only shape memory since only shear phenomena are taken into account because of the physical problem treated. The general theory deals with several microscopic phenomena giving rise to anelastic strains (slip, dislocations, etc.) so that the k -th one is characterized by a tensor $\gamma^{(k)}$. Then the *total deformation gradient* γ is given by the sum of an elastic part characterized by the tensor $\gamma^{(el)}$ and an anelastic one given as the sum of the k anelastic strains as in the following:

$$\gamma = \gamma^{(el)} + \sum_{k=1}^n \gamma^{(k)}. \quad (1)$$

Moreover a flow analogous to the viscous flow of ordinary fluids is assumed. Since the n microscopic phenomena are assumed to be different, the n anelastic strains are introduced as internal degrees of freedom (“internal variables”) (Maugin and Muschik 1994a; Kluitenberg 1962). The specific entropy s will depend on the specific internal energy u , the elastic strain tensor and the k anelastic strains

$$s = s(u, \gamma^{(el)}, \gamma^{(1)}, \dots, \gamma^{(n)}). \quad (2)$$

Because of the decomposition (1) we can assume the dependance of s in term of the total strain tensor:

$$s = s(u, \gamma, \gamma^{(1)}, \dots, \gamma^{(n)}). \quad (3)$$

From (3), we can write the differential of s

$$ds = \partial_u s du + \partial_\gamma s d\gamma + \sum_{k=1}^n \partial_{\gamma^{(k)}} s d\gamma^{(k)}. \quad (4)$$

We make the following positions

$$T^{-1} = \partial_u s(u, \gamma, \gamma^{(1)}, \dots, \gamma^{(n)}) \quad (5)$$

where T is the absolute temperature;

$$-\frac{1}{\rho} T^{-1} \sigma^{(eq)} = \partial_\gamma s(u, \gamma, \gamma^{(1)}, \dots, \gamma^{(n)}) \quad (6)$$

where $\sigma^{(eq)}$ is defined as the equilibrium stress tensor which is of a thermoelastic nature and ρ is the mass density; moreover

$$\frac{1}{\rho} T^{-1} \sigma^{(k)} = \partial_{\gamma^{(k)}} s(u, \gamma, \gamma^{(1)}, \dots, \gamma^{(n)}) \quad (7)$$

where $\sigma^{(k)}$, $k = 1, \dots, n$ is the affinity-stress tensors conjugate to $\gamma^{(k)}$, $k = 1, \dots, n$.

With positions (5)-(7) the differential of the entropy (3) can be written as

$$Tds = du - v\sigma^{(eq)} : d\gamma + v \sum_{k=1}^n \sigma^{(k)} : d\gamma^{(k)} \quad (8)$$

where $v = \rho^{-1}$ is the specific volume and the colon represents total contraction over the indexes of tensors. Using eq.(2) one may also write eq. (8) as

$$Tds = du - v\sigma^{(eq)} : d\gamma^{(el)} + v \sum_{k=1}^n \sigma_{(m)}^{(k)} : d\gamma^{(k)} \quad (9)$$

where the following tensor has been introduced:

$$\sigma_{(m)}^{(k)} = \sigma^{(k)} - \sigma^{(eq)} \quad (10)$$

which will be called the memory-stress tensor conjugate to $\gamma^{(k)}$.

2.2. Thermodynamical balance equations. In this section we will derive the entropy balance equation; its general form is

$$\rho \dot{u} = \nabla \cdot \mathbf{J}^{(q)} + \sigma : \dot{\gamma} \quad (11)$$

where $\mathbf{J}^{(q)}$ is the heat flux, σ is the mechanical stress tensor and the superimposed dot stands for the total derivative with respect to time. From eq.(8) one can deduce the Gibbs relation

$$T\dot{s} = \dot{u} - v\sigma^{(eq)} : \dot{\gamma} + v \sum_{k=1}^n \sigma^{(k)} : \dot{\gamma}^{(k)}. \quad (12)$$

Eq. (11), together with (12), gives:

$$\rho T\dot{s} = -\nabla \cdot \mathbf{J}^{(q)} + \sigma^{(vi)} : \dot{\gamma} + \sum_{k=1}^n \sigma^{(k)} : \dot{\gamma}^{(k)} \quad (13)$$

where we have introduced the *viscous stress tensor*:

$$\sigma^{(vi)} = \sigma - \sigma^{(eq)}. \quad (14)$$

which is analogous to the viscous stresses in ordinary (Newtonian) fluids. By using the vectorial identity $\nabla \cdot \left(\frac{\mathbf{J}^{(q)}}{T} \right) = \frac{1}{T} \nabla \cdot \mathbf{J}^{(q)} + \mathbf{J}^{(q)} \cdot \nabla T^{-1}$, eq.(13) can be written in the usual balance form:

$$\rho \dot{s} = -\nabla \cdot \left(\frac{\mathbf{J}^{(q)}}{T} \right) + \Sigma_s \quad (15)$$

where the entropy production Σ_s is given by:

$$\Sigma_s = T^{-1} \left(-T^{-1} \mathbf{J}^{(q)} \cdot \nabla T + \sigma^{(vi)} : \dot{\gamma} + \sum_{k=1}^n \sigma^{(k)} : \dot{\gamma}^{(k)} \right) \quad (16)$$

under the form of the usual bilinear form given as a sum of products of fluxes and affinities (Maugin and Muschik 1994a).

By using the usual split of a generic tensor \mathbf{A} in deviator $\tilde{\mathbf{A}}$ and scalar part A

$$\mathbf{A} = \tilde{\mathbf{A}} + A\delta \quad (17)$$

where $A = \frac{1}{3} \sum_{\gamma} A_{\gamma\gamma}$ and δ is the Kronecher tensor, one can write the entropy production in a form which is especially useful for the splitting of shear and volumetric phenomena

$$\Sigma_s = T^{-1} \left(-T^{-1} \mathbf{J}^{(q)} \cdot \nabla T + \tilde{\boldsymbol{\sigma}}^{(vi)} : \dot{\boldsymbol{\gamma}} + \sum_{k=1}^n \tilde{\boldsymbol{\sigma}}^{(k)} : \dot{\boldsymbol{\gamma}}^{(k)} + 3\boldsymbol{\sigma}^{(vi)} \dot{\boldsymbol{\gamma}} + 3 \sum_{k=1}^n \boldsymbol{\sigma}^{(k)} \dot{\boldsymbol{\gamma}}^{(k)} \right). \quad (18)$$

A medium with the entropy of the form (2) will be called viscoanelastic of order n . It will be said without memory if one of the memory-stress tensor vanishes identically.

2.3. Phenomenological equations and stress-strain relation. In this section we introduce the phenomenological equations for the medium under consideration and derive the stress-strain relation, in the particular case of linearized equations of state and by regarding the phenomenological coefficients as constant. For the general case, i.e. anisotropic media of order n (see Kluitenberg 1968).

According to the usual procedures of non-equilibrium thermodynamics, one can write the phenomenological equations relating fluxes and affinities in the entropy production. In case of isotropic media of order 1, i.e. $n = 1$ (one internal variable only is considered), the phenomenological equations take the form (Maugin and Muschik 1994b)

$$\dot{\boldsymbol{\gamma}}^{(1)} = \eta_s^{(1,1)} \tilde{\boldsymbol{\sigma}}^{(1)} + \eta_s^{(1,0)} \dot{\boldsymbol{\gamma}} \quad (19)$$

$$\tilde{\boldsymbol{\sigma}}^{(vi)} = \eta_s^{(0,1)} \tilde{\boldsymbol{\sigma}}^{(1)} + \eta_s^{(0,0)} \dot{\boldsymbol{\gamma}} \quad (20)$$

$$\mathbf{J}^{(q)} = -\lambda \nabla T \quad (21)$$

The first two equations account for irreversible processes which are *shear anelasticity* (anelastic flow) and *shear viscosity* (viscous flow), together with the heat conduction.

For our purposes no volume anelasticity, viscosity and memory are taken into account. This assumption can be introduced also via the incompressibility condition

$$\text{tr} \boldsymbol{\gamma} = 0. \quad (22)$$

By using the phenomenological equations (19)-(21) the following stress-strain relation is obtained (details of the calculations can be found in Kluitenberg (1968)):

$$R_{(d)0}^{(\sigma)} \tilde{\boldsymbol{\sigma}} + \frac{d\tilde{\boldsymbol{\sigma}}}{dt} = R_{(d)0}^{(\gamma)} \tilde{\boldsymbol{\gamma}} + R_{(d)1}^{(\gamma)} \frac{d\tilde{\boldsymbol{\gamma}}}{dt} + R_{(d)2}^{(\gamma)} \frac{d^2\tilde{\boldsymbol{\gamma}}}{dt^2} \quad (23)$$

where “d” stands for “distortional”. The term in the first derivative of the displacement field with the rheological coefficient ($R_{(d)0}^{(\gamma)}$) is responsible for the memory effects. The term in the second derivative of the displacement field with the rheological coefficient ($R_{(d)2}^{(\gamma)}$) is responsible for the viscosity. The term with rheological coefficient $R_{(d)1}^{(\gamma)}$ is connected with

possible cross-effects between viscous and anelastic flow.

In previous papers the following inequalities are derived (see Ciancio *et al.* 2008):

$$R_{(d)0}^{(\sigma)} = a^{(1,1)} \eta_s^{(1,1)} \geq 0 \quad (24)$$

$$R_{(d)0}^{(\gamma)} = a^{(0,0)} [a^{(1,1)} - a^{(0,0)}] \eta_s^{(1,1)} \geq 0 \quad (25)$$

$$R_{(d)1}^{(\gamma)} = \{a^{(0,0)}(1 + 2\eta_s^{(0,1)}) + a^{(1,1)}[\eta_s^{(0,0)} \eta_s^{(1,1)} + (\eta_s^{(0,1)})^2]\} \geq 0 \quad (26)$$

$$R_{(d)2}^{(\gamma)} = \eta_s^{(0,0)} \geq 0 \quad (27)$$

The rheological model introduced above covers the following particular cases

- Anelastic media with memory (Poynting-Thomson or Standard Linear Solid)
- Viscoanelastic media without memory \Rightarrow Jeffreys media;
- Anelastic media without memory (Maxwell media);
- Viscoelastic media (Kelvin-Voigt media);
- Elastic media (Hooke media);
- Newtonian fluids.

3. Instability of small perturbations

We consider a homogeneous deformation characterized by a *deformation gradient* γ^0 and a *Cauchy stress tensor* σ^0 . We introduce small time and space fluctuations with respect to the homogeneous state by setting:

$$\gamma = \gamma^0 + \gamma'(\mathbf{x}, t) \quad (28)$$

with $\|\gamma'\| \ll 1$; here (\mathbf{x}, t) stand for position and time variable respectively. Perturbations are assumed of the form:

$$\gamma' = \gamma^* \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t] \quad (29)$$

where \mathbf{n} is the unit normal to the band, q is the wave number, γ^* is a constant amplitude, ω is the speed of growth and i is the imaginary unit. By using eq. (28) and making the position $\sigma' = \sigma - \sigma^0$, the rheological equation (23) in the perturbed state takes the form:

$$R_{(d)0}^{(\sigma)} \tilde{\sigma}' + \frac{d\tilde{\sigma}'}{dt} = R_{(d)0}^{(\gamma)} \tilde{\gamma}' + R_{(d)1}^{(\gamma)} \frac{d\tilde{\gamma}'}{dt} + R_{(d)2}^{(\gamma)} \frac{d^2\tilde{\gamma}'}{dt^2} \quad (30)$$

In the following we will omit the tilde and the symbol (d) in the coefficients for the sake of simplicity. By simple calculations one obtains the first derivative of the perturbed deformation gradient form eq.(29) as given by:

$$\frac{d\gamma'}{dt} = \gamma^* \omega \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t] + \mathbf{v} \cdot \{iq \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t]\} \gamma^* \otimes \mathbf{n}. \quad (31)$$

In our approximation (small disturbances) we can neglect the second term on the right hand side of (31), so that $\frac{d\gamma'}{dt} \equiv \frac{\partial \gamma'}{\partial t}$. In this approximation the second derivative of the displacement gradient reads:

$$\frac{d^2\gamma'}{dt^2} = \gamma^* \omega^2 \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t] \equiv \frac{\partial^2 \gamma'}{\partial t^2}. \quad (32)$$

By substituting (31) and (32) into the perturbed rheological equation (30) we obtain:

$$\frac{d\sigma'}{dt} = -R_0^{(\sigma)} \sigma' + [R_2^{(\gamma)} \omega^2 + R_1^{(\gamma)} \omega + R_0^{(\gamma)}] \gamma^* \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t]. \quad (33)$$

Now, at any given point of space (i.e., at fixed \mathbf{x}) we have obtained the following non-homogeneous linear system:

$$\frac{d\sigma'}{dt} = -R_0^{(\sigma)} \mathbf{I} \sigma' + \mathbf{f}(t) \quad (34)$$

where the “forcing” term $\mathbf{f}(t)$ is given by:

$$\mathbf{f}(t) = [R_2^{(\gamma)} \omega^2 + R_1^{(\gamma)} \omega + R_0^{(\gamma)}] \gamma^* \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t]. \quad (35)$$

The solution of the associated homogeneous equation reads

$$\sigma' = e^{-R_0^{(\sigma)} t} \delta \sigma_0 = e^{-R_0^{(\sigma)} t} \delta \sigma_0. \quad (36)$$

It is simple to test that the particular solution (36) of the dynamical system (34) is stable because of inequality (24). By using the variation of arbitrary constants method one obtains the general solution of the dynamical system (34) under the form:

$$\begin{aligned} \sigma &= e^{-R_0^{(\sigma)} t} \sigma_0 + e^{-R_0^{(\sigma)} t} \delta \int e^{R_0^{(\sigma)} t} \mathbf{f}(t) dt = \\ &= \int [R_2^{(\gamma)} \omega^2 + R_1^{(\gamma)} \omega + R_0^{(\gamma)}] \gamma^* \exp[iq(\mathbf{n} \cdot \mathbf{x}) + (\omega R_0^{(\sigma)} t)] dt \end{aligned} \quad (37)$$

By analyzing the second order equation in ω , i.e.:

$$R_2^{(\gamma)} \omega^2 + R_1^{(\gamma)} \omega + R_0^{(\gamma)} = 0 \quad (38)$$

we notice that when the following inequality is satisfied:

$$[R_1^{(\gamma)}]^2 - 4R_2^{(\gamma)} R_0^{(\gamma)} \leq 0 \quad (39)$$

then the real part of both solutions for ω is given by:

$$\omega_{real} = -\frac{R_1^{(\gamma)}}{2R_2^{(\gamma)}} < 0 \quad (40)$$

which is negative because of inequalities (26) and (27). If, on the contrary, the following inequality is satisfied:

$$[R_1^{(\gamma)}]^2 - 4R_2^{(\gamma)} R_0^{(\gamma)} > 0 \quad (41)$$

then the solutions for ω are both real and positive because of the inequalities (25), (26) and (27). One can then deduce that the values in the range

$$R_1^{(\gamma)} < -2\sqrt{R_2^{(\gamma)} R_0^{(\gamma)}}; R_1^{(\gamma)} > 2\sqrt{R_2^{(\gamma)} R_0^{(\gamma)}} \quad (42)$$

correspond to unstable solutions where inhomogeneous disturbances grow in time possibly giving rise to the formation of shear bands.

4. The case of planar deformations

In the case of planar deformations, the constant amplitude reduces to:

$$\gamma_{11}^* = -\gamma_{22}^*, \quad \gamma_{33}^* = 0 \quad (43)$$

and we can introduce an orientation angle θ for the shear band (Zbib and Aifantis 1988) (whose normal is \mathbf{n}) so that:

$$(n_1, n_2) = (-\sin\theta, \cos\theta) \quad (44)$$

where $\mathbf{n} = (n_1, n_2)$. Then the non-homogeneous dynamical system (34) reduces to:

$$\begin{aligned} \frac{d\sigma_{11}}{dt} &= -R_0^{(\sigma)} \sigma_{11} + \gamma_{11}^* [R_2^{(\gamma)} \omega^2 + R_1^{(\gamma)} \omega + R_0^{(\gamma)}] \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t], \\ \frac{d\sigma_{22}}{dt} &= -R_0^{(\sigma)} \sigma_{22} - \gamma_{11}^* [R_2^{(\gamma)} \omega^2 + R_1^{(\gamma)} \omega + R_0^{(\gamma)}] \exp[iq(\mathbf{n} \cdot \mathbf{x}) + \omega t]. \end{aligned} \quad (45)$$

Depending on the phenomenological assumptions on the rheological coefficients ($R_0^{(\sigma)}$, $R_0^{(\gamma)}$, $R_1^{(\gamma)}$, $R_2^{(\gamma)}$) one can find algebraic conditions under which the “eigenvalue” ω of equations (45) has a real positive part for given wave number q and orientation θ . These, in turn, ensures via (29) that the homogeneous state is unstable and that a periodic perturbation can possibly grow into a shear band instability. Maximizing the positive root ω with respect to θ allows then one to determine, for fixed q , the critical angle θ_c around which the instabilities grow.

The rheological coefficients of the introduced stress-strain relation are not yet known for specific materials; in the following we propose, basing on an example regarding a specific material, a simple methodology which would allow to calculate them. Experiments on aged maraging steel (Anand and Spitzig 1980) give the following results: in tension the critical angle is given by:

$$\theta_c = \pm(38 \pm 2)^0 \quad (46)$$

and in compression it is given by:

$$\theta_c = \pm(55 \pm 2)^0 \quad (47)$$

Therefore, any reasonable choice of the rheological coefficients should be such to be compatible with (46) and (47) (for the example of aged maraging steel used for the sake of illustration); the resulting conditions can be interpreted as physical restrictions on (45). The illustrated procedure should apply for other material choices.

5. Research perspectives

The aim of the theoretical model presented in the paper is of proposing a methodology to use the physical phenomenon of the initiation of shear bands in real materials to calculate the rheological coefficients of specific materials.

Acknowledgments

M. D. wishes to remember the deep and uncommon cleverness of Prof. Mauro Francaviglia, always surprising and stimulating, and expresses her personal sorrow and gratitude.

References

- Anand, L. and Spitzig, W. A. (1980). “Initiation of localized shear bands in plane strain”. *J. Mech. Phys. Solids* **28**(2), 113–128. DOI: [10.1016/0022-5096\(80\)90017-4](https://doi.org/10.1016/0022-5096(80)90017-4).
- Atkinson, J. (1997). *Geotecnica, meccanica delle terre e delle fondazioni*. McGraw-Hill.
- Ciancio, V., Dolfin, M., Francaviglia, M., and Preston, S. (2008). “Uniform materials and multiplicative decomposition of the deformation gradient in finite elasto-plasticity”. *J. Non-Equilib. Therm.* **33**(3), 199–234. DOI: [10.1515/JNETDY.2008.009](https://doi.org/10.1515/JNETDY.2008.009).
- Dolfin, M., Francaviglia, M., Preston, S., and Restuccia, L. (2010). “Material element model and the geometry of the entropy form”. *Int. J. Geom. Methods Mod. Phys.* **7**(6), 1021–1042. DOI: [10.1142/S0219887810004695](https://doi.org/10.1142/S0219887810004695).
- Dolfin, M., Francaviglia, M., Preston, S., and Restuccia, L. (2012). “Entropy form and the contact geometry of the material point model”. *Int. J. Geom. Methods Mod. Phys.* **9**(1), 1–33. DOI: [10.1142/S0219887812500132](https://doi.org/10.1142/S0219887812500132).
- Dolfin, M., Francaviglia, M., and Restuccia, L. (2012). “Material element model and dissipative processes for ferrimagnetic crystals with a non-Euclidean structure”. *Int. J. Geom. Methods Mod. Phys.* **9**(3), 1–18. DOI: [10.1142/S0219887812500259](https://doi.org/10.1142/S0219887812500259).
- Kluitenberg, G. A. (1962). “On rheology and thermodynamics of irreversible processes”. *Physica* **23**(11), 1173–1183. DOI: [10.1016/0031-8914\(62\)90064-2](https://doi.org/10.1016/0031-8914(62)90064-2).
- Kluitenberg, G. A. (1968). “A thermodynamic derivation of the stress-strain relations for Burgers media and related substances”. *Physica* **38**(4), 513–548. DOI: [10.1016/0031-8914\(68\)90002-5](https://doi.org/10.1016/0031-8914(68)90002-5).
- Lambe, T. W. and Whitman, R. V. (2005). *Meccanica dei terreni*. Flaccovio Ed.
- Maugin, G. A. (1990). “Internal variables and dissipative structures”. *J. Non-Equilib. Thermodyn.* **15**(2), 173–192. DOI: [10.1515/jnet.1990.15.2.173](https://doi.org/10.1515/jnet.1990.15.2.173).
- Maugin, G. A. and Muschik, W. (1994a). “Thermodynamics with internal variables. Part I. General Concepts”. *J. Non-Equilib. Thermodyn.* **19**(3), 217–249. DOI: [10.1515/jnet.1994.19.3.217](https://doi.org/10.1515/jnet.1994.19.3.217).
- Maugin, G. A. and Muschik, W. (1994b). “Thermodynamics with internal variables. Part II. Applications”. *J. Non-Equilib. Thermodyn.* **19**(3), 250–289. DOI: [10.1515/jnet.1994.19.3.250](https://doi.org/10.1515/jnet.1994.19.3.250).
- Terzaghi, K. and Peck, R. B. (1984). *Geotecnica*. U.T.E.T.
- Zbib, H. M. and Aifantis, E. C. (1988). “On the localization and postlocalization behavior of plastic deformation. I. On the initiation of shear bands”. *Res. Mechanica, Int. J. Struct. Mech. Mater. Sc.* **23**, 261–277.

^a Università degli Studi di Messina
Dipartimento di Ingegneria Civile, Informatica, Edile, Ambientale e Matematica Applicata
Contrada Di Dio (S. Agata), 98166 Messina, Italy

* Email: mdolfin@unime.it

^b Università degli Studi di Torino
Dipartimento di Matematica
Via Carlo Alberto, 10123 Torino, Italy

Communicated 30 May 2013; published online 5 September 2013

This article is an open access article licensed under a [Creative Commons Attribution 3.0 Unported License](https://creativecommons.org/licenses/by/3.0/)

© 2013 by the Author(s) – licensee *Accademia Peloritana dei Pericolanti* (Messina, Italy)