DOI: 10.1478/AAPP.91S1A11

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 91, Suppl. No. 1, A11 (2013)

THE DYNAMICS OF A LANDSLIDE

MARSHALL J. LEITMAN [a] AND PIERO VILLAGGIO [b*]

ABSTRACT. The onset of slip of a landslide may be mathematically modeled as the sudden detachment of a block of softer material initially bonded to the side of a mountain. Its subsequent motion is slowed by friction and by energy dissipation in the block due to its change of shape, which commences when the block reaches the bottom of the mountain. Since the bottom is a horizontal plane, the block will continue to slide along it until its kinetic energy is exhausted or it collides with an obstacle. A numerical example shows that the front of even a relatively small landslide can travel far from the base of the mountain.

To Giuseppe Grioli in recognition
of his mechanical taste and musical vocation

1. Introduction

For millennia humans have observed the onset of landslides: the sudden detachment of a layer of earth or mud situated on an incline and its subsequent downward slide until it is stopped by an obstacle in its path or by natural damping. Often the sliding earthy mass is slow and its volume small. Its flow, though steady, can be blocked effectively by artificial embankments, such as those that flank the roads in alpine regions. But the sudden fall of a huge mass of earth at high velocity can be catastrophic, resulting in the obliteration of a town or the damming of a river valley. Natural causes of landslides include the weakened resistance of the earthy mass by variation in cohesiveness, trains of seismic waves, and massive deforestation of the mountainside.

There is a huge literature on this subject, beginning with the classical memoir by Coulomb [1] (masterfully commented upon by Heyman [3]) and extended in the 19th Century by Rankine [5] and Culmann [2]. They, however, were concerned with establishing the critical equilibrium of an earth-mass before its detachment but not its subsequent descent and progressive disgregation. Studies of the dynamics of landslides are more recent. A suggestive account of the onset, development and arrest of some impressive landslides in the first half of the 20th Century was written by Terzaghi [7] but it is merely descriptive.

We here propose a simple, one-dimensional model of the creation and evolution of a landslide regarded as a block bonded to a layer on an inclined plane. When the bond is broken the block starts to slip down the incline until it reaches the horizontal bottom and continues to slide along the horizontal base. If the constitutive properties of the material and the coefficients of cohesion and friction are known, it is possible to describe the complete motion of the block from its initial detachment to its final configuration at standstill.

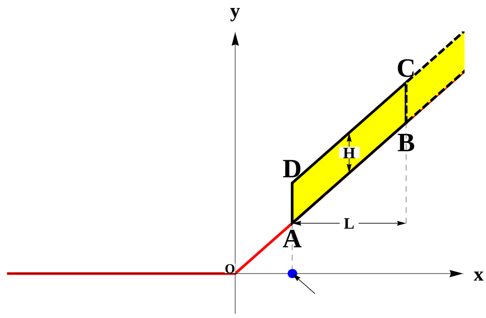

Figure 1. The sudden detachment of the landslide along $\overline{BC}$. The front is at x_0.

2. The Model and the Critical State

One of the most common geological situations giving rise to a landslide occurs when the flank of a mountain, regarded as a rigid inclined plane, is partially covered by a semi-infinite layer of soft material, the *front* of which, $\overline{AD}$, is initially located at a horizontal distance x_0 from the origin, O (Fig. 1). To define the geometry of the problem we introduce a Cartesian $(x, y)-$coordinate system as shown in Fig. 1. The half-line $(x > 0,\ y = x \tan\alpha)$ emanating from the origin with angle α $(0 < \alpha < \pi/2)$, from the positive $x-$axis, is the *flank* of the mountain and a plane of slip; the half-strip of height H and unit thickness starting at $\overline{AD}$ is the *region of potential detachment;* and the half-line $(x < 0,\ y = 0)$ emanating from the origin along the negative $x-$axis is the *base* of the mountain and also a plane of slip.

We first assume that the layer of material on the flank of the mountain is unbroken and consider the equilibrium of the sub-region $ABCD$ of horizontal length L according to slab theory (see Thomsen *et al* [8]). Refer now to the free-body diagram in Fig 2. The (vertical) weight of the block $ABCD$ is $W = \gamma HL$, where γ denotes the specific weight of the material. The downward vertical force W can be resolved into a tangential component, $W \sin\alpha$, and a normal component, $W \cos\alpha$. The latter induces a frictional tangential force $fW \cos\alpha$, where f is the coefficient of friction. Hence, tangential equilibrium of the block is achieved provided that a traction

$$N = W(\sin\alpha - f \cos\alpha) = \gamma HL(\sin\alpha - f \cos\alpha) \qquad (1)$$

is transmitted across the section $\overline{BC}$. We assume that the coefficient of friction and the slope of the incline are such that $N \geq 0$; thus,

$$f \leq \tan\alpha. \qquad (2)$$

Denote the horizontal and vertical components of N by $S_x = N \cos\alpha$ and $S_y = N \sin\alpha$. According to slab theory, the component S_y induces a uniform tangential stress in the slab of magnitude

$$\tau_{xy} = \frac{S_y}{H}. \qquad (3)$$

For sufficiently small values of τ_{xy} the layer remains in equilibrium on the incline. But, as soon as τ_{xy} reaches a critical value c, the *cohesion*, on some vertical section, a sudden

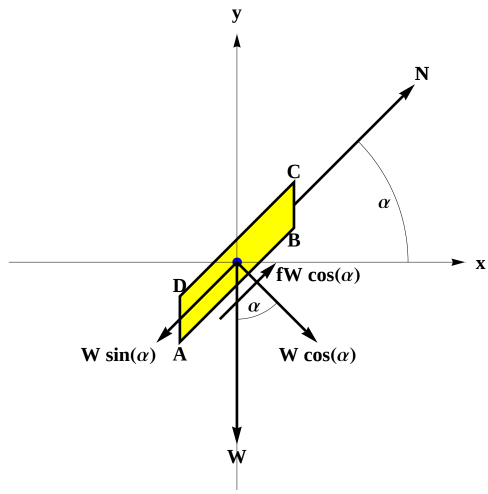

Figure 2. Free body diagram

detachment will occur there. If $\overline{BC}$ (Fig. 1) is such a section, the entire block, the parallelogram $ABCD$, is no longer attached to the rest of the layer and will begin to slide down the slope. This is the onset of the landslide.

From, Eq. (3) the condition of first detachment is

$$\tau_{xy} = \frac{S_y}{H} = \frac{N \sin \alpha}{H} = c, \tag{4}$$

called the *special critical state* by Sokolovsi [6]. From Eq. (1), the length, L, of the detaching block can be determined in terms of the other physical quantities: $\alpha,\ f,\ \gamma,$ and c.

3. The Descent of the Block

After the sudden detachment of the block at the section $\overline{BC}$ the motion of the block $ABCD$ will be that of a rigid body sliding down an inclined plane. Locate the block by its front, its forward face $\overline{AD}$, at a horizontal distance x from the origin. At the instant of detachment, $t_0 = 0$, the location is x_0. From this instant, until the block reaches the base of the mountain at $x = 0$, the motion is governed by the dynamical equation

$$M\ddot{x} = -N \cos \alpha, \tag{5}$$

where M is the mass of the block and the initial conditions are

$$x(0) = x_0, \qquad \dot{x}(0) = v_0 = 0. \tag{6}$$

Since the mass of the block is $M = \frac{\gamma}{g} HL$, where g is the acceleration of gravity, Eq. (5) becomes

$$\ddot{x} = -g(\sin \alpha - f \cos \alpha) \cos \alpha, \tag{7}$$

Quadrature of Eq. (7), using the initial data in Eq. (6), is immediate:

$$x(t) = x_0 - [g(\sin \alpha - f \cos \alpha) \cos \alpha] \frac{t^2}{2}. \tag{8}$$

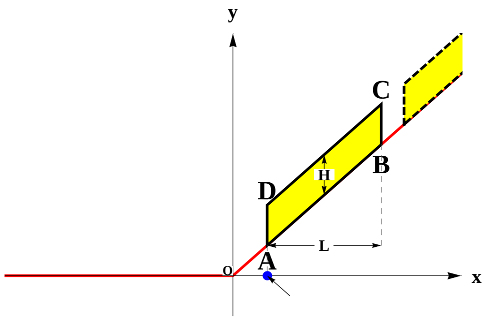

Figure 3. The landslide in progress on the mountainside

From Eq. (8), the time, t_1, at which the forward vertex A reaches the origin, O, and the corresponding velocity, $v_1 = \dot{x}(t_1)$ are

$$t_1 = \sqrt{\frac{2x_0}{g(\sin\alpha - f\cos\alpha)\cos\alpha}} \tag{9}$$

and

$$v_1 = \dot{x}(t_1) = -\sqrt{2x_0 g(\sin\alpha - f\cos\alpha)\cos\alpha}. \tag{10}$$

The assumption in Eq. (2) guarantees that these expressions are well-defined.

4. The Transition to the Plane

At time t_1, when the block makes first contact with the base (Fig. 4), the block begins a progressive permanent strain of pure shear as shown in Fig. 5. This process, due to decohesion, will stop at time t_2 when the rear vertex B reaches the origin, O (Fig. 6). The block $ABCD$, initially a parallelogram, has been permanently deformed into the rectangular block $ABCD$ given by $-L \leq x \leq 0$, $0 \leq y \leq H$. We assume that the velocity $v_1 = \dot{x}(t_1)$ is so high that the rear vertex B reaches the origin.

Decohesion and the resulting permanent strain is necessarily accompanied by energy dissipation. In our case we can compute the energy, $D_c(x)$, dissipated in terms of the location, x, of the forward vertex A by the formula (consistent with slab theory):

$$D_c(x) = -c\alpha HL\left(\frac{x}{L}\right),\ -L \leq x \leq 0, \tag{11}$$

where the angle α is measures the strain.

In addition to the energy dissipated by decohesion, there are frictional losses. Omitting the details, this energy loss, $D_f(x)$, is

$$D_f(x) = f\gamma HL^2\left(\frac{x}{L}\right)^2,\ -L \leq x \leq 0. \tag{12}$$

Energy is also gained during the transition: potential energy is is converted to kinetic energy as the massive block descends. The gain in kinetic energy, $G_{ke}(x)$, is found by computing the change in height of the center of mass of the block in terms of x and multiplying

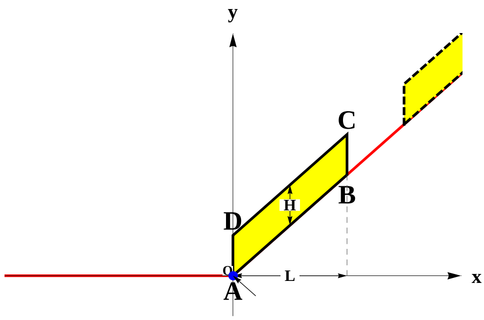

Figure 4. The onset of the transition

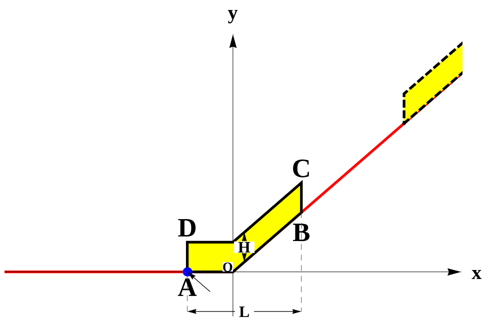

Figure 5. The Transition in progress

by the weight of the block. Thus,

$$G_{ke}(x) = -\frac{\gamma}{2}HL^2\left(\frac{x}{L}\right)\left(2 + \left(\frac{x}{L}\right)\right)\tan\alpha, \; -L \le x \le 0. \tag{13}$$

We assume that the total change in kinetic energy during transition is entirely associated with these losses and gains. If $v(x)$ is the velocity when the vertex A is at x, then

$$\frac{1}{2}Mv_1^2 - \frac{1}{2}Mv^2(x) = D_c(x) + D_f(x) - G_{ke}(x), \; -L \le x \le 0, \tag{14}$$

Where $M = \frac{\gamma}{g}LH$ is the mass of the block. Thus,

$$v(x) = -\sqrt{v_1^2 + 2c\alpha\frac{g}{\gamma}\frac{x}{L} - \frac{g}{L}\Big((2L\tan\alpha)x + (2f + \tan\alpha)x^2\Big)}, \; -L \le x \le 0. \tag{15}$$

Our assumption that v_1 was large enough to complete the transition is equivalent to assuming that the radicand in Eq. (15) is always non-negative. Let t_2 be the time when the transition is complete, that is, when the forward vertex A is at $x = -L$. Write $v_2 = v(-L)$ for the velocity at that time. Then, from Eq. (15),

$$v_2 = -\sqrt{v_1^2 - 2c\alpha\frac{g}{\gamma} - gL(2f - \tan\alpha)}. \tag{16}$$

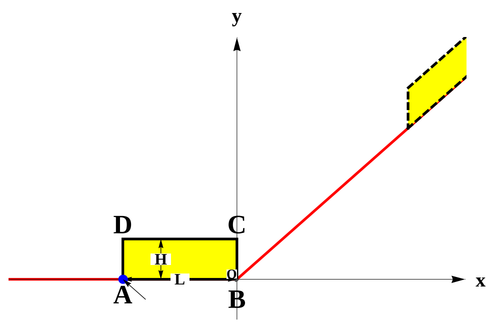

Figure 6. The end of transtition

The formula in Eq. (15) prompts two observations. First, the volume of the block, HL, does not appear. This is not surprising, since the energy dissipation and gain terms are all proportional to the volume. Second, it can be used to compute the duration of the transition, the time it takes for the vertex A to go from 0 to x. If $\hat{t}(x)$ is the time when A is at x,

$$\hat{t}(x) - t_1 = \int_0^x \frac{d\xi}{v(\xi)}, \; -L \le x \le 0. \tag{17}$$

Observe that the third term in the radicand of the expression in Eq. (16) depends only on the frictional energy losses and the gain in kinetic energy. If it is small in comparison with the first two terms and neglected, the magnitude of the velocity v_2 will be little changed. The same argument applies to the velocity function $v(x)$ of Eq. (15). The third term under the radicand also represents the contribution of frictional energy losses and the gain in kinetic energy. If its contribution is small in comparison with the first two terms, it can be neglected. (We will see that this is indeed the case for the typical example discussed in Section **6**.) Henceforth, we replace the function $v(x)$ of Eq. (15) by the approximation

$$\hat{v}(x) = -\sqrt{v_1^2 + 2c\alpha \frac{g}{\gamma}\frac{x}{L}}, \; -L \le x \le 0, \tag{18}$$

thereby neglecting the combined effects of frictional energy losses and the gain in kinetic energy while retaining the effects of decohesion. To be consistent, we also replace the value of v_2 in Eq. (16) by

$$\hat{v}_2 = -\sqrt{v_1^2 - 2c\alpha \frac{g}{\gamma}}. \tag{19}$$

Now if we use $\hat{v}(x)$ in place of $v(x)$ in the integral for the elapsed time of Eq. (17) we get the compellingly simple result:

$$\hat{t}(x) - t_1 = \frac{\gamma L}{c\alpha g}(\hat{v}(x) - v_1), \; -L \le x \le 0. \tag{20}$$

Hence, retaining only the effect of decohesion, the duration of the complete transition is

$$\hat{t}_2 - t_1 = \frac{\gamma L}{c\alpha g}(\hat{v}_2 - v_1) \tag{21}$$

and the change in speed during the transition is approximately linear in the time lapse.

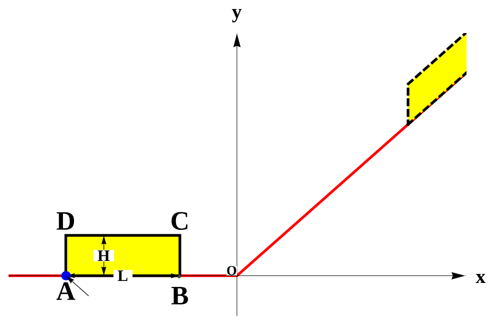

Figure 7. The landslide in progress after the transition

5. The Plane Shift of the Block

Once the transition is completed the block $ABCD$, now rectangular, lies on the interval $-L \leq x \leq 0$. The subsequent motion of the vertex A for $t \geq t_2$ is described by the equation

$$\ddot{x} = gf, \, t > t_2, \tag{22}$$

with the initial conditions

$$x(t_2) = -L \text{ and } \dot{x}(t_2) = v_2. \tag{23}$$

After a first quadrature we get

$$v(t) = \dot{x}(t) = v_2 + gf(t - t_2), \, t \geq t_2. \tag{24}$$

And after a second we have

$$x(t) = \frac{1}{2} gf(t - t_2)^2 + v_2(t - t_2) - L, \, t \geq t_2. \tag{25}$$

The block comes to rest at time t_3 when $v(t_3) = 0$, namely when

$$t_3 - t_2 = -\frac{v_2}{gf}. \tag{26}$$

Its rest position, $x(t_3)$, is now completely determined:

$$x(t_3) = -\Big(L + \frac{v_2^2}{2gf}\Big). \tag{27}$$

6. A Numerical Example

In order to get some insight into the magnitudes of the physical quantities involved in a landslide, we will assume the following values for the angle of inclination α, the distance x_0, and the thickness of the layer H:

$$\alpha = \frac{\pi}{4} \, radians, \qquad x_0 = 100 \, m, \qquad \text{and} \qquad H = 10 \, m. \tag{28}$$

Note that the length L cannot be prescribed but must be computed. For the constitutive properties of the material we take the cohesion c, the friction factor f and the specific weight γ to be (Jenne [4]):

$$c = 5 \times 10^3 \, \frac{kg}{m^2}, \qquad f = \tan \frac{\pi}{6} \approx 0.58 \qquad \text{and} \qquad \gamma = 2 \times 10^3 \, \frac{kg}{m^3}. \tag{29}$$

Replacing N from Eq. (1) into Eq. (4) yields

$$\gamma L (\sin \alpha - f \cos \alpha) \sin \alpha = c, \tag{30}$$

whence, after using the values from Eqs. (28) and (29), $L = 11.83\, m$. This landslide has the volume $HL = 118.30\, m^3$ and weighs $\gamma HL = 2.37 \times 10^5\, kg$, a relatively small landslide.

From Eqs. (9) and (10) we recover the values

$$t_1 = 8.26\, sec \qquad \text{and} \qquad v_1 = -24.20 \, \frac{m}{sec}. \tag{31}$$

Next, from Eqs. (18), (21), and (31), Eqs. (19) and (21) yield the values

$$\hat{v}_2 = -23.39 \, \frac{m}{sec} \qquad \text{and} \qquad \hat{t}_2 - t_1 = 0.50\, sec. \tag{32}$$

Finally, from Eqs. (26) and (27), we have from Eqs. (31) and (32)

$$t_3 - \hat{t}_2 = 4.13\, sec \qquad \text{and} \qquad x(t_3) = -60.19\, m. \tag{33}$$

The last numerical value, $x(t_3)$, is the point on the negative $x-$axis at which the front of the landslide comes to rest.

These numerical values were obtained assuming that the transition depends only on decohesion. Had we included all the energy terms to compute transition values, the velocity v_2 from Eq. (16) would be $v_2 = -23.01 \, \frac{m}{sec}$ and, from Eq. (17), the time lapse would be $t_2 - t_1 = 0.48\, sec$. The transition period is very short in comparison with the slide down the mountain and the slide along the base and the velocity change during transition is small. *A posteriori,* at least in this case, the entire transition process can be ignored.

7. Concluding Comments

From the numerical values in Section **6**, our very simple model predicts that a relatively small landslide can travel far from the base of the mountain in a short time. The simplifying assumption that the decohesion is dominant during the transition phase of the landslide's progress leads to a good estimate for the given data. Our numerical example leads us to conclude that, for the given data, the entire transition can be ignored. Of course, this might not obtain if other data are used. Villages located far from the base of the mountain may not be safe from the devastating effects of a landslide.

References

[1] C. A. Coulomb, "Essai sur une application des règles de maximis & minimis à quelques problèmes de statique relatifs à l'architecture", *Mém. Divers Savants* **7**, 343–382 (1773).

[2] K. Culmann, *Die Graphische Statik* (Zürich, 1866).

[3] J. Heyman, *Coulomb's Memoir on Statics* (Imperial College Press, London, 1997).

[4] G. Jenne, "Erdruck", in *Beton Kalender*; W. Ernst, Ed. (Berlin/München, 1967); pp. 310–343.

[5] W. J. Rankine, "On the stability of loose earth", *Phil. Trans. Royal Soc.* **147** (I), 9–27 (1857).

[6] V. V. Sokolovski, *Statics of Soil Media* (Butterworths, London, 1960).
[7] K. Terzaghi, “Mechanism of landslides”, *The Geological Soc. of America*, 83–123 (1960).
[8] E. G. Thomsen, C. T. Yang and S. Kobayashi, *Mechanics of Plastic Deformation in Metal Processing* (Macmillan, New York, 1965).

---

[a] Case Western Reserve University
Department of Mathematics
10900 Euclid Avenue
Cleveland, OH 44106-7058
USA

[b] Università di Pisa
Dipartimento di Ingegneria Civile, Sede di Strutture
Via Largo Lazzarino, 1
56126 Pisa
Italia

* Email: dis@ing.unip.it

Article contributed to the Festschrift volume in honour of Giuseppe Grioli on the occasion of his 100th birthday.

Received 1 August 2012; published online 29 January 2013
