DISSIPATIVE ENTROPY
MAKES THE HEAT EQUATION HYPERBOLIC

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ABSTRACT. Infinite disturbance speed, an undesirable feature of the standard heat equation, is usually avoided by assuming a Boltzmann-Volterra dependence of the heat influx vector on the history of the temperature gradient. It is shown in this paper how to attain the same *desideratum* by postulating a constitutive splitting of entropy into standard and dissipative parts.

Dedicato al Professor Giuseppe Grioli
con l'ammirazione di sempre
nel giorno del suo centesimo compleanno

1. Introduction

In this paper, I present a mathematical theory of heat conduction in which the spatial and temporal dependence of temperature is ruled by a *hyperbolic* partial differential equation. Such a theory is at variance with the standard model, which, as is well known, presumes that the temperature field $\vartheta = \widehat{\vartheta}(x, t)$ is a solution of the *parabolic* equation:

$$\lambda\dot{\vartheta} = \kappa\Delta\vartheta, \tag{1}$$

which is arrived at by combining the energy balance:

$$\dot{\varepsilon} = -\text{div}\,\mathbf{q}, \tag{2}$$

where ε is the specific (that is, per unit volume) *internal energy* and $\mathbf{q}$ the *heat-flux vector*, with the constitutive prescriptions:

$$\varepsilon = \lambda\vartheta, \quad \mathbf{q} = -\kappa\nabla\vartheta, \tag{3}$$

where both the *heat capacity* λ and the *conductivity* κ are positive constants.

The parabolic nature of the diffusion equation (1) implies that thermal signals propagate at infinite speed as a consequence, say, of a sudden change in temperature at the boundary. In the account [1, 2] of the literature up to 1990 due to Joseph and Preziosi, we read how this undesired feature can be eliminated by coupling (2) not with Fourier's law $(3)_2$ but rather with one or another version of Cattaneo's proposal [3] of 1948:

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\kappa\nabla\vartheta, \tag{4}$$

where $\tau > 0$ is the relaxation time for the solutions of the hyperbolic equation one arrives at doing this,

$$\lambda\tau\ddot{\vartheta} + \lambda\dot{\vartheta} = \kappa\Delta\vartheta. \tag{5}$$

In 1968, Gurtin and Pipkin [4] noticed that Cattaneo's assumption constitutes a special dependence of the heat flux on the history of the temperature gradient, whose general integral representation of Boltzmann-Volterra type is:

$$\widehat{\mathbf{q}}(x,t) = -\int_{-\infty}^{t} k(t-s)\nabla\vartheta(x,s)ds;^{1} \tag{6}$$

interestingly, an equivalent statement of (6) is that the heat flux is equal to minus the spatial gradient of the weighted temperature history up to time t:

$$\widehat{\mathbf{q}}(x,t) = -\nabla\Big(\int_{-\infty}^{t} k(t-s)\vartheta(x,s)ds\Big).$$

Apparently, ever since the paper by Gurtin and Pipkin appeared, the continuum mechanical research on heat waves somehow concentrated on choosing one or another weighted space of temperature (or temperature gradient) histories; however, different proposals were advanced by Coleman, Fabrizio, and Owen [5] and by Morro and Ruggeri [6]; an analytical study of a model generalizing those assembled in [5] and [6] is found in [7]. The path to (5) I here propose is, to my knowledge, new.

My derivation is consistent with the accepted principles and methods of modern continuum thermodynamics. The main elements of novelty I introduce are that, when it comes to specify constitutive dependencies, I let the entropy depend on the temperature's time derivative and I introduce an additive splitting of the entropy mapping into an 'equilibrium' part and a 'dissipative' part: the former turns out to be determined by the free-energy mapping and hence to depend only on temperature, as is quite standard; the latter may depend on the temperature's time derivative and induces hyperbolicity when it does.

2. Classic Heat Conduction, Review and Commentary

We begin by a quick commented review of the classic theory of *rigid heat conductors*, consisting of two parts: the balance and imbalance principles for, respectively, energy and entropy; and the constitutive restrictions dictated by the consequent upper bound for the free-energy growth in time.

2.1. Energy balance and entropy imbalance. For $\mathcal{P}$ a typical body part, the internal-energy balance:

$$\frac{d}{dt}\Big(\int_{\mathcal{P}}\varepsilon\Big) = -\int_{\partial\mathcal{P}}\mathbf{q}\cdot\mathbf{n} + \int_{\mathcal{P}}r \tag{7}$$

associates the energy's time rate with the energy's *external influx* $\mathbf{q}$ and *external source* r (here $\mathbf{n}$ denotes the outer normal to the part boundary $\partial\mathcal{P}$). By localization, the point-wise

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[1]Cattaneo's prescription (4) is recovered for

$$k(t-s) = (\kappa/\tau)\exp\big(-(t-s)/\tau\big).$$

version of (7) is arrived at:

$$\dot{\varepsilon} = -\operatorname{div}\mathbf{q} + r \tag{8}$$

(cf. (2)).

Remark 1. The modifiers 'internal' and 'external' refer to the part under attention; hereafter, we drop them when we talk about an extensive field like energy and its *inflow*, that is, its associated (influx, source) pair.

The part-wise *entropy imbalance* is:

$$\frac{d}{dt}\left(\int_{\mathcal{P}}\eta\right) \geq -\int_{\partial\mathcal{P}}\mathbf{h}\cdot\mathbf{n} + \int_{\mathcal{P}}s\,, \tag{9}$$

and has the point-wise version:

$$\dot{\eta} \geq -\operatorname{div}\mathbf{h} + s, \tag{10}$$

with η the specific *entropy*, $\mathbf{h}$ its influx, and s its source.

We see from (8) that the *energy inflow* $(\mathbf{q}, r)$ determines the energy's time rate; from (10), that the entropy growth in time is bounded below in terms of its inflow $(\mathbf{h}, s)$. Equations (8) and (10) may be interpreted as the statement, respectively, that

$$\text{\textit{the energy-production} } \gamma_{\varepsilon} := \dot{\varepsilon} - \big(-\operatorname{div}\mathbf{q} + r\big) \text{ \textit{is null}} \tag{11}$$

and that

$$\text{\textit{the entropy-production} } \gamma_{\eta} := \dot{\eta} - \big(-\operatorname{div}\mathbf{h} + s\big) \text{ \textit{is nonnegative}}. \tag{12}$$

Implicit in the use of an inequality in the latter case are both a prejudice about the nature of entropy and a confession of ignorance: while the role we ascribe to entropy is to furnish a macroscopic measure of microscopic disorder in matter, we cannot exclude that entropy changes have *internal* causes. Indeed, the standard constitutive prescription, which associates entropy with temperature – that is, with a macroscopic measure of velocity fluctuations in a large system of particles at statistical equilibrium – suggests a purely 'agitational' notion of disorder, whereas a 'positional' notion might be appropriate to deal with, say, phase transition by atom re-arrangement in crystalline solids, where the role of temperature is played by the inverse of the chemical potential [8] (one can also think of physical circumstances where an 'orientational' notion of entropy would be in order).

2.2. Free-Energy growth and Coleman-Noll procedure.
Considering the third state function of importance, the *Helmholtz free energy*

$$\psi := \varepsilon - \eta\vartheta, \quad \vartheta > 0, \tag{13}$$

where ϑ is the absolute temperature, from the First and Second Laws (8) and (10), and definition (13), it follows that its growth in time is bounded above:

$$\dot{\psi} \leq -\eta\dot{\vartheta} - \mathbf{h}\cdot\nabla\vartheta + \big(-\operatorname{div}(\mathbf{q} - \vartheta\mathbf{h}) + (r - \vartheta s)\big). \tag{14}$$

In the archetypal theory we are reviewing, the entropy and energy inflows are deemed proportional, through the *coldness* ϑ^{-1}:

$$(\mathbf{h}, s) = \vartheta^{-1}(\mathbf{q}, r). \tag{15}$$

This assumption reflects, among others, the experimental fact that an energy inflow has different entropic effects according to the temperature at which it takes place, effects that

are less and less pronounced as that temperature gets bigger and bigger.[2] An immediate consequence of (15) is that (14) reduces to the following familiar *dissipation inequality*:

$$\dot{\psi} \leq -\eta\dot{\vartheta} - \vartheta^{-1}\,\mathbf{q}\cdot\nabla\vartheta. \tag{16}$$

Remark 2. It is customary to accept (15), and to refer to

$$\delta := \vartheta\gamma_\eta \tag{17}$$

as to the *internal dissipation* [11]; in view of (10) and the second of (13), we have that

$$\delta \geq 0 \quad \Leftrightarrow \quad (16) \text{ holds}. \tag{18}$$

A message implicit in the dissipation inequality (16) is that the quantities in the need of constitutive specifications are free energy, entropy, and heat flux; and that, in addition to temperature, also its space and time derivatives may count to determine how those quantities evolve in a process. In a justly famous paper appeared almost fifty years ago [12], Coleman and Noll proposed to discard as 'unphysical' whatever constitutive choices would not satisfy (16) along all admissible processes; their procedure quickly became part of the vulgate of continuum thermodynamics.

Remark 3. Assumption (15) is expedient to reduce the dissipation principle to an algebraic inequality to be satisfied identically, whence the facileness of the Coleman-Noll procedure. In this connection, many questions come to mind. One may ask whether it would possible to find a rationale for assigning (15) the status of a general feature of all thermodynamical theories of continuous media. I am convinced that this is not the case, for more than one reason that I plan to discuss elsewhere. But then it is incumbent on me to ask how to characterize the most inclusive constitutive class for which (15) holds true. One may also wonder whether a modified Coleman-Noll line of reasoning could be based on (14) and some less drastic assumption on energy and entropy inflows than (15), given that (15) seems too simplistic to be applied for all material bodies different from rigid heat conductors.

As a matter of fact, assumption (15) has been questioned by Müller [13], who has developed a thermodynamic theory where it is replaced by making the entropy influx the object of a specific constitutive prescription, just as is customarily done for the energy influx, and by letting the entropy source be in principle independent from the energy source. In a theory *à la* Müller, the Coleman-Noll procedure has no role: a case-dependent set of relationships between the entropy and energy inflows follow from requiring consistency with the balance and imbalance principles (8) and (10) of all constitutive prescriptions for a given material class. E.g., for isotropic materials and fluids whose response depends on the standard list of state variables (that is to say, deformation gradient and its time rate, plus temperature and its time rate) it can be proved that

$$\mathbf{h} = \tilde{\vartheta}^{-1}\mathbf{q}, \tag{19}$$

with the *empirical temperature* $\tilde{\vartheta}$ a function of the absolute temperature (see [14], p. 165, and [15, 16]).

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[2]This fact was evident to Gibbs: on p. 7 of [9] we read the following pertinent quotation of Gibbs: "...heat received at one temperature is by no means the equivalent of the same amount of heat received at another temperature ..."; see also the lively discussion by Antman, in [10], Sect. **12**.14.

Remark 4. Let $\chi := \dot{\psi} + \eta\dot{\vartheta} + \mathbf{h} \cdot \nabla\vartheta$; moreover, let

$$\mathcal{E}(\mathcal{P}) := -\int_{\partial\mathcal{P}} (\mathbf{q} - \vartheta\mathbf{h}) \cdot \mathbf{n} + \int_{\Re} (r - \vartheta s) \tag{20}$$

be the *excess energy-inflow* of body part $\mathcal{P}$. Then, it follows from (14) that

$$\int_{\mathcal{P}} \chi \leq \mathcal{E}(\mathcal{P}), \tag{21}$$

for every body part $\mathcal{P}$. This part-wise statement of the dissipation principle is frame-indifferent, and as such irreducible. It has an interesting implication for $\mathcal{P}$ a pillbox of thickness ρ and orientation $\mathbf{r}$ centered at a point $x \in \mathcal{P}$: in the limit as $\rho \to 0$, it follows from (21) that

$$[\![\mathbf{q} - \vartheta\mathbf{h}]\!](x) \cdot \mathbf{r} = 0,$$

for all x and $\mathbf{r}$.

Remark 5. It would seem that the energy and entropy source fields r and s should *not* be thought of as mutually independent control fields, at our disposal to induce in a constitutively described material element whatever local continuation of a given admissible process we please. If r and s were mutually independent, given that (21) incorporates balance of energy whatever r, then it would be possible to falsify that inequality for some choice of r itself, whatever s. Hence, r and s must be mutually dependent: the simplest selection is $r = \vartheta s$.

3. An Odd Use of the Coleman-Noll Procedure

With the conceptual background recapitulated so far in mind, we take three steps along the Coleman-Noll path, of which the first two are unusual.

Firstly, we lay down the following provisional constitutive assumptions:

$$\psi = \widehat{\psi}(\vartheta), \quad \eta = \widehat{\eta}(\vartheta, \nabla\vartheta, \dot{\vartheta}). \quad \mathbf{q} = \widehat{\mathbf{q}}(\vartheta, \nabla\vartheta, \dot{\vartheta}), \tag{22}$$

with a view toward filtering their unphysical contents, if any, the Coleman-Noll way. Note that, at variance with what is done in the classic theory of heat conduction and in hopes of generalizing the parabolic heat equation (1), we have let both entropy and heat flux depend on the temperature's time derivative.[3]

Secondly, we split the entropy mapping $\widehat{\eta}$ into a 'standard' part and a 'dissipative' part:

$$\widehat{\eta}(\vartheta, \nabla\vartheta, \dot{\vartheta}) = \widehat{\eta}^s(\vartheta, \nabla\vartheta) + \widehat{\eta}^d(\vartheta, \nabla\vartheta, \dot{\vartheta}), \tag{23}$$

with

$$\widehat{\eta}^s(\vartheta, \nabla\vartheta) := \widehat{\eta}(\vartheta, \nabla\vartheta, 0), \quad \widehat{\eta}^d(\vartheta, \nabla\vartheta, \dot{\vartheta}) := \widehat{\eta}(\vartheta, \nabla\vartheta, \dot{\vartheta}) - \widehat{\eta}(\vartheta, \nabla\vartheta, 0). \tag{24}$$

Thirdly, we require, *à la* Coleman-Noll, that the dissipation inequality (16) be satisfied for $\widehat{\psi}, \widehat{\eta}$ and $\widehat{\mathbf{q}}$ as in (22)−(24) whatever the local continuation of any admissible process, that is, in the present case, for any choice of $(\dot{\vartheta}, \nabla\dot{\vartheta})$ at any given state $(\vartheta, \nabla\vartheta)$. We quickly

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[3]Had we assumed that $\psi = \widehat{\psi}(\vartheta, \nabla\vartheta, \dot{\vartheta})$, we would have quickly reached the conclusion that whatever dependence of the free energy on either $\nabla\vartheta$ or $\dot{\vartheta}$ is incompatible with a use of (16) in the manner of [12].

find that this requirement is satisfied if and only if the standard entropy is determined by the free energy:

$$\widehat{\eta}^s(\vartheta, \nabla\vartheta) = -\partial_\vartheta \widehat{\psi}(\vartheta) \tag{25}$$

and, in addition, the *reduced dissipation inequality*:

$$\widehat{\eta}^d(\vartheta, \nabla\vartheta, \dot{\vartheta})\dot{\vartheta} + \vartheta^{-1}\,\widehat{\mathbf{q}}(\vartheta, \nabla\vartheta, \dot{\vartheta}) \cdot \nabla\vartheta \leq 0 \quad \text{for all } \vartheta \text{ and } \nabla\vartheta. \tag{26}$$

is satisfied along all admissible processes.

Let us now make the classic constitutive choice for the specific free energy:

$$\widehat{\psi}(\vartheta) = -\lambda\,\vartheta(\log\vartheta - 1), \tag{27}$$

whence, in view of (25),

$$\widehat{\eta}^s(\vartheta, \nabla\vartheta) =: \widetilde{\eta}(\vartheta) = \lambda\log\vartheta$$

for the standard part of the entropy and, in view of (13),

$$\widehat{\varepsilon}(\vartheta, \nabla\vartheta, \dot{\vartheta}) = \lambda\vartheta + \vartheta\,\widehat{\eta}^d(\vartheta, \nabla\vartheta, \dot{\vartheta}) \tag{28}$$

for the internal energy. Moreover, let us pick

$$\widehat{\eta}^d(\vartheta, \nabla\vartheta, \dot{\vartheta}) = \eta_0\vartheta^{-1}\dot{\vartheta}, \quad \eta_0 \geq 0, \tag{29}$$

for the 'dissipative' part of entropy, and

$$\widehat{\mathbf{q}}(\vartheta, \nabla\vartheta, \dot{\vartheta}) = -q_0\nabla\vartheta, \quad q_0 > 0, \tag{30}$$

for the energy influx. Then, provided the source term r is null, the energy balance (2) takes the form:

$$\eta_0\ddot{\vartheta} + \lambda\dot{\vartheta} = q_0\Delta\vartheta, \tag{31}$$

which coincides with (5) provided one sets:

$$\eta_0 = \lambda\tau, \quad q_0 = \kappa.$$

The standard heat equation (1) is recovered by setting $\eta_0 = 0$, that is, for $\tau = 0$.

Remark 6. Under the constitutive circumstances stipulated just above, inequality (26) takes the form:

$$\eta_0\dot{\vartheta}^2 - q_0|\nabla\vartheta|^2 \leq 0, \tag{32}$$

a restriction on the type of processes that the present theory admits, heat waves included (in particular, temperature must be constant in time where and when it is constant in space). Moreover, definitions (12) and (17) yield the following expression for the internal dissipation:

$$\delta = -\eta_0\dot{\vartheta}^2 + q_0|\nabla\vartheta|^2;$$

thus, the internal dissipation associated with the standard theory of heat conduction (namely, $\delta_s = q_0|\nabla\vartheta|^2$) is in the present theory mitigated because of the introduction of a 'dissipative' part of the entropy.

Finally, it follows from (31) that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{P}}\left(\eta_0\dot{\vartheta}^2 + q_0|\nabla\vartheta|^2\right)\right) + \int_{\mathcal{P}}\lambda\dot{\vartheta}^2 = -\int_{\partial\mathcal{P}}\dot{\vartheta}\,\mathbf{q}\cdot\mathbf{n},$$

whence, provided $\dot{\vartheta}\,\mathbf{q}\cdot\mathbf{n} \equiv 0$ over the boundary of $\mathcal{P}$, we deduce that the theory's *Liapunov function* is:

$$t \mapsto \Phi_{\mathcal{P}}(t) := \frac{1}{2}\int_{\mathcal{P}} \left(\eta_0\dot{\vartheta}^2(x,t) + \kappa|\nabla\vartheta|^2(x,t)\right)dx\,.$$

4. Scholion

There are continuum mechanical theories where the state variables have psychologically different statuses: although they all belong to one and the same list, some are meant to carry gross information about a large collection of 'substates'. For example, in classical thermoelasticity, *temperature* has a psychological status different from displacement gradient: the latter measures *macroscopic* local changes in shape, volume, or orientation, the former is meant to account for infinitely many, grossly equivalent substates of *microscopic agitation*.[4] In my opinion, it would help clarity and definiteness to introduce the habit of distinguishing between *state* and *substate variables* (such as, respectively, displacement gradient and temperature in thermoelasticity), while referring collectively to all fields of constitutive relevance for a given material class as to the *constitutive variables*. An alternative nomenclature that was quite popular in the good old days, *intermediate variables*, has a motivation not so easy to figure out: it alludes to the fact that those variables mediate the space-time dependence of the fields they determine through constitutive prescriptions (for example, in thermoelasticity the stress field depends on space x and time t via the values taken at (x,t) by displacement gradient, temperature, and temperature gradient). Thanks to Walter Noll, who introduced it in his thesis, the more appropriate term 'constitutive' has supplanted 'intermediate'.

Acknowledgments

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