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A NOTE ON ANTIPLANE MOTIONS IN NONLINEAR ELASTODYNAMIC

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ABSTRACT. We determine a simple class of exact solutions for the anti-plane shear motion problem of fourth-order elasticity. These exact solutions, computed by reducing the governing equations to an autonomous system of ordinary differential equations, may be used to provide some examples of global existence in time for the Cauchy problem of nonlinear elastodynamic.

Al Maestro Giuseppe Grioli
con stima ed affetto

1. Introduction and Basic Equations

Consider an incompressible isotropic elastic body which in its undeformed state, occupies a simply connected cylindrical region $\mathcal{R}$. We introduce a system of Cartesian coordinates (X_1, X_2, X_3) such that the X_3-axis is parallel to the generators of $\mathcal{R}$ and we denote by (x_1, x_2, x_3) the Cartesian coordinates, aligned with (X_1, X_2, X_3), of the current position $\boldsymbol{x}$ in the deformed configuration.

The solid body which occupies $\mathcal{R}$ in its undeformed configuration is said to be deformed to a state of *anti-plane* shear if the displacement of each particle is parallel to the generators of the cylinder and independent of the axial position of the particle. Hence

$$\boldsymbol{x} = X_1\boldsymbol{E}_1 + X_2\boldsymbol{E}_2 + [X_3 + u(X_1, X_2; t)]\boldsymbol{E}_3. \tag{1}$$

where $u(X_1, X_2; t)$, the anti-plane motion, is a smooth function of all its arguments. Here $\boldsymbol{E}_i$ $(i = 1, 2, 3)$ are the unit vectors of the Cartesian frame introduced in the reference configuration. Motion (1) is isochoric.

As usual $\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$ denotes the associated deformation gradient tensor, and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ the left Cauchy-Green strain tensor. The first and second principal invariant are defined as $I_1 \equiv \mathrm{tr}\boldsymbol{B}$ and $I_2 \equiv \mathrm{tr}(\boldsymbol{B}^{-1})$.

The Cauchy stress tensor $\boldsymbol{T}$, derived from the incompressible strain-energy function $W = W(I_1, I_2)$, is

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\frac{\partial W}{\partial I_1}\boldsymbol{B} - 2\frac{\partial W}{\partial I_2}\boldsymbol{B}^{-1}, \tag{2}$$

where p is a Lagrange multiplier introduced by the constraint of incompressibility, $\det \boldsymbol{F} = 1$.

In the absence of body forces, the equations of motion are

$$\operatorname{div} \boldsymbol{T} = \rho \boldsymbol{a}_{tt}$$

where ρ is the (constant) density of the material and $\boldsymbol{a}$ is the acceleration vector. Equivalently, using the nominal or first Piola stress tensor $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{F}^{-T}$, we have

$$\operatorname{Div} \boldsymbol{P} = \rho \boldsymbol{x}_{tt}.$$

For the class of motions (1) the balance equations reduce to an *overdetermined system* of partial differential equations

$$q_j + 2\left(\frac{\partial W}{\partial I_2} u_j u_i\right)_i = 0, \quad j = 1, 2, \tag{3}$$

and

$$2\left[\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) u_i\right]_i = \rho u_{tt}, \tag{4}$$

where summation over the repeated index $i = 1, 2$ is assumed and a subscript indicates differentiation with respect to the corresponding coordinate.

The auxiliary quantity q in (3) is related to the pressure field $p(X_1, X_2, X_3; t) = \Gamma(t)X_3 + \hat{p}(X_1, X_2, t)$ via the relation

$$q(X_1, X_2; t) = \hat{p}(X_1, X_2, t) - 2\frac{\partial W}{\partial I_1} - 2(3 + u_1^2 + u_2^2)\frac{\partial W}{\partial I_2}.$$

Here the derivatives of the strain-energy density function are functions of $\Omega \equiv u_1^2 + u_2^2$ and in the following for the sake of simplicity we set $\Gamma \equiv 0$.

The system composed of equations (3) and (4) is the dynamic extension of the *Knowles overdetermined system* [7].

It is clear from equations (3) and (4) that rectilinear shear is atypical in the nonlinear theory of elasticity and is possible only for special materials and/or for special cross sectional shapes of $\mathcal{R}$, when $\partial W / \partial I_2 \neq 0$.

There exists a large literature about this problem in the framework of the static theory of nonlinear elasticity (see for example [1], [7]); a review paper on this topic has been written by Horgan [6] and a recent contribution is given in [9].

Following Knowles [7] a material, characterized by the strain-energy density $W(I_1, I_2)$, will be said to admit *nontrivial* states of anti-plane shear if, for every domain and for every solution $u(X_1, X_2)$ of the static version of (4), there is a function $q(X_1, X_2)$ such that (3) holds. Knowles shows that this is possible if and only if the strain-energy satisfies

$$\frac{\partial W}{\partial I_2} = D\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) \tag{5}$$

where D is an arbitrary constant and the derivatives of the strain-energy function are evaluated for $I_1 = I_2 = 3 + u_1^2 + u_2^2$. It is easy to show that this result also holds true in elastodynamics [9].

On the other hand, the overdetermined system (3) and (4) is compatible for any material only for special functional forms of the antiplane motion as

$$u(X_1, X_2; t) = \phi\left(X_1^2 + X_2^2; t\right), \quad u(X_1, X_2; t) = \psi\left(K_1 X_1 + K_2 X_2; t\right), \tag{6}$$

where K_1, K_2 are real constants such that $K_1^2 + K_2^2 = 1$. The complete list of such special functional forms is given in [5].

Here we restrict our attention to the special strain-energy density function

$$W(I_1, I_2) = H(I_1) + E(I_2 - 3), \tag{7}$$

where $E(\neq 0)$ is a constitutive parameter and $H(I_1)$ a material function. It is clear that (7) satisfies (5) if and only of $H(I_1)$ is a linear function.

In the case of the material (7) the compatibility equation obtained from (3) is given by

$$[\left(u_1^2\right)_1 + (u_1 u_2)_2]_2 - [(u_2 u_1)_1 + \left(u_2^2\right)_2]_1 = 0, \tag{8}$$

hence

$$\nabla^2 u = \mathcal{F}(u), \tag{9}$$

$\mathcal{F}(u)$ being an arbitrary function. Using (7) it is possible to rewrite (4) as

$$2(H' + E)\nabla^2 u + 2H'' \left[\Omega_1 u_1 + \Omega_2 u_2\right] = \rho u_{tt}. \tag{10}$$

The aim of this note is to consider solutions of the overdetermined system (9) and (10) that are not listed in [5]. These solutions are interesting for several reasons. First of all they are simple exact solutions of nonlinear elastodynamic. Second, some of these solutions are obtained via the reduction of the overdetermined system (9), (10) to standard systems of Classical Mechanics. A nice and interesting analogy. Finally, some of the solutions we found are examples of global existence for nonlinear hyperbolic equations.

2. Fourth-order Elasticity

The fourth order weakly nonlinear theory of elasticity, [4], is characterized by the strain-energy density

$$W = \mu \mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{A}}{3}\mathrm{tr}(\boldsymbol{E}^3) + \mathcal{D}[\mathrm{tr}(\boldsymbol{E})]^2, \tag{11}$$

where $\boldsymbol{E} = (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})/2$ is the Green strain tensor and μ, $\mathcal{A}$ and $\mathcal{D}$ are second, third-, and fourth-order elasticity constants. The (11) may be related to the strain-energy density (7).

Choosing $H(I_1) = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2$, where C_{10}, C_{20} are constitutive parameters. In such a way, we obtain the strain-energy density

$$W(I_1, I_2) = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + E(I_2 - 3), \tag{12}$$

which is equivalent, up to the fourth order in the amplitude of the displacement, to (11) if

$$\mu = 2(C_{10} + E), \quad \mathcal{A} = -8(C_{10} + 2E), \quad \mathcal{D} = 2(C_{10} + 3E + 2C_{20}).$$

Here we assume that all the constitutive parameters, C_{10}, C_{20} and E, are positive.

For the strain-energy (12) equation (10) is given by

$$\mu\nabla^2 u + 4C_{20}\left[\Omega\nabla^2 u + \Omega_1 u_1 + \Omega_2 u_2\right] = \rho u_{tt}. \tag{13}$$

If we consider $u = \phi(\xi; t)$, $\xi = K_1 X_1 + K_2 X_2$ as in $(6)_2$, the (8) is an identity and the (10) reduces to the one dimensional wave equation

$$\left(\mu + 12C_{20}\phi_\xi^2\right)\phi_{\xi\xi} = \rho\phi_{tt}. \tag{14}$$

It is well known that the solutions of the Cauchy problem for this wave equation shows blow-up in finite time (see for example [2]). This is the rule for nonlinear hyperbolic equations, but in nonlinear elastodynamics, if we consider the Cauchy problem in the whole space, there are some notable exceptions of global existence in time as for example the circularly polarized shear waves determined by Carroll in [3].

Consider $\mathcal{F}(u) \equiv 0$ in (9). In this case the overdetermined system (3) and (4) for the strain-energy density (12) becomes

$$\nabla u = 0, \quad 4C_{20}\left[\Omega_1 u_1 + \Omega_2 u_2\right] = \rho u_{tt}. \tag{15}$$

It is remarkable that the polynomial solutions of the two-dimensional Laplace equation

$$u = \frac{1}{2}A(t)\left(X_1^2 - X_2^2\right) + B(t)X_1X_2, \tag{16}$$

introduced in the second equation of (15) gives the coupled nonlinear system of ordinary differential equations

$$A_{tt} = k^2(A^2 + B^2)A, \quad B_{tt} = k^2(A^2 + B^2)B, \tag{17}$$

where $k^2 = 4C_{20}/\rho$.

We start considering $A = B$. In this case, the system (17) reduces to the single ordinary differential equation $A_{tt} = 2k^2A^3$. It is convenient to rescale this equation as

$$\tilde{A}_{tt} = 2\tilde{A}^3, \tag{18}$$

where $\tilde{A}(t) = A(t)/|k|$. For the (18) it is possible to derive the ”energy” integral as

$$\tilde{A}_t^2 = \tilde{A}^4 + 2E_0, \tag{19}$$

and when $E_0 = 0$ the solutions of (18) are

$$\tilde{A}(t) = \pm\frac{1}{t + k_1},$$

where k_1 is a constant of integration.

In this way we obtain the exact solutions

$$u(X_1, X_2; t) = \pm\frac{|k|}{t + k_1}\left[\frac{1}{2}\left(X_1^2 - X_2^2\right) + X_1X_2\right],$$

in the whole space for the family of Cauchy problems with smooth initial data

$$u(X_1, X_2; 0) = \pm\frac{|k|}{k_1}\left[\frac{1}{2}\left(X_1^2 - X_2^2\right) + X_1X_2\right],$$

$$u_t(X_1, X_2; 0) = \mp\frac{|k|}{k_1^2}\left[\frac{1}{2}\left(X_1^2 - X_2^2\right) + X_1X_2\right].$$

We point out that for $k_1 > 0$ the solution of this Cauchy problems exists globally in time, whereas for $k_1 < 0$ it blows up for $t^* = |k_1| < \infty$.

It is interesting to note, using a dynamical system interpretation of the solutions here proposed, that, from (19), the ”motion” is confined to the ”configurations” where $\tilde{A}^4 + 2E_0 \geq 0$ and the special case where $\tilde{A}^4 + 2E_0 = 0$ is a *barrier* because it may be not crossed by the motion. The barriers that are single roots are inversion points whereas the barriers that are double or multiple roots are asymptotic limits because it takes an infinite time to reach it.

This analysis is due to the fact that (19) may be solved by quadratures as

$$t = \pm \int_{\tilde{A}(0)}^{\tilde{A}} \frac{d\tilde{A}}{\sqrt{\tilde{A}^4 + 2E_0}}.$$

When $E_0 = 0$ we have that $\tilde{A} = 0$ is an asymptotic limit. If we start from $\tilde{A}(0) = A^+ > 0$ or $\tilde{A}(0) = A^- < 0$ and with an initial "speed" pointing toward $A = 0$, we have that $A(t)$ must be a bounded function such that $\lim_{t\to\infty} A(t) = 0$. Therefore, we have a global existence result for our problem. On the other hand if the initial "speed" points out from $\tilde{A} = 0$ we have blow up in finite time. In fact,

$$t = \pm \int_{\tilde{A}(0)}^{\infty} \frac{d\tilde{A}}{\tilde{A}^2} \equiv \pm \frac{1}{A^{\pm}},$$

and thus $\tilde{A} \to \infty$ in finite time.

When $E_0 \neq 0$ it is possible to obtain the exact solution of (18) in terms of the inverse of elliptic functions, but the qualitative character of such solutions may-be always analyzed in terms of the roots of $A^4 + 2E_0$ and therefore we have to discuss $E_0 > 0$ and $E_0 < 0$.

If $E_0 > 0$ we have no barriers; thus $A(t)$ is a monotone function and in any case we have blow-up of the solution. If $E_0 < 0$, i.e. $A_t^2(0) < A^4(0)$, we have barriers, but they are always inversion points. Therefore, $A(t)$ maybe non-monotone, but in any case, we have always blow-up in finite time.

If we go back to the system (17) it is easy to show that when $A \neq B$, the system admits the first integrals of the "areal velocity" and of the "energy"

$$A_t\, B - A\, B_t = E_1, \quad A_t^2 + B_t^2 = \frac{k}{2}\left(A^2 + B^2\right) + E_2,$$

where $E_1 \neq 0, E_2$ are the first integral constants. Therefore we are dealing with a classical central motion problem and it is convenient to introduce the change of dependent variables

$$A(t) = r(t)\sin\theta(t), \quad B(t) = r(t)\cos\theta(t),$$

obtaining

$$\frac{1}{2} r^2 \theta_t = E_1, \quad r^2\theta_t^2 + r_t^2 = \frac{1}{2} r^4 + E_2,$$

i.e.

$$r_t^2 = \frac{k}{2} r^4 + E_2 - 4E_1^2 r^{-2}. \tag{20}$$

The discussion of this system is a classical one similar to the previous one ($E_1 = 0$). By considering in details all the opportunities, we see that solutions with global existence in time are not possible.

REMARK I The examples of global existence we have provided are different from the Carroll waves with are periodic solutions in time and space. It is interesting to note that if we consider antiplane shear motions in the form

$$u = A(t)v(X_1, X_2) \tag{21}$$

where $\nabla v = 0$ the compatibility problem (15) reduces to determine the solutions the Laplace equation shares with the partial differential equation

$$4C_{20}A^3\left[\Omega_1^v v_1 + \Omega_2^v v_2\right] = \rho A_{tt} v, \tag{22}$$

where $\Omega^v = v_1^2 + v_2^2$.

A necessary conditions for the existence of periodic solutions in time is $A_{tt} = -\kappa^2 A^3$. Using this ansatz and the general solution of the 2D Laplace equation as $v = f(z) + g(\zeta)$ where $z = x + iy$ and $\zeta = x - iy$ we obtain from (22) the equation

$$(k^2 + \kappa^2)(f + g) + 8(f_z^2 g_{\zeta\zeta} + g_\zeta^2 f_{zz}) = 0.$$

No real solutions of this equation exists and therefore periodic (in time) solutions of the form (21) do not exists.

REMARK II It is possible to ask if there is some hope to find more special exact solutions of the Knowles overdetermined system for different special families of strain-energies different from (12). For example, considering once again $\mathcal{F}(u) \equiv 0$ in (9) and going back to the overdetermined system

$$\nabla u = 0, \quad 2H''\left[\Omega_1 u_1 + \Omega_2 u_2\right] = \rho u_{tt}, \tag{23}$$

we remark that for the following solution of the Laplace equation:

$$u(X_1, X_2; t) = \exp\left(\pm\mu X_1\right)\left[A(t)\sin(\mu X_2) + B(t)\cos(\mu X_2)\right] \tag{24}$$

we have

$$\Omega = (A^2 + B^2)\mu^2 \exp(2\mu X_1),$$

and the second equation in (23) is reduced to

$$2H''\Omega\mu^2\left[A\sin(\mu X_2) + B\cos(\mu X_2)\right] = \rho\left[A_{tt}\sin(\mu X_2) + B_{tt}\cos(\mu X_2)\right].$$

Therefore if we require

$$H''\Omega = C_3, \tag{25}$$

and the time dependent functions $A(t)$ and $B(t)$ solutions of

$$A_{tt} = 2\mu^2\frac{C_3}{\rho}A, \quad B_{tt} = 2\mu^2\frac{C_3}{\rho}B, \tag{26}$$

we obtain a compatible special case of the Knowles system.

From (25) we compute

$$H(I_1) = C_{10}(I_1 - 3) + C_3(I_1 - 3)\left[\log(I_1 - 3) - 1\right],$$

where C_3 is the nonlinearity constant. The strain energy associated with this functional form is not meaningful, because the Cauchy stress tensor blows-up for $\boldsymbol{F} = \boldsymbol{I}$.

3. Concluding Remarks

The simple exact solutions we have considered are interesting for several reasons but first of all because they may be used to have explicit examples of global in time existence for the Cauchy problem of nonlinear elastodynamics. It is well known that for the equations of motions of nonlinear elasticity we may have an existence theorem for small data. The *local* in time existence theorem is the well known John's *almost global existence result* [8]. When the nonlinearities of the elastodynamic model fulfill a *null condition* it turns out that a global existence theorem for small initial data is possible [11]. The null condition is formulated in the unconstrained (compressible) case and on the third order elasticity constants.

In the incompressible case the third order elasticity is equivalent to the Mooney-Rivlin material and wave propagation becomes quite special. Indeed, it is well known that waves in Mooney-Rivlin materials are essentially governed by linear equations. This result has been obtained by many authors in different forms, for example considering the class of elastic potentials such that exceptional waves may propagate in every direction [10].

Thanks to the Carroll waves solutions [3], it is evident that the global existence in time for the Cauchy problem of incompressible elastodynamics is not satisfactory. The Carroll waves are examples of global solutions in time for general incompressible elastic materials and for initial data of any amplitude. Here we have provided other examples of solutions of the Cauchy problem in the whole space that exists globally in time. The mathematical nature of our examples is completely different from the one of Carroll waves because we have solutions polynomial in the space variables and with decay in time. The mathematics beyond these special exact solutions is still a puzzling problem that needs to be understood in more details.

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