DOI: 10.1478/AAPP.91S1A6

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 91, Suppl. No. 1, A6 (2013)

NONLINEAR WAVE INTERACTIONS FOR QUASILINEAR HYPERBOLIC 2×2 SYSTEMS

CARMELA CURRÒ [a*] AND DOMENICO FUSCO [a]

ABSTRACT. A reduction approach based upon the combined use of differential constraints theory and hodograph method is developed in order to determine closed form solutions for 2×2 hyperbolic quasilinear nonhomogeneous models. The problem of integrating the standard linear hodograph system associated with 2×2 homogeneous models is also considered. Along the lines of the proposed reduction approach different examples of 2×2 governing models are analyzed in order to highlight the flexibility of the provided solutions to describe hyperbolic wave processes.

Dedicated to Prof. G. Grioli on occasion of his 100th birthday.

1. Introduction

Quasilinear hyperbolic nonhomogeneous and autonomous systems of first order PDEs involving two dependent and two independent variables (2×2) play a prominent role in several physical and engineering applications where dissipative effects must be also taken into account in the hyperbolic wave dynamics. A striking feature of these mathematical models is that, under assumption of strict hyperbolicity, they can be recast into a form which expresses the evolution of a privileged set of field variables, the Riemann variables, along the related characteristic curves. As well known the homogeneous (non dissipative) 2×2 models can be linearized through the classical hodograph transformation although the solution of the resulting pair of linear equations as expressed in terms of the Riemann function is of very limited use in describing one dimensional wave processes. Hence over the years attention was focused on developing appropriate reduction methods for integrating the hodograph system in a closed form [1, 2, 3]. If the governing balance laws are not conservative then for the resulting 2×2 non homogeneous system of PDEs direct reduction to linear form through the standard hodograph transformation is no longer possible so that alternative linearizing approaches have been proposed through the use of direct methods [4, 5] or of group methods [6, 7]. In view of determining exact wave-like solutions to nonhomogeneous 2×2 models which incorporate all the features of the classical Riemann invariants and of the classical simple waves as those satisfying the linear hodograph equations associated to homogeneous systems, recently in [8] there was proposed an approach based on the combined use of the hodograph method and of the differential constraints technique. This approach proves to be an effective tool for integrating the highly nonlinear hodograph system which arises from the non conservative set of field balance equations.

As for many reduction techniques, an inherent aspect in this method is the possibility of selecting structural forms of the response functions involved in the constitutive laws (model constitutive laws) related to a given system of balance equations. In the latter context the resulting exact solutions also provide a mathematical vehicle for validating governing systems of interest in physical applications. It is to be also noticed that the leading lines of the method in point permit a complete and accurate description of the wave interaction to classes of 2×2 strictly hyperbolic and homogeneous systems which encompass a number of relevant mathematical models to wave propagation (see [9]).

Here we consider the following quasi-linear hyperbolic system of first order PDEs

$$\mathbf{U}_t + A(\mathbf{U})\,\mathbf{U}_x = \mathbf{B}(\mathbf{U}) \tag{1}$$

where $\mathbf{U} = \begin{pmatrix} u_1, & u_2 \end{pmatrix}^T$, A, $\mathbf{B}$ are matrix coefficients while x and t denote, respectively, space and time coordinates. The strict hyperbolicity of (1) requires the matrix A to admit two real distinct eigenvalues λ and ν with corresponding left $\mathbf{l}^{(\lambda)}, \mathbf{l}^{(\nu)}$ and right $\mathbf{d}^{(\lambda)}, \mathbf{d}^{(\nu)}$ eigenvectors. Hereafter we assume $\mathbf{l}^{(\lambda)} \cdot \mathbf{d}^{(\lambda)} = \mathbf{l}^{(\nu)} \cdot \mathbf{d}^{(\nu)} = 1$. Along the lines of the investigation worked out in [10, 11] we append to (1) a first order differential constraint of the form

$$\mathbf{l}^{(\lambda)} \cdot \mathbf{U}_x = p(\mathbf{U}, x, t) \tag{2}$$

where the function $p(\mathbf{U}, x, t)$ is to be determined and in fact defines the class of searched solutions. The paper is organized as follows. In Section **2** we write the over-determined system (1) and (2) in terms of Riemann field variables and then we use the standard hodograph transformation in order to recast the resulting hodograph system into a quasi-linear form. The main steps of consequent resulting consistency process are illustrated. Within the proposed theoretical framework later it is also revisited the problem of integrating in a closed form the standard linear hodograph system arising from the classical 2×2 homogeneous models. Next in Section **3** attention is focused on the traffic flow model proposed in [12]. Here we consider two simple waves travelling along different families of characteristic curves and we illustrate in detail the alteration in the profiles as well as in the wave time distortion of the emerging pulses caused by the interaction process. Furthermore in Section **4** it is considered a special class of nonhomogeneous 2×2 models for which the reduction approach under interest provides exact solutions which result to be also appropriate to describe simple wave-like evolution. The interaction of quasi-simple waves is investigated thoroughly for the governing system of nonlinear transmission lines [13] for a special form of the material response functions therein involved. It is shown that soliton-like superposition [2] of pulses travelling in opposite direction may also occur. Section **5** is mainly devoted to point out that the reduction theoretical framework outlined in this paper can be also relevant to investigate wave problems and especially wave interactions ruled hyperbolic models more general than those 2×2. Actually we consider multicomponent quasilinear homogeneous systems which can also involve more space variables. As well known, despite generalized hodograph methods have been proposed for diagonalizable and semi-Hamiltonian $(1 + 1)$ homogeneous systems with an arbitrarily large number of components [14], in general it is a hard task to determine solutions in a closed form to initial value problems in order to achieve an exact description of wave interaction processes. Here, along the lines of the general procedure worked out in [15] as well as of the reduction method developed in Section **2**, we are able to describe special wave processes by determining exact solutions to the full set of governing field equations through integrating an auxiliary strictly hyperbolic 2×2 homogeneous subsystem which arises as an intermediate step from the resulting reduction process. Finally Section **6** consists of conclusions and some general remarks.

2. Exact solutions via hodograph method and differential constraints

In this Section we outline the reduction procedure recently developed in [8, 9] by the authors in order to find exact solutions to quasilinear hyperbolic systems of form (1). The method combines the hodograph transformation and the differential constraints theory and also provides a useful tool for generating wave-like solutions in a closed form which result to be appropriate for describing the interaction of nonlinear waves. By means of a standard procedure [16] for the system (1) the *Riemann variables* R and S can be introduced

$$R(\mathbf{U}) = \int \mathbf{l}^{(\lambda)} \cdot d\mathbf{U} \qquad S(\mathbf{U}) = \int \mathbf{l}^{(\nu)} \cdot d\mathbf{U} \tag{3}$$

whereupon the system (1) along with the differential constraint (2) can be recast into the following characteristic form

$$R_t + \lambda R_x = \mathbf{l}^{(\lambda)} \cdot \mathbf{B} = C(R,S)\,, \quad S_t + \nu S_x = \mathbf{l}^{(\nu)} \cdot \mathbf{B} = D(R,S) \tag{4}$$

$$R_x = p(R,S,x,t) \tag{5}$$

By direct inspection, it is easy to ascertain that the compatibility of (4) and (5) requires the following conditions to be satisfied

$$\begin{aligned} & p(C - \lambda p)_R = p_t + \lambda p_x + (C - \lambda p) p_R + D p_S \\ & (C - \lambda p)_S + \nu p_S = 0. \end{aligned} \tag{6}$$

In [8] there has been proved the following

Proposition. Let

$$\mathbf{U}_t + A(\mathbf{U})\,\mathbf{U}_x = \mathbf{B}(\mathbf{U}) \tag{1}$$

be a quasilinear strictly hyperbolic 2×2 system endowed with the differential constraint (2), then the solution of (1) in the hodograph plane (R,S) satisfies the following quasilinear first order system

$$\begin{aligned} & x_S = \left(\lambda - \frac{C}{p}\right) t_S, \qquad x_R = \frac{D}{p} t_S + \nu t_R \\ & D t_S + (p(\nu - \lambda) + C)\, t_R = 1 \end{aligned} \tag{7}$$

or alternatively when $p = 0$

$$
\begin{aligned}
& t\,(R) = \int \frac{1}{\mathbf{C}\,(R)} dR \\
& Cx_R + Dx_S = \nu .
\end{aligned}
\tag{8}
$$

■

Proof. Through the standard hodograph transformation

$$
x = x\,(R,S)\,, \qquad t = t\,(R,S)\,, \qquad J = \frac{\partial\,(x,t)}{\partial\,(R,S)} \neq 0 \tag{9}
$$

the system of equations (4)-(5) reduces to

$$
x_S = \lambda t_S - JC, \qquad x_R = \nu t_R + JD \tag{10}
$$

$$
t_S = pJ \tag{11}
$$

so that owing to (6) the two possible forms (8) or (7) arise. □

It is worth noticing that in a different way from the constraint free case where the hodograph transformation reduces the original 2×2 nonhomogeneous quasilinear model to the highly nonlinear first order pair of hodograph equations (10), within the present reduction approach appending the constraint equation (2) to (1) in fact defines a strategy for integrating the resulting hodograph system in a closed form.

Recently in [9] the 2×2 homogeneous systems associated to (1) $(\mathbf{B} = 0)$ has been considered. Here, despite the inherent linearity of the associated hodograph system, it is well known that the classical Riemann method provides in the hodograph plane exact solutions of direct use in wave problems only in a few cases so that several reduction methods have been developed for solving initial or boundary value problems. Hence the problem of generating closed form wave-like solutions to the standard linear hodograph system has been faced in a systematic way by means of the approach we developed hitherto. In particular we proved the following

Proposition. Let

$$
\mathbf{U}_t + A\,(\mathbf{U})\,\mathbf{U}_x = \mathbf{0} \tag{12}
$$

be a quasilinear strictly hyperbolic 2×2 homogeneous system with eigenvalues λ and ν satisfying the condition

$$
\begin{aligned}
& \nabla \left(\frac{\nabla\lambda\, \mathbf{d}^{(\lambda)}}{\nu - \lambda} + \frac{\nabla \left(\nabla\lambda\, \mathbf{d}^{(\lambda)}\right)\, \mathbf{d}^{(\nu)}}{\nabla\lambda\, \mathbf{d}^{(\nu)}} \right) \mathbf{d}^{(\nu)} + \\
& \frac{(\nabla\lambda \mathbf{d}^{(\nu)})(\nabla\lambda \mathbf{d}^{(\lambda)})}{(\nu - \lambda)^2} + \frac{\nabla \left(\nabla\lambda \mathbf{d}^{(\lambda)}\right) \mathbf{d}^{(\nu)}}{\nu - \lambda} = 0
\end{aligned}
\tag{13}
$$

then the solution to (12) in the hodograph plane (R, S) is given by

$$t(R,S) = \Phi(R,S)\left\{\Lambda(R,S)\left(Z(S) - M(R,S)\right) + \frac{dZ}{dS} - M_S\right\}$$
$$x(R,S) = \Phi(R,S)\left\{\left(\lambda\Lambda(R,S) - \lambda_S\right)\left(Z(S) - M(R,S)\right) + \lambda\left(\frac{dZ}{dS} - M_S\right)\right\} \tag{14}$$

where

$$\Phi(R,S) = \exp\left(\int \frac{\lambda_R}{\nu - \lambda}\, dR\right), \quad \Lambda(R,S) = \frac{\lambda_{SS}}{\lambda_S} + \int \left(\frac{\lambda_R}{\nu - \lambda}\right)_S dR$$
$$M(R,S) = \int \frac{\Psi(R,S)}{\gamma(R)\,\lambda_S \Phi(R,S)}\, dR, \quad \Psi(R,S) = \exp\left(\int \frac{\lambda_S}{\lambda - \nu}\, dS\right) \tag{15}$$

with $Z(S)$ and $\gamma(R)$ being arbitrary functions. ■

Proof. Here by limiting ourselves to the non trivial case $\lambda_S \neq 0$ where the governing system (12) written in terms of the Riemann variables decouples, we recast relation (13) in the characteristic form

$$\left(\frac{\lambda_{RS}}{\lambda_S} + \frac{\lambda_R}{\nu - \lambda}\right)_S + \frac{\lambda_{RS}}{\nu - \lambda} + \frac{\lambda_R \lambda_S}{(\nu - \lambda)^2} = 0. \tag{16}$$

Then, by taking (16) into account the equations (6) can be easily solved so that we get

$$p(R,S,x,t) = \gamma(R)\left\{\gamma(R)\left(\frac{\lambda_{RS}}{\lambda_S} + \frac{\lambda_R}{\nu - \lambda}\right)x + \right. \tag{17}$$
$$\left. \gamma(R)\left(\lambda_R - \lambda\left(\frac{\lambda_{RS}}{\lambda_S} + \frac{\lambda_R}{\nu - \lambda}\right)\right)t + \exp\left(\int \frac{\lambda_S}{\lambda - \nu} dS\right)\right\}^{-1}.$$

Furthermore integration of equations (7) along with (17) gives rise to

$$x - \lambda t = \lambda_S \Phi(R,S)\left\{M(R,S) - Z(S)\right\} \tag{18}$$

$$t(R,S) = \Phi(R,S)\left\{\Lambda(R,S)\left(Z(S) - M(R,S)\right) + \frac{dZ}{dS} - M_S\right\}. \tag{19}$$

Then the relations (18) and (19) give the solution (14) provided that condition (13) holds. □

The solution (14) involves the arbitrary functions $Z(S)$ and $\gamma(R)$ and this is useful for solving initial and/or boundary value problems so that the study of nonlinear wave interactions can be carried on. As well known, an exhaustive description of nonlinear wave interactions can be performed only if the explicit evaluation of the characteristic wavelets $\alpha(x,t) = \text{const}$ and $\beta(x,t) = \text{const}$ associated to (1) and ruled by the following equations

$$C^{(\lambda)}: \alpha_t + \lambda\alpha_x = 0\,, \qquad C^{(\nu)}: \beta_t + \nu\beta_x = 0 \tag{20}$$
$$\alpha(x,0) = \beta(x,0) = x$$

is obtained. Within the latter theoretical framework in [8] a special class of nonhomogeneous models allowing for simple wave-like interactions has been considered, whereas in a recent paper [9] the wave interaction problem for 2×2 homogeneous systems has been exhaustively investigated by means of exact solutions to initial value problems. In line with the systematic analysis developed in [8, 9], in the next sections we'll consider exact solution to special initial value problems for different examples of 2×2 governing models.

3. Traffic flow: Rascle model

We consider the 2×2 hyperbolic system of balance laws introduced in [12] for describing traffic flow

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v\right) = 0 \\ &\frac{\partial y}{\partial t} + \frac{\partial}{\partial x}\left(y v\right) = 0 \end{aligned} \tag{21}$$

where $\rho\left(x,t\right)$ and $v\left(x,t\right)$ are, respectively, the density and the velocity of the cars on the roadway while $y = \rho\left(v + P\left(\rho\right)\right)$. The model (21) has been first introduced by Aw & Rascle in order to preserve the anisotropic character of the traffic flow. The function $P\left(\rho\right)$ is smooth and strictly increasing and it satisfies

$$P\left(0\right) = 0, \quad \lim_{\rho \to 0} \rho P'\left(\rho\right) = 0, \quad \rho P''\left(\rho\right) + 2P'\left(\rho\right) > 0 \quad \forall \rho > 0. \tag{22}$$

The last assumption ensures strict hyperbolicity. A standard form for the function $P\left(\rho\right)$ is $P\left(\rho\right) = \rho^{\gamma},\ \gamma > 0$. The characteristic wave speeds associated to (21) are

$$\lambda = v, \qquad \nu = v - \rho P'\left(\rho\right) \tag{23}$$

with corresponding left eigenvectors

$$\mathbf{l}^{(\lambda)} = \begin{pmatrix} P'\left(\rho\right); & 1 \end{pmatrix}; \qquad \mathbf{l}^{(\nu)} = \begin{pmatrix} 0; & 1 \end{pmatrix} \tag{24}$$

If $\rho > 0$ the $\nu-$wave is genuinely nonlinear and the $\lambda-$wave is linearly degenerate [17]. The greatest eigenvalue λ is equal to the flow speed v so that the anisotropic character of the traffic is strictly preserved. The homogeneous governing system (21) does not take into account any dissipation which is inherent in these problems. Upgraded models were considered in later papers [18, 20] and recently in [19] a nonhomogeneous model has been investigated within the context of differential constraints theory and several generalized Riemann problems have been solved. However it has to be remarked that, owing to the basic modelling assumptions (22), the hyperbolic governing system in point possesses a wave speed which is always exceptional (as the material wave speed in fluid dynamics) while the other wave speed never does. Because of (24) the Riemann variables (3) specialize to

$$R = v + P\left(\rho\right), \qquad S = v \tag{25}$$

The integration of the associated hodograph system yields the following classes of closed form solutions in terms of Riemann invariants

$$
\begin{aligned}
t(R,S) &= \int \frac{f'(R-S)}{\Gamma(R)}\, dR + Z'(S) \\
x(R,S) &= \int \frac{Sf'(R-S) + f(R-S)}{\Gamma(R)}\, dR + SZ'(S) - Z(S)
\end{aligned} \tag{26}
$$

where $f(R-S) = \dfrac{1}{\rho}$, $Z(S)$ is an arbitrary function and

$$
p(R,S) = \frac{\Gamma(R)}{f(R-S)} \tag{27}
$$

is the constraint $p-$function. In view of describing in the $(x,t)-$ plane the interaction of two simple waves travelling along characteristic curves of different type , we assume that at $t=0$, the $C^{(\lambda)}$ travelling pulse occupies the region $x_l \le x \le -x_f$ and the $C^{(\nu)}$ travelling pulse the region $x_f \le x \le x_r$ (see Figure 1). Both pulses separate regions of constant states, furthermore we suppose $\lambda > \nu > 0$ and we assume the initial data smooth functions as follows

$$
\begin{aligned}
R(x,0) = \mathcal{R}(x) &= \begin{cases} R_1 & x < x_l \\ \omega(x) & x_l \le x \le -x_f \\ R_2 & x > -x_f \end{cases} \\
S(x,0) = \mathcal{S}(x) &= \begin{cases} S_1 & x < x_f \\ \zeta(x) & x_f \le x \le x_r \\ S_2 & x > x_r \end{cases}
\end{aligned} \tag{28}
$$

$$
\omega(x_l) = R_1, \qquad \omega(-x_f) = R_2, \qquad \zeta(x_f) = S_1, \qquad \zeta(x_r) = S_2.
$$

Moreover, owing to the differential constraint (5) along with (27), the initial data $\mathcal{R}(x)$ and $\mathcal{S}(x)$ must satisfy the condition

$$
\Gamma(\mathcal{R}(x)) = \mathcal{R}'(x)\, f(\mathcal{R}(x) - \mathcal{S}(x)).
$$

In the $(x,t)-$plane there are several distinct simple wave regions where the characteristic wavelets $x - \lambda t$, $x - \nu t$ can be explicitly calculated so that we are able to achieve an exhaustive analysis of the nonlinear wave interaction in point by investigating the alteration in the profile as well as in the wave time distortion of the emerging pulses. Here we consider firstly the regular pulse propagating along the characteristic $C^{(\lambda)}$ associated to the linearly degenerate eigenvalue λ and we have (see Figure 1)

$$\text{REGION II}\left\{\begin{array}{c} R=\omega\left(\alpha\right), \quad S=S_1 \qquad x_l\leq\alpha\leq -x_f, \quad x_l\leq\beta\leq x_f \\ \\ x-\lambda\left(\omega\left(\alpha\right),S_1\right)t=\alpha \end{array}\right. \tag{29}$$

$$\text{REGION V}\left\{\begin{array}{c} R=\omega\left(\alpha\right), \quad S=S_2 \qquad x_l\leq\alpha\leq -x_f, \quad \beta\geq x_r \\ \\ x-\lambda\left(\omega\left(\alpha\right),\, S_2\right)t=\alpha+I_r\left(\alpha\right) \\ \\ I_r\left(\alpha\right)=\int_\alpha^{x_r}\left(1-\frac{f(\mathcal{R}(\xi)-S_2)}{f(\mathcal{R}(\xi)-\mathcal{S}(\xi))}\right)d\xi \end{array}\right. \tag{30}$$

It has to be remarked that, as it is expected and according to the general analysis developed in [9], from (30) it follows that the travelling simple wave associated to the exceptional eigenvalue λ never evolves into a shock. Furthermore, in the special case $S_1=S_2=S_0$ (red lines in Figure 1) we find $I_r=\displaystyle\int_{x_f}^{x_r}\left(1-\frac{f\left(R_2-S_0\right)}{f\left(R_2-\zeta\left(\xi\right)\right)}\right)d\xi=$ const so that the only effect of the interaction (region IV) on the right emerging wave is to produce a profile which would correspond to the initial conditions at $t=0$

$$\mathcal{R}\left(x\right)=\left\{\begin{array}{cc} \omega\left(x-I_r\right) & x_l+I_r\leq x\leq -x_f+I_r \\ R_0 & \text{otherwise} \end{array}\right.$$
$$\tag{31}$$
$$\mathcal{S}\left(x\right)=\left\{\begin{array}{cc} \zeta\left(x\right) & x_f\leq x\leq x_r \\ S_0 & \text{otherwise} \end{array}\right.$$

that is a change in the origin of x in the original pulse (dashed red lines) and the interaction is in fact soliton-like [2, 21].
Next we consider the simple wave corresponding to a regular pulse propagating along the characteristic $C^{(\nu)}$ and in this case we have (see Figure 1)

$$\text{REGION III}\left\{\begin{array}{c} R=R_2, \quad S=\zeta\left(\beta\right), \quad -x_f\leq\alpha\leq x_r, \quad x_f\leq\beta\leq x_r \\ \\ x-\nu t=\beta \end{array}\right. \tag{32}$$

$$\text{REGION VI}\left\{\begin{array}{c} R=R_1, \quad S=\zeta\left(\beta\right), \qquad \alpha\leq x_l, \quad x_f\leq\beta\leq x_r \\ \\ x-\nu t=\beta+I_l\left(\beta\right) \\ \\ I_l\left(\beta\right)=\int_{x_l}^{\beta}\left(\frac{f(R_1-\mathcal{S}(\beta))}{f(\mathcal{R}(\xi)-\mathcal{S}(\xi))}\,\frac{f'(\mathcal{R}(\xi)-\mathcal{S}(\beta))}{f'(R_1-\mathcal{S}(\beta))}-\frac{f(\mathcal{R}(\xi)-\mathcal{S}(\beta))}{f(\mathcal{R}(\xi)-\mathcal{S}(\xi))}\right)d\xi \end{array}\right. \tag{33}$$

Here the critical time associated to the $C^{(\nu)}$ travelling pulse in absence of interaction ($t_c^{(III)}$) and the critical time after the interaction process ($t_c^{(VI)}$) are related as follows

[9]

$$t_c^{(VI)} = t_c^{(III)} - \min_{\beta} \left\{ \frac{1}{\nu_S \zeta'(\beta)} \frac{dI_l(\beta)}{d\beta} \right\} \tag{34}$$

with $\nu_S = \dfrac{2P'(\rho) + \rho P''(\rho)}{P'(\rho)} > 0$. From (34) it follows that the wave time distortion due to the interaction is strictly related to the choice of the initial data $R(x, 0)$ and $S(x, 0)$. For the particular choice $P(\rho) = \rho^\gamma$, $R_1 = R_2 = R_0$ we have

$$t_c^{(VI)} = -\frac{1}{\nu_S} \left\{ \frac{1}{\zeta'(\beta)} + \frac{\gamma + 1}{\gamma} \int_{x_l}^{-x_f} \frac{\rho(\omega(\xi), S_1)(R_0 - \omega(\xi))}{(\rho(\omega(\xi), \zeta(\beta)))^{1+2\gamma}} d\xi \right\}. \tag{35}$$

Finally, in order to illustrate the interaction process as well as the wave behavior described hitherto, we consider the numerical solutions of the system (21) with initial data simulating two simple waves travelling along different families of characteristic curves. In particular the simple wave travelling along the $C^{(\lambda)}$ curve, obtained when S =const, is a contact discontinuity corresponding to situations where each car just follows the leading car at the same speed. The simple wave travelling along the $C^{(\nu)}$ curve, obtained when R =const, describes the more realistic situation of a regular pulse (if $t < t_c$) with speed dependence of car density $v = v(\rho)$. Here we have considered a travelling simple wave connecting smoothly the left constant state (R_0, S_1) with the right constant state (R_0, S_2) and simulating a rarefaction-like wave ($S_1 < S_2$) so that, as well known and in line with (35), in absence of interaction ($\omega(x) = R_0$) it never evolves into a shock wave. In Figure 2 and in Figure 3 we show the resulting interaction process which simulates a situation where an amount of fast cars running towards slow cars results in the fast cars slowing down and the slow cars speeding up. We notice that, as it is expected from (35), no critical time occurs for the rarefaction-like wave.

4. Nonlinear transmission lines: quasi-simple waves interactions

In a different way from 2×2 quasilinear hyperbolic homogeneous systems of PDEs, for governing models involving source-like terms as in (1) the Riemann variables are no longer invariant along the associated family of characteristic curves so that direct use of standard hodograph transformation does not permit to investigate thoroughly the interaction of waves.

Here we consider the special case where the system (4) takes the form:

$$R_t + \lambda R_x = C(R), \qquad S_t + \nu S_x = D(S). \tag{36}$$

Let α and β be the left and right propagating characteristic wavelets defined in (20); we set

$$\widehat{R}(\alpha) = \int \frac{dR}{C(R)} - t, \quad \widehat{S}(\beta) = \int \frac{dS}{D(S)} - t \tag{37}$$

so that it is straightforward to ascertain that along the characteristics associated to (36) $\widehat{R}(\alpha)$ and $\widehat{S}(\beta)$ behave as Riemann invariants.

Next we prove that the combined use of the involutive differential constraint (2) and of the hodograph transformation allows us to construct explicit solutions to (36) in the entire domain where existence is expected to hold. These solutions expressed in terms of

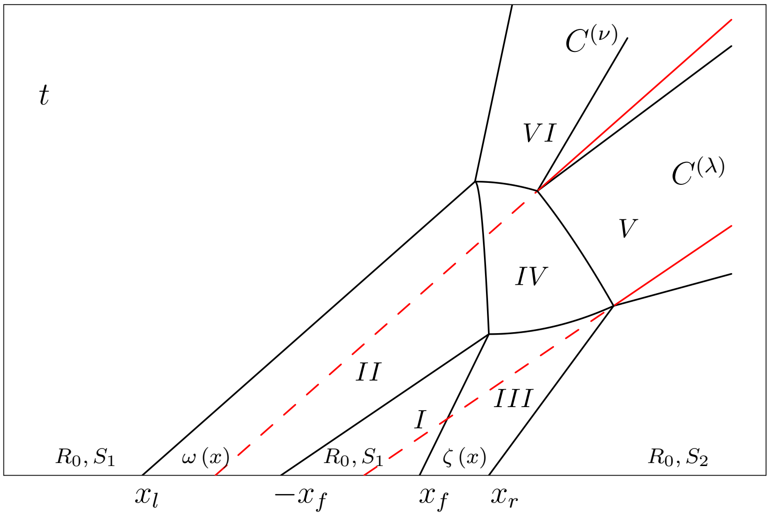

Figure 1. Qualitative behavior in the (x, t) −plane of the interaction between two simple waves travelling along different characteristic curves. The initial data for $\mathcal{R}(x)$ and $\mathcal{S}(x)$ are as in (28) with $R_1 = R_2 = R_0$. The red lines would correspond to the choice $S_1 = S_2 = S_0$ when the interaction is soliton-like [2]. The dashed lines evidentiate that the emerging pulse in fact behaves as it results to be generated by the initial data (31).

the "Riemann-like invariants" (37) turn out to be useful to investigate nonlinear wave processes.

Through the change of variables

$$R = R\left(\widehat{R}, \widehat{S}\right), \quad S = S\left(\widehat{R}, \widehat{S}\right) \tag{38}$$

the relations (7) specialize to

$$x_{\widehat{S}} = \lambda t_{\widehat{S}}, \qquad x_{\widehat{R}} = \nu t_{\widehat{R}} \tag{39}$$

$$t_{\widehat{R}} = \frac{C}{p(\nu - \lambda)}. \tag{40}$$

Hence the set of equations (6) selects all possible structural conditions to be obeyed by λ, ν, $C(R)$, $D(S)$ and consequently the compatible forms of the constraint function p in order to make consistent the reduction process in point. Once p is determined, insertion in (40) allows us to obtain special classes of solutions in a closed form to the nonlinear governing model in point. Because of their inherent wave features these solutions can be of direct use in studying nonlinear superposition of regular pulses.

For a complete list of all resulting cases we refer to [8]; here, by way of illustration, we consider a possible model of physical interest. Within mathematical modelling of nonlinear dissipative transmission lines, in applications where electromagnetic pulses propagating at finite wave speeds have to be taken into account the following hyperbolic system is

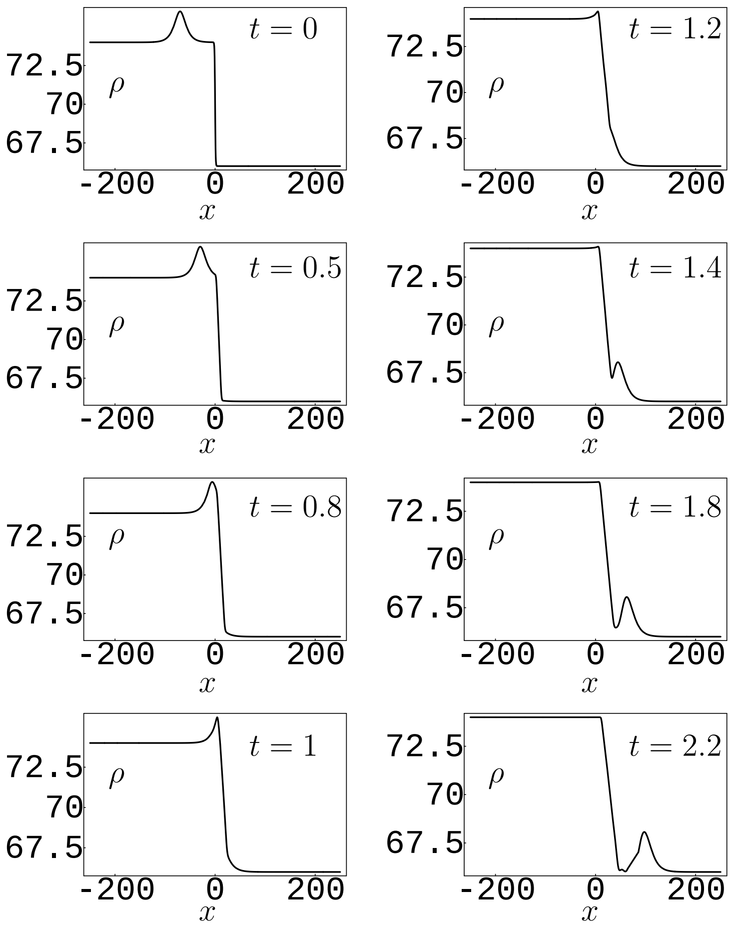

Figure 2. Simulation of two interacting simple waves travelling in the positive x-direction. The numerical solution of (21) with model laws $P(\rho) = \rho$ is obtained with initial data $\mathcal{S}(x) = 85 + 4\tanh(x)$, $\mathcal{R}(x) = 155 + 2\,\mathrm{sech}\,(0.1\,(x+70))$.

assumed [13]

$$L\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = -ru$$

$$g(v)\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = f(v) \tag{41}$$

where L is the constant inductance, r the constant resistance, $g(v)$ the capacitance, $f(v)$ the conductance whereas $u(x,t)$ and $v(x,t)$ denote, respectively, current and voltage.

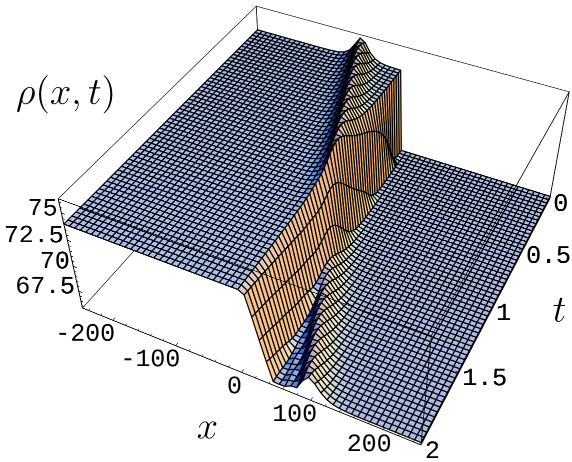

Figure 3. 3D-plots of nonlinear wave interaction of two simple waves. The profile for density $\rho\,(x,t)$ is obtained with the same initial data as in Figure 2.

The model (41) has been recently investigated in [22] where several Generalized Riemann Problems have been solved by a combined use of the differential constraints approach and of the method of characteristics. Furthermore, functional forms for the material response functions $f\,(v)$ and $g\,(v)$ have been also characterized.

The characteristic wave speeds associated to (41) are

$$\lambda = \frac{1}{\sqrt{Lg\,(v)}}; \qquad \nu = -\frac{1}{\sqrt{Lg\,(v)}} \tag{42}$$

with corresponding left eigenvectors

$$\mathbf{l}^{(\lambda)} = \left(\ 1; \quad \sqrt{\tfrac{g(v)}{L}}\ \right); \qquad \mathbf{l}^{(\nu)} = \left(\ 1; \quad -\sqrt{\tfrac{g(v)}{L}}\ \right). \tag{43}$$

Because of (43), the Riemann variables (3) specialize to

$$R = u + \frac{1}{\sqrt{L}}\int \sqrt{g\,(v)}dv, \qquad S = u - \frac{1}{\sqrt{L}}\int \sqrt{g\,(v)}dv \tag{44}$$

whereas system (4) becomes

$$\begin{aligned} R_t + \frac{1}{\sqrt{Lg\,(v)}}R_x &= -\frac{r}{L}R + \frac{1}{\sqrt{L}}\left(\frac{r}{L}\int \sqrt{g\,(v)}dv + \frac{f\,(v)}{\sqrt{g\,(v)}}\right) \\ S_t - \frac{1}{\sqrt{Lg\,(v)}}S_x &= -\frac{r}{L}S - \frac{1}{\sqrt{L}}\left(\frac{r}{L}\int \sqrt{g\,(v)}dv + \frac{f\,(v)}{\sqrt{g\,(v)}}\right). \end{aligned} \tag{45}$$

Bearing in mind the structure of the source-like term involved in (36), in the sequel we focus our attention on material response functions satisfying

$$\frac{r}{L}\int\sqrt{g\left(v\right)}dv+\frac{f\left(v\right)}{\sqrt{g\left(v\right)}}=0. \tag{46}$$

Therefore if the structural condition (46) is obeyed by $f\left(v\right)$ and $g\left(v\right)$, then the Riemann-like invariants $\widehat{R}\left(\alpha\right)$ and $\widehat{S}\left(\beta\right)$ are defined through the relations

$$\widehat{R}\left(\alpha\right)=R\left(x,t\right)\exp\left(\frac{r}{L}t\right);\quad \widehat{S}\left(\beta\right)=S\left(x,t\right)\exp\left(\frac{r}{L}t\right). \tag{47}$$

Although the Riemann variables in the present dissipative case are not invariant along the characteristic curves $C^{(\lambda)}$ and $C^{(\nu)}$ respectively, nevertheless we will show later that the representation of the general solution of the system (45) can be obtained in terms of the characteristic parameters α and β. Hence, through a suitable choice of initial conditions there can be generated pulse-like solutions to the nonhomogeneous system under consideration whose behavior extends that of classical simple waves ("quasi-simple waves").

The relation (46) defines the classes of governing models which are consistent with (47). In the sequel we limit ourselves to the following constitutive case also considered in [22]

$$g\left(v\right)=g_0 v^{-4},\qquad f\left(v\right)=-3\frac{r}{L}g_0 v^{-3} \tag{48}$$

where g_0 is a material constant. Bearing in mind (20), here we have

$$\begin{aligned}
C^{(\lambda)}:\ \left.\frac{dx}{dt}\right|_{\alpha=const}&=c_0\left(\widehat{R}\left(\alpha\right)-\widehat{S}\left(\beta\right)\right)^{-2}\exp\left(\frac{2r}{L}t\right)\\
C^{(\nu)}:\ \left.\frac{dx}{dt}\right|_{\beta=const}&=-c_0\left(\widehat{R}\left(\alpha\right)-\widehat{S}\left(\beta\right)\right)^{-2}\exp\left(\frac{2r}{L}t\right)
\end{aligned} \tag{49}$$

being $c_0=\frac{4}{L}\sqrt{\frac{g_0}{L}}$. In this case from (6) the following form for the p function is obtained

$$p\left(x,t,\widehat{R},\widehat{S}\right)=\frac{\left(\widehat{S}-\widehat{R}\right)K_0\left(\widehat{R}\right)\exp\left(-\frac{r}{L}t\right)}{2\left(xK_0\left(\widehat{R}\right)+1\right)} \tag{50}$$

with $K_0\left(\widehat{R}\right)$ an arbitrary function. Owing to (50), the first order nonlinear system of PDEs (39) and (40) allows for explicit integration and the following representation of the solution in terms of the characteristic parameters α and β is easily obtained

$$\exp\left(\frac{2r}{L}t(\alpha,\beta)\right) = 1 + \frac{2r}{c_0 L}\left\{\left(\widehat{S}(\beta) - \widehat{R}(\alpha)\right)\left(M\left(\widehat{R}(\alpha)\right) - N\left(\widehat{S}(\beta)\right)\right) + m\left(\widehat{R}(\alpha)\right) + n\left(\widehat{S}(\beta)\right)\right\}$$

$$x(\alpha,\beta) = \frac{M\left(\widehat{R}(\alpha)\right) + N\left(\widehat{S}(\beta)\right)}{\widehat{R}(\alpha) - \widehat{S}(\beta)} \tag{51}$$

$$\frac{dm}{d\alpha} = 2M\left(\widehat{R}(\alpha)\right)\widehat{R}'(\alpha), \quad \frac{dn}{d\beta} = 2N\left(\widehat{S}(\beta)\right)\widehat{S}'(\beta)$$

$$M\left(\widehat{R}\right) = -\int \frac{d\widehat{R}}{K_0\left(\widehat{R}\right)}$$

with $N\left(\widehat{S}\right)$ arbitrary function.
As far as the Cauchy problem is concerned, we choice initial data for R and S as follows

$$R(x,0) = \widehat{R}(x,0) = \mathcal{R}(x), \quad S(x,0) = \widehat{S}(x,0) = \mathcal{S}(x) \qquad -\infty < x < +\infty$$

so that from (51) the functions $M(\alpha), N(\beta), m(\alpha), n(\beta)$ are determined

$$\begin{aligned}
M(\alpha) &= \frac{1}{2}\int_{x_0}^{\alpha}\left(\mathcal{R}(\xi) + \mathcal{S}(\xi)\right)d\xi - \alpha\mathcal{R}(\alpha)\\
N(\beta) &= \frac{1}{2}\int_{x_0}^{\beta}\left(\mathcal{R}(\xi) + \mathcal{S}(\xi)\right)d\xi - \beta\mathcal{S}(\beta)\\
m(\alpha) &= \alpha\mathcal{R}(\alpha)\int_{x_0}^{\alpha}\left(\mathcal{R}(\xi) + \mathcal{S}(\xi)\right)d\xi - \int_{x_0}^{\alpha}\mathcal{R}(\xi)\,\mathcal{S}(\xi)\,d\xi - \alpha\mathcal{R}^2(\alpha)\\
n(\beta) &= \beta\mathcal{S}^2(\beta) - \mathcal{S}(\beta)\int_{x_0}^{\beta}\left(\mathcal{R}(\xi) + \mathcal{S}(\xi)\right)d\xi + \int_{x_0}^{\beta}\mathcal{R}(\xi)\,\mathcal{S}(\xi)\,d\xi
\end{aligned} \tag{52}$$

while, because of (5) and (50), $\mathcal{R}(x)$ and $\mathcal{S}(x)$ must satisfy the following differential condition

$$\left[1 + xK_0(\mathcal{R}(x))\right]\mathcal{R}'(x) = \frac{\mathcal{S}(x) - \mathcal{R}(x)}{2}K_0(\mathcal{R}(x)). \tag{53}$$

As usual in order to describe the interaction of two opposite travelling quasi-simple waves in the (x,t) −plane we are led to the following choice (see Figure 4)

$$\mathcal{R}(x) = \begin{cases} \omega(x) & x_l \le x \le -x_f \\ R_0 & \text{otherwise} \end{cases} \quad ; \quad \mathcal{S}(x) = \begin{cases} \zeta(x) & x_f \le x \le x_r \\ S_0 & \text{otherwise} \end{cases} \tag{54}$$

$$\omega(x_l) = \omega(-x_f) = R_0, \qquad \zeta(x_f) = \zeta(x_r) = S_0.$$

In (54) S_0 is arbitrary whereas, according to (53), R_0 is such that $K_0(R_0) = 0$. In the (x,t) −plane there are several distinct regions where the characteristic wavelets can be explicitly calculated. In particular for the right travelling quasi-simple wave we have

$$\text{REGION II} \begin{cases} \widehat{R} = \widehat{R}(\alpha), \quad \widehat{S} = S_0 \qquad x_l \le \alpha \le -x_f, \quad x_l \le \beta \le x_f \\ x - \frac{c_0 L}{2r}\left(\widehat{R}(\alpha) - S_0\right)^{-2}\left(\exp\left(\frac{2r}{L}t\right) - 1\right) = \alpha \end{cases}$$

$$\text{REGION V} \begin{cases} \widehat{R} = \widehat{R}(\alpha), \quad \widehat{S} = S_0 \qquad x_l \le \alpha \le -x_f, \quad \beta \ge x_r \\ x - \frac{c_0 L}{2r}\left(\widehat{R}(\alpha) - S_0\right)^{-2}\left(\exp\left(\frac{2r}{L}t\right) - 1\right) = \alpha + I_r(\alpha) \\ I_r(\alpha) = \frac{R_0 - S_0}{\left(\widehat{R}(\alpha) - S_0\right)^2}\int_{x_f}^{x_r}\left(\zeta(\xi) - S_0\right) d\xi \end{cases} \tag{55}$$

we can observe that the right travelling pulse traverses region II where it is a quasi-simple wave, it interacts with the left travelling pulse in region IV, emerges in the region V as a quasi-simple wave identical with that produced by the initial conditions (54) at

$$t = t^* = \frac{L}{2r}\ln\left(1 + \frac{2r}{c_0 L}(S_0 - R_0)\int_{x_f}^{x_r}\left(\zeta(\xi) - S_0\right) d\xi\right) \tag{56}$$

Thus, the only effect of the interaction on the right travelling emerging wave is to produce a profile which would correspond to change the origin of t in the original pulse. A similar argument holds for the left travelling pulse which traverses the region III as a quasi-simple wave, interacts in the region IV with the right travelling wave and emerges with unaltered profile in region VI. The interaction process is qualitatively outlined in Figure 4. The dashed lines evidentiate that the emerging pulse in fact behaves as it results to be generated by the initial data (54) at $t = t^*$. Therefore these pulses evolve as hyperbolic waves but in the interaction process exhibit a behavior which is in fact similar to that of solitons. The results concerning the interaction of two quasi-simple waves have been also illustrated by numerical integration of the system (41) endowed with the model laws (48) to which there

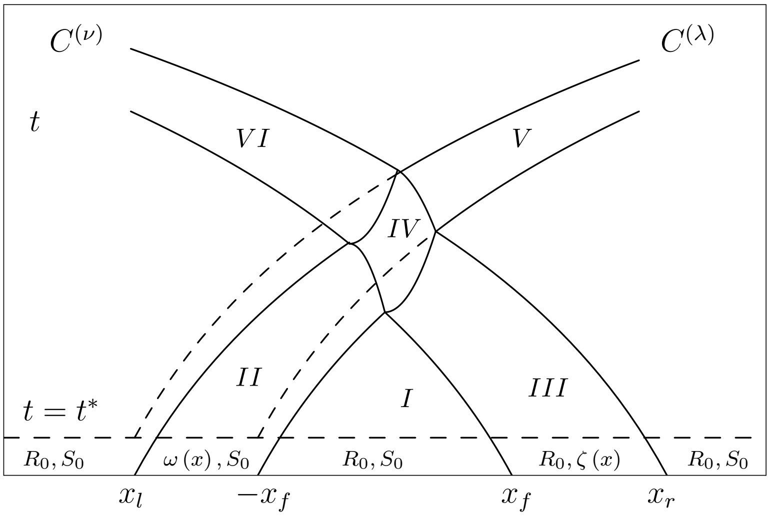

Figure 4. Qualitative behavior in the (x, t) −plane of the interaction between two quasi-simple waves travelling in opposite directions. The initial data for $\mathcal{R}(x)$ and $\mathcal{S}(x)$ are as in (54). The dashed lines point out that the right travelling emerging pulse in fact behaves as it results to be generated by the initial data (54) at $t = t^*$ (dashed horizontal line).

correspond for current u and voltage v, respectively, the initial data

$$u(x,0) = \frac{\mathcal{R}(x) + \mathcal{S}(x)}{2}$$
$$v(x,0) = \sqrt{\frac{g_0}{L}}\frac{2}{\mathcal{S}(x) - \mathcal{R}(x)}. \tag{57}$$

In particular we consider the numerical solution of (41) obtained with initial conditions simulating two localized pulses originating at $t = 0$ from data which are not of quasi-simple wave. Later they propagate through regions which are adjacent to quasi-constant states so that the pulses in point, after a finite time, become quasi-simple waves. In Figure 5 and Figure 6 it is depicted the resulting interaction process.

5. Quasilinear hyperbolic systems of multicomponent field PDEs

In this section we consider a multicomponent quasi-linear hyperbolic homogeneous system of first-order PDEs

$$A^\gamma \frac{\partial \mathbf{U}}{\partial x_\gamma} = 0 \tag{58}$$

where $x_\gamma (\gamma = 0, ...m)$ denote the independent variables, $\mathbf{U} = \mathbf{U}(x_\gamma) \in \mathrm{R}^N$ is a column vector representing the field while A^γ are $N-$order square matrices. In general for systems of form (58) it is a hard task to determine solution in a closed form to initial value

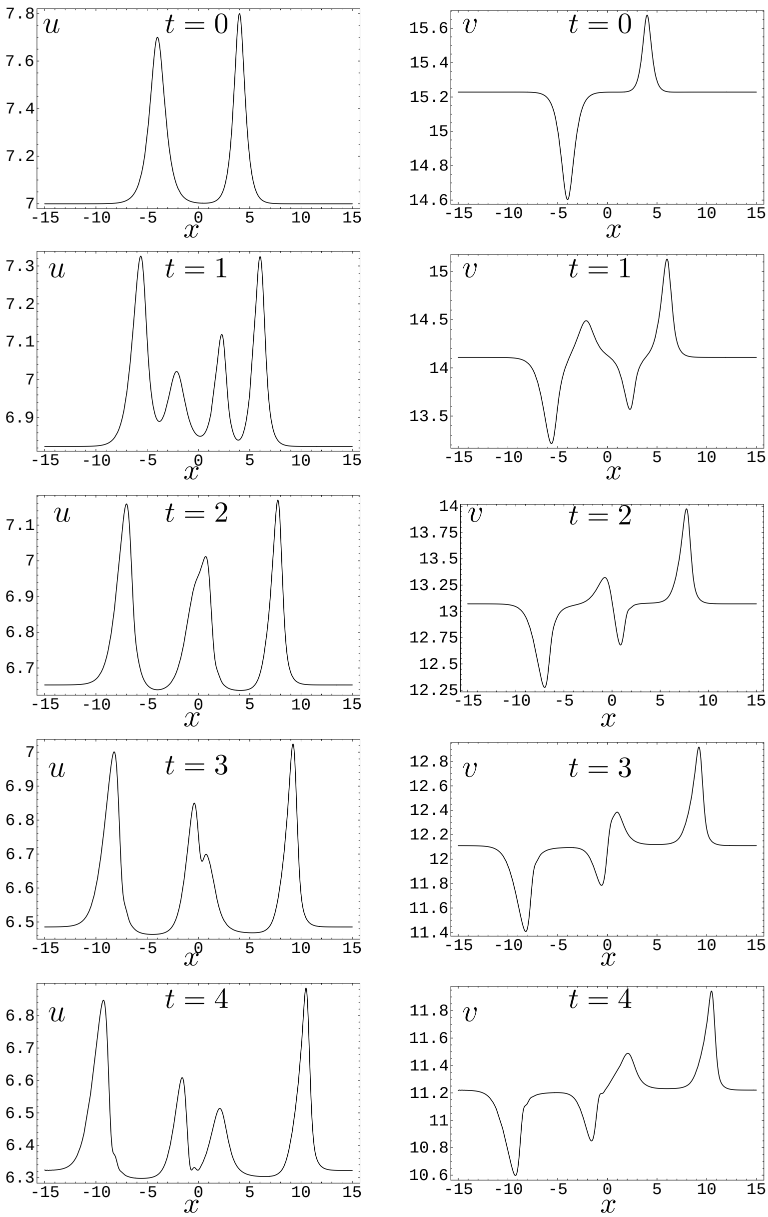

Figure 5. Simulation of two interacting quasi-simple waves emerging from localized pulses. The numerical solution of equations (41) with model laws (48) is obtained with parameters $L = 1.1, r = 0.28, g_0 = 10^4$ and initial data from (44) with $\mathcal{R}(x) = \operatorname{sech}(2.2(x-4)) + 0.4 \operatorname{sech}(1.4(x+4)), \quad \mathcal{S}(x) = 14 + \operatorname{sech}(1.6(x+4)) + 0.6 \operatorname{sech}(2(x-4))$.

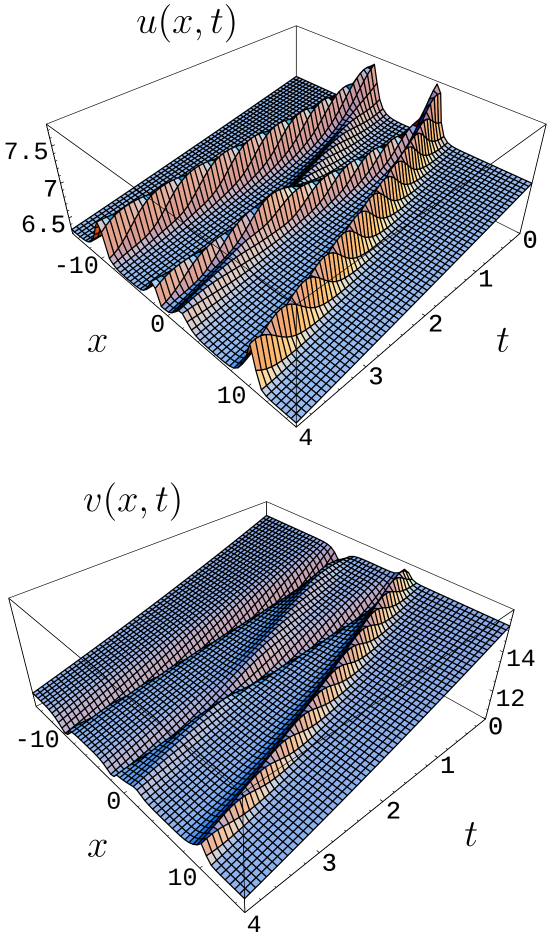

Figure 6. 3D-plots of nonlinear wave interaction of the two quasi-simple waves emerging from two localized pulses. The profiles for $u\,(x,t)$ and $v\,(x,t)$ are obtained with the same initial data as in Figure 5.

problems in order to achieve an exact description of wave interaction processes. Although the approach worked out in the previous Sections concerns wave analysis for 2×2 strictly hyperbolic systems, here we would like to underline that it can be also viewed within the broader context of investigating wave interaction processes ruled by quasilinear hyperbolic homogeneous systems of $N > 2$ first-order PDEs eventually involving more space variables. Actually, along the leading lines of investigation proposed by the authors in [15] in the sequel we outline the main steps of a reduction method worked out for determining exact solutions with inherent wave features to the class of models (58) through the integration of an auxiliary 2×2 subsystem. We assume (58) to be hyperbolic with respect to the time coordinate $x_0 = t$, namely for each unit space vector $\mathbf{n} \equiv (n_i)$, $i = 1..m$ the equation

$$\det\left(-\lambda A^0 + A_n\right) = 0, \qquad A_n = \sum_{i=1}^{m} A^i n_i \tag{59}$$

has N real roots λ_i and the matrix A_n has a complete set of left and right eigenvectors. The autonomous system (58) is invariant with respect to the one-parameter (ϵ) infinitesimal translation group

$$x'_{\overline{\gamma}} = x_{\overline{\gamma}}, \qquad x'_\sigma = x_\sigma + \epsilon w_\sigma \qquad \sigma \neq \overline{\gamma} \tag{60}$$

where w_σ are constants and $x_{\overline{\gamma}}$ is a chosen space or time coordinate. The invariant solutions of (58)

$$\mathbf{U} = \mathbf{U}\left(x_{\overline{\gamma}}, \xi\right), \qquad \xi = \sum_{\sigma \neq \overline{\gamma}} w_\sigma x_\sigma \tag{61}$$

corresponding to (60) are defined by the system

$$A^{\overline{\gamma}} \frac{\partial \mathbf{U}}{\partial x_{\overline{\gamma}}} + \widetilde{A} \frac{\partial \mathbf{U}}{\partial \xi} = 0, \qquad \widetilde{A} = \sum_{\sigma \neq \overline{\gamma}} w_\sigma A^\sigma \,. \tag{62}$$

Owing to the invariance, the characteristic surfaces of the system (58) corresponding to the class of solutions (61) transform into characteristic curves of the system (62) [23]. Now we search for the following class of solutions

$$u_1 = u_1\left(x_{\overline{\gamma}}, \xi\right), \qquad u_2 = u_2\left(x_{\overline{\gamma}}, \xi\right), \qquad u_k = u_k\left(u_1, u_2\right), \quad k = 3, ..N. \tag{63}$$

Insertion of (63) into (62) yields the over determined system

$$\widetilde{B} \frac{\partial \mathbf{V}}{\partial x_{\overline{\gamma}}} + \widetilde{C} \frac{\partial \mathbf{V}}{\partial x_\xi} = 0 \tag{64}$$

$$\overline{B} \frac{\partial \mathbf{V}}{\partial x_{\overline{\gamma}}} + \overline{C} \frac{\partial \mathbf{V}}{\partial x_\xi} = 0 \tag{65}$$

where the matrix coefficients $\widetilde{B}\,, \widetilde{C}\,, \overline{B}\,, \overline{C}$ are defined in terms of the field vector $\mathbf{V} = \begin{pmatrix} u_1\,; & u_2 \end{pmatrix}^T$ as follows

$$A^{\overline{\gamma}} \nabla_{\mathbf{V}} \mathbf{U} = \begin{pmatrix} \widetilde{B}\,; & \overline{B} \end{pmatrix}^T, \quad \widetilde{A} \nabla_{\mathbf{V}} \mathbf{U} = \begin{pmatrix} \widetilde{C}\,; & \overline{C} \end{pmatrix}^T, \qquad \nabla_{\mathbf{V}} = \partial / \partial \mathbf{V}.$$

Next we assume the two equations (64) to be independent while we require the remaining $N-2$ equations (65) to be identically satisfied by any solution $\mathbf{V} = \mathbf{V}\left(x_2, \xi\right)$ of the system (64). Hence the following compatibility conditions must be fulfilled

$$L\widetilde{B} = \overline{B}, \qquad L\widetilde{C} = \overline{C} \tag{66}$$

where the matrix L is a Lagrange multiplier. The over determined set of $4\left(N-2\right)$ conditions (66) characterizes the $N-2$ functions $u_k = u_k\left(u_1, u_2\right)$ along with the $2\left(N-2\right)$ components of the matrix L in order that any solution $\mathbf{V} = \mathbf{V}\left(x_{\overline{\gamma}}, \xi\right)$ of (64) provides through (63) a solution of the original system (58). The structure of the solutions to system (58) considered herein resembles that of the partially invariant (double wave) solutions studied in [23, 24, 25], with the functions u_1 , u_2 playing the role of wave parameters. However, on this subject we observe first that the solutions (61) are in fact invariant solutions of the system (58). Furthermore, unlike the procedure worked out to determine double-wave solutions here we treat (65) as a set of supplementary equations [26, 27] which must be identically satisfied by any solution u_1 , u_2 of the system (64). That permits us to construct solutions to systems of form (58) by means of the reduction approach proposed for integrating 2×2 hyperbolic systems and which are of more direct use for solving problems of interest in non-linear wave propagation. As far as the hyperbolicity of the auxiliary 2×2 system (64) is concerned, in [15] it has been proved the following

Proposition. The hyperbolicity of the system (62) induces the hyperbolicity of the system (64) with respect to the $\xi-$direction. ■

In particular two of the characteristic speeds of the spectrum of $\lambda' s$, solutions of (59) are the eigenvalues of $\widetilde{B}$ with respect to the matrix $\widetilde{C}$ selected by the relations (63) and by the pair of equations (64). Furthermore, the reduction process developed in [15] preserves the genuine nonlinearity or the exceptionality of an eigenvalue. So that by requiring (64) to be strictly hyperbolic, the combined use of hodograph method and differential constraints theory allows us to obtain exact solutions to (58) which exhibit wave behavior and can be used for studying nonlinear wave interactions of double plane waves.

By way of illustration, in the following we consider the system of balance laws for isotropic ferromagnetic materials. In absence of electric charge and current, the Maxwell field equations of electrodynamics are

$$\dot{\mathbf{B}} + \nabla \times \mathbf{E} = \mathbf{0}, \qquad \dot{\mathbf{D}} - \nabla \times \mathbf{H} = \mathbf{0} \tag{67}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0 \tag{68}$$

$$\mathbf{B} = \mu \left(H^2\right) \mathbf{H}, \qquad \mathbf{D} = \varepsilon \mathbf{E} \tag{69}$$

where the overdot denotes time derivative whereas $\mathbf{x} = (x_1, x_2, x_3)^T$ is the position vector. $\mathbf{E}(\mathbf{x}, t) = (E_1, E_2, E_3)^T$ and $\mathbf{H}(\mathbf{x}, t) = (H_1, H_2, H_3)^T$ are the electric and magnetic fields respectively, ε is the constant electric permeability whereas the magnetic permeability μ satisfies the following conditions [28]

$$\mu > 0, \qquad \mu + 2\mu' H^2 > 0 \tag{70}$$

so that the system of equations (67) turns out to be hyperbolic. Nonlinear wave propagation for system (67) has been exhaustively investigated by several authors. In particular in [29] the propagation of acceleration waves has been studied and the characteristic speeds have been determined, it has been proved that two eigenvalues are always exceptional whereas, under suitable constitutive assumptions, the system is fully exceptional. Setting

$$\xi = \mathbf{w} \cdot \mathbf{x} - st, \qquad \mathbf{w} \equiv (w_1, w_2, 0)^T \tag{71}$$

in the present case (62) specializes to

$$A^3 \frac{\partial \mathbf{U}}{\partial x_3} + \widetilde{A} \frac{\partial \mathbf{U}}{\partial \xi} = 0 \tag{72}$$

where

$$\mathbf{U}=\left[\begin{array}{c}\mathbf{H}\\ \mathbf{E}\end{array}\right],\quad \widetilde{A}=\left[\begin{array}{cc}P & Q\\ -Q & W\end{array}\right],\quad A^3=\left[\begin{array}{cc}0 & \overline{Q}\\ -\overline{Q} & 0\end{array}\right] \tag{73}$$

$$P=-s\left(\mu\mathbf{I}+2\mu'\mathbf{H}\otimes\mathbf{H}\right),\quad W=-s\varepsilon\mathbf{I},$$

$$\mathbf{I}=\|\delta_{ij}\|\,,\quad \mathbf{H}\otimes\mathbf{H}=\|H_iH_j\| \qquad i,j=1,2,3$$

$$Q\mathbf{v}=\mathbf{w}\times\mathbf{v},\quad \overline{Q}\mathbf{v}=\mathbf{k}\times\mathbf{v},\quad \forall\mathbf{v}\in R^3,\quad \mathbf{k}\equiv(0,0,1)^T\,.$$

Furthermore from (68) we obtain

$$s\mu H_3=(\mathbf{w}\times\mathbf{E})\cdot\mathbf{k},\qquad s\varepsilon E_3=-\left(\mathbf{w}\times\mathbf{H}\right)\cdot\mathbf{k}. \tag{74}$$

The characteristic speeds, $\lambda_{(\pm)}$ and $\lambda_{1,2}$, associated to (72) are defined by

$$\begin{gathered}\left(s^2\varepsilon\mu-\mathbf{w}\cdot\mathbf{w}\right)\lambda^2=1 \qquad \text{double exceptional waves}\\ \lambda^2\left(s^2\varepsilon\mu\left(2\mu'H^2+\mu\right)-\mu\mathbf{w}\cdot\mathbf{w}-2\mu'\left(\mathbf{w}\cdot\mathbf{H}\right)^2\right)+\\ 4\mu'\left(\mathbf{w}\cdot\mathbf{H}\right)H_3\lambda-\left(\mu+2\mu'H_3^2\right)=0\end{gathered} \tag{75}$$

with corresponding left eigenvectors

$$\mathbf{l}^{(\pm)}=\left(\ \mathbf{q}_{(\pm)}\times\mathbf{H}\,,\quad \frac{1}{s\varepsilon\lambda_{(\pm)}}\mathbf{q}_{(\pm)}\times\left(\mathbf{q}_{(\pm)}\times\mathbf{H}\right)\ \right)^T$$

$$\mathbf{l}^{(1,2)}=\left(\ s^2\varepsilon\mu\lambda_{1,2}^2\mathbf{H}-\left(\mathbf{q}_{1,2}\cdot\mathbf{H}\right)\mathbf{q}_{1,2}\,,\quad s\mu\lambda_{1,2}\left(\mathbf{q}_{1,2}\times\mathbf{H}\right)\ \right)^T$$

being $\mathbf{q}=-\lambda\mathbf{w}+\mathbf{k}$. Next we search for solutions of (72) of the following form

$$\mathbf{U}=\mathbf{U}\left(\mathbf{V}\right),\qquad \mathbf{V}=\left(E_1,H_2\right)^T \tag{76}$$

so that the resulting 2×2 system (64) can be recast in the following characteristic form

$$R_\xi+SR_{x_3}=0,\qquad S_\xi+\lambda_1S_{x_3}=0 \tag{77}$$

where the associated Riemann invariants R and S are defined by

$$R=\lambda_{(+)},\quad \nabla_{\mathbf{V}}S=\mathbf{l}^{(1)}\widetilde{\mathbf{A}}\nabla_{\mathbf{V}}\mathbf{U}.$$

Moreover from (65) the following compatibility conditions arise

$$\mathbf{l}^{(-)}\widetilde{A}\nabla_{\mathbf{V}}\mathbf{U}=\mathbf{l}^{(2)}\widetilde{A}\nabla_{\mathbf{V}}\mathbf{U}=0,\qquad \nabla_{\mathbf{V}}\lambda_{(+)}=\mathbf{l}^{(+)}\widetilde{A}\nabla_{\mathbf{V}}\mathbf{U}. \tag{78}$$

Finally we observe that, once the conditions (78) are satisfied, closed form solutions to (77) can be found by means of the general reduction procedure developed in Section **2**. These solutions through the relations (74) and (76) generate in turn a class of exact solutions to the original system (67) which describe, for suitable initial value problems, two nonlinear waves initiated as plane waves and exhibiting a soliton-like interaction.

6. Conclusions and final remarks

As well known the crucial aspect of investigating hyperbolic wave processes and especially wave interactions is strictly related to understanding in detail the behavior of the solutions along the different families of characteristic curves which are associated to a given governing model. In that context quasilinear hyperbolic and autonomous (2×2) systems play a prominent role in several engineering applications because, under assumption of strict hyperbolicity, they can be recast into a form which expresses the evolution of a privileged set of field variables, the Riemann variables, along the related characteristic curves. Hence the Riemann variables have an intrinsic wave feature and they prove to be appropriate to describing nonlinear hyperbolic wave evolution whose mathematical modelling usually requires solution in a closed form to initial and/or boundary value problems. That is a hard task in general. Actually, direct use of the classical hodograph transformation to search for exact solutions to 2×2 quasilinear hyperbolic systems of first order PDEs is very limited, even in the case of a non dissipative wave motion where the governing model is homogeneous (source free case) and the associated hodograph system turns out to be linear, so that in principle the solution can be expressed in terms of the Riemann's function. In recent papers [8, 9] a reduction approach has been developed in order to define an appropriate framework where the hodograph transformation may be an effective and successful tool for obtaining classes of exact solutions to 2×2 systems either homogeneous or non-homogeneous. The proposed method combines differential constraints theory and hodograph transformation and also provides a mathematical vehicle for determining classes of quasilinear hyperbolic models allowing for exact solution to initial or boundary value problems. In Section **2** we have outlined the main steps of the resulting consistency process and later in order to highlight the flexibility of the provided solutions in the description of nonlinear wave interactions we have considered different examples of governing models. Furthermore in Section **3** we have considered the traffic flow model proposed by AW & Rascle in [12] as an example of 2×2 homogeneous system which turns out to be appropriate for illustrating interaction processes of nonlinear hyperbolic waves travelling along different characteristic curves. In line with the analysis which recently was worked out in [9] where several traffic flow situations have been simulated, here we have investigated the alteration in the profile as well as the time distortion associated to the interaction between a simple wave which connects smoothly two different constant states (rarefaction-like wave) and a simple wave which simulates a situation where each car follows the leading car at the same speed. Next in Section **4** attention has been focused on a special class of 2×2 non-homogeneous models which admits suitable field dependent quantities behaving like the classical Riemann invariants associated to the homogeneous governing systems. In particular for the pair of equations describing nonlinear dissipative transmission lines, under suitable constitutive model assumptions, closed form solutions have been determined and nonlinear wave interaction of two regular pulses has been studied. These pulses evolve as hyperbolic waves but exhibit a soliton-like behavior in interaction. In Section **5** we have considered quasilinear hyperbolic homogeneous systems of $N > 2$ first-order PDEs eventually also involving more space variables. In order to extend the approach herein proposed to investigate more general wave processes, by means of "ad hoc" reduction technique we have characterized classes of models which are consistent with special evolution processes ruled by an auxiliary 2×2 hyperbolic subsystem arising as intermediate step in searching for exact wave solutions to the original set of field equations. Within this context the system of balance laws for isotropic ferromagnetic materials has been studied.

Acknowledgments

This work was supported by research project PRA 2008 *Metodi di riduzione per problemi di evoluzione in Meccanica dei Continui* of University of Messina.

References

[1] R. Courant and K. O. Friedrichs, *Supersonic Flows and Shock Waves* (Interscience Publ., New York, 1962).

[2] B. R. Seymour and E. Varley, "Exact solutions describing soliton-like interactions in a non dispersive medium", *SIAM J. Appl. Math.* **42**, 804-821 (1982).

[3] C. Currò and D. Fusco, "Reduction to linear canonical forms and generation of conservation laws for a class of quasilinear hyperbolic systems", *International Journal of Nonlinear Mechanics* **23** (1), 25-35 (1988).

[4] C. Currò and G. Valenti, "A linearization procedure for quasi-linear nonhomogeneous and non-autonomous 2×2 first-order systems", *International J. Non-linear Mechanics* **31**, 377-386 (1996).

[5] C. Currò, D. Fusco and N. Manganaro, "Wave features of a class of nonhomogeneous and nonautonomous models", *Applicable Analysis* **57**, 47-62 (1995).

[6] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer Verlag, Berlin, 1989).

[7] C. Currò and F. Oliveri, "Reduction of nonhomogeneous quasilinear 2×2 systems to homogeneous and autonomous form", *J. Math. Phys.* **49**, 103504-11 (2008).

[8] C. Currò, D. Fusco and N. Manganaro, "Hodograph transformation and differential constraints for wave solutions to 2×2 quasilinear hyperbolic nonhomogeneous systems", *J. Phys. A: Math. Theor.* **45**, 195207 (19p) (2012).

[9] C. Currò, D. Fusco and N. Manganaro ,"An exact description of nonlinear wave interaction processes ruled by 2×2 hyperbolic systems", in print on ZAMP, DOI: 10.1007/s00033-012-0282-0.

[10] N. Manganaro and S. V. Meleshko,"Reduction procedure and generalized simple waves for systems written in Riemann variables", *Nonlinear Dynamics* **30**, 87-102 (2002).

[11] D. Fusco and N. Manganaro, "A method for finding exact solutions to hyperbolic systems of first-order PDEs", *IMA J. Appl. Math.* **57**, 223-42 (1996).

[12] A. Aw and M. Rascle, "Resurrection of second order models of traffic flow", *SIAM Journal Applied Math.* **60**, 916-938 (2000).

[13] A. C. Scott, *Active and Nonlinear Wave Propagation in Electronics* (New York: Wiley-Interscience, 1970).

[14] S.P. Tsarev, "The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method", *Math. USSR Izvestiya* **37** No. 2, 397-419 (1991).

[15] C. Currò and D. Fusco , "A reduction method for quasilinear hyperbolic systems of multicomponent field PDEs with application to wave interaction", *International Journal Nonlinear Mechanics* **37**, 281-295 (2002).

[16] A. Jeffrey, *Quasilinear Hyperbolic Systems and Waves*, Research Notes in Math. **5** (Pitman Publ., London, 1976).

[17] G. Boillat," Nonlinear Hyperbolic Fields and Waves", in *Recent Mathematical Methods in Nonlinear Wave Propagation*; CIME course; Lecture Notes in Mathematics, T. Ruggeri ed. (Springer, Berlin Heidelberg, New York, 1995); Vol. 1640.

[18] J.M. Greenberg," Extension and amplification of a traffic model of AW and Rascle", *SIAM Journal Appl. Math.* **62** (3), 729-745 (2001).

[19] C. Currò, D. Fusco and N. Manganaro , "Differential Constraints and Exact Solution to Riemann Problems for a Traffic Flow Model", *Acta Appl. Math.* **122** (1), 167-178 (2012). DOI 10.1007/s10440-012-9735-x.

[20] M. Rascle, "An improved macroscopic model of traffic flow: derivation and links with the Lighthill-Whitham model", *Math. and Computer Modelling* **35**, 581-590 (2002).

[21] C. Currò and D. Fusco, "On a class of quasi-linear hyperbolic reducible systems allowing for special wave interactions", *ZAMP J. Appl. Math. Phys.* **38**, 580-594 (1987).

[22] C. Currò, D. Fusco and N. Manganaro, “A reduction procedure for generalized Riemann problems with applications to nonlinear transmission lines”, *J. Phys. A: Math. Theor.* **44**, 335205 (19 p) (2011).
[23] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York, 1982).
[24] S.V. Meleshko,“Homogeneous autonomous systems with three independent variables”, *J. Appl. Math. Mech.* **58**, 857-863 (1994).
[25] A.M. Grundland and J. Tafel, “Nonclassical symmetry reduction and Riemann wave solutions”, *J. Math. Anal. Appl.* **198**, 879-892 (1996).
[26] G. Boillat, “Sur l’existence et la recherche d’equations de conservation supplementaires pour les systèmes hyperboliques”, *C. R. Acad. Sci. Paris* **278** A, 909-912 (1974).
[27] I. Müller and T. Ruggeri, *Extended Thermodynamics* Vol. **37** (Springer, New York, 1993).
[28] S. Graffi, “Problemi non lineari nella teoria del campo elettromagnetico”, *Acc. Naz. Sc. Lett. ed Arti Modena* se. VI, 9 (1967).
[29] A. Donato and T. Ruggeri, “Onde di discontinuità e condizioni di eccezionalità per materiali ferromagnetici”, *Rend. Acad. Naz. Lincei, Classe Sc. Fis. Mat. Nat.* serie VIII Vol **LIII**, 208-214 (1972).

[a] University of Messina
Department of Mathematics
Viale F. Stagno D’Alcontres 31
98166 MESSINA, Italy

* Email: ccurro@unime.it

Article contributed to the Festschrift volume in honour of Giuseppe Grioli on the occasion of his 100th birthday.

Communicated 13 April 2012; published online 29 January 2013