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THINKING ABOUT RANDOM EVENTS FROM A LOGICAL POINT OF VIEW

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ABSTRACT. The purpose of this essay is to identify a meaningful property of random events by using a theorem introduced in an unconventional symbolic language, which we shall call $\mathscr{L}_u$. More precisely, we attempt to show that every random event must occur at least once. The method, which we use, consists in defining, by means of sentences of the language $\mathscr{L}_u$, the concept of random event, after showing that some statements about the structural proprieties of the sentences of $\mathscr{L}_u$ can be translated into sentences of the language $\mathscr{L}_u$ itself. Thanks to this peculiar feature of $\mathscr{L}_u$, we achieve an important gain in facilitating the identification of the propriety looked for. In fact, it is easier to deal with sentences of a formal system, free of concealed assumptions and possibly misleading associations of meaning, than with true and false statements.

1. Introduction

The basic idea underlying this work is the following one:

If we transform inexact concept of random event familiar to ordinary folk (i.e., something which can be characterized by a statement(event) of which, under certain conditions, we do not know whether it is true or false) into exact one, expressed by sentences of a symbolic language, then the task may be made helpful in investigating any its properties.

From the intuitive notion of random event presented above, it is clear that:

(1) When we assert that, under certain conditions, the truth-value of any statement is undecidable, we means that the set C of these conditions completely reflects all of the necessary and sufficient reasons to conclude that, realized the set of conditions C, no methods exist for predicting whether the given statement is true or false;
(2) When we speak of the randomness of any event, we shall always mean that it is random with respect to some definite set of conditions.[1]

Following these indications, the paper aims at proving that any random event g must occur at least once, i.e., it must exist at least one realization of the set of conditions, with respect to which g is random, that yields the occurrence of event g. This statement is a strong claim, especially, if put into the context of probability theory. In fact, in probability theory, we only

[1]An analogous position about the nature of random event can be found in Gnedenko (2005).

can define events as subsets from a sample space, and unless an event set is not empty, we can obtain a positive probability, *which, nevertheless, does not imply that the event occurs*. Upon closer examination, from the nature of an event in this sense, it does not even follow that it is meaningful to talk about its probability as though it were a definite number. Indeed, the concept of mathematical probability deserves a thorough philosophical study. The basic specific philosophical question (still unsolved) raised by the very existence of probability theory and by its successful application to real phenomena is the following one:
is there objective meaning in the quantitative estimate of the probability of a random event and what is that meaning?
A clear understanding of the interrelation between the notion of random event and reality is thus an inevitable prerequisite for the serious analysis of the concept of mathematical probability.
However, in the following pages, we will not attempt to deal with probability theory, that is, we will not turn our attention to the question of how do components of the symbolic language $\mathscr{L}_u$ or of the $\mathscr{L}_u$ semantics relate to classical notions of probability theory like, for example, sample space or random variable; although, for the importance of our thesis, we will examine a problem which present a co-occurrence of both approaches.
The paper is organized in three sections. In the first, we describe the structure of the symbolic language $\mathscr{L}_u$ by specifying its rules. In the second, we introduce some fundamental properties concerning the sentences of $\mathscr{L}_u$. In the third section, we present a way of defining, through sentences of $\mathscr{L}_u$, the concept of random event, and we prove the theorem, which is the aim that we have in view. In the paper, it will be illustrated an interesting application of this theorem.

2. Language $\mathscr{L}_u$ (unconventional language)

The language $\mathscr{L}_u$ consists in setting up sentences concerning objects of a certain structure and specifically in ascribing a certain relation to objects in question. The basic objects treated in the language $\mathscr{L}_u$ are called *individuals* of the system; and their totality, the *domain*.
It is thus necessary that the language $\mathscr{L}_u$ contains at least two types of symbols:

(1) *names for the individuals* of the domain; we call these (designations) ***individual constants***;
(2) *a name for the unique relation* predicated of the individuals; we call this (designation) ***predicative constant***.

Precisely, $\mathscr{L}_u$ is constituted in the following way:

Axiom 2.1. $\mathscr{L}_u$ contains two kinds of individual constants:

(1) 'a_{11}', 'a_{12}', 'a_{21}', 'a_{22}', 'a_{13}', 'a_{31}', 'a_{14}', 'a_{23}', 'a_{32}', 'a_{41}', 'a_{33}', 'a_{34}', ..., 'a_{mn}', ..., (countable infinity). The set of all these constants will be indicated by '**E**'.
(2) 'Θ', 'O', 'O_1', 'O_2', ..., 'O_n', ..., (countable infinity). We agree to associate with each of these constants, ***observers*** of $\mathscr{L}_u$ say, a unique subset of **E** freely selected; we call this *patrimony of observer*. The set of all observers of $\mathscr{L}_u$ will be indicated by 'Ω'.

Axiom 2.2. Each member of the set **E** belongs to one of two disjoint sets Y and $-Y$, i.e., **E** is a subset of union of the pair $(Y, -Y)$.

Axiom 2.3. The patrimony of observer Θ, say $P(\Theta)$, is the set of all elements of **E** which also belong to the set Y.

Axiom 2.4. For every x in **E**, the set (singleton) $\{x\}$ is the patrimony of at least one observer of $\mathscr{L}_u$, i.e., of at least one element of Ω.

Axiom 2.5. For every element x of **E**, there is a unique member y of **E** such that y is in Y if and only if x is in $-Y$. Given an x, that unique member is denoted by '$\sim x$' and called *the opposite of x*.

Interpretation 1. The two sets Y and $-Y$, used in this paper, may be construed as the sets of true and false statements, respectively. In this way, the exposition is more easily grasped.

Axiom 2.6. The unique predicative constant of $\mathscr{L}_u$ is the following one:

'...KNOW(...)'.

It is short of '...knows the argument...'.

Axiom 2.7. In $\mathscr{L}_u$ there are some individual functions, that is, expressions formed by the combination of two signs:
one constant, say *known term of individual function*;
the other variable, say *variable of* $\mathscr{L}_u$;
so that, when a numeral (i.e., a numerical sign which designates a natural number) is assigned to this last constituent, the resulting string is an element of **E**.

The individual functions of $\mathscr{L}_u$ are the following signs:

'a_{1h}', 'a_{2h}', 'a_{3h}', ..., 'a_{nh}', ...

The known terms of these functions are indicated by

'a_1', 'a_2', 'a_3', ..., 'a_n', ...,

respectively.
The variable of $\mathscr{L}_u$ will be obviously designated by 'h'.
The variable of $\mathscr{L}_u$ is, therefore, a numerical variable for which one of following numerals

'1', '2', '3', ..., 'n', ...

can be substituted.
'**T**' will indicate the set of all these numerals.
It is thus clear that by substituting in any individual function of $\mathscr{L}_u$ (e.g., in 'a_{2h}') the corresponding variable (i.e., the variable 'h') by a member of **T** (e.g., the numeral '2'), we obtain an element of **E** (in this case, the individual constant 'a_{22}').

Interpretation 2. In connection with interpretation 1, each individual function of $\mathscr{L}_u$ may be construed as a propositional function, that is, as an expression containing one or more undetermined constituents, such that, when values (in this case, numerals) are assigned to these constituents, the expression becomes a true or false statement.

Axiom 2.8. Let α_h is any individual function of the language $\mathscr{L}_u$ whose known term is α (i.e., one of the signs 'a_1', 'a_2', ..., 'a_n', ...), and h is the variable of $\mathscr{L}_u$ that appears in it. There is at least one element t of the set **T** such that the constant α_t of **E** also falls in set Y.

Axiom 2.9. $\mathscr{L}_u$ contains the following connective signs:

'$\neg$' which is short for 'not' and is called *negation*;
'$\vee$' which is short for 'or' and is called *disjunction*.

Definition 2.1. We say that A is an *elementary sentence* of $\mathscr{L}_u$ if and only if A is of type

$${}_{\bar{O}}KNOW(\varepsilon),$$

for some ε element of **E** and for some $\bar{O}$ element of Ω.

Combining the elementary sentences of $\mathscr{L}_u$ with the connective signs, we have the sentences of the language $\mathscr{L}_u$.

Definition 2.2. We say that A is a sentence of $\mathscr{L}_u$ if and only if one of the following holds:

(1) A is an elementary sentence of $\mathscr{L}_u$;
(2) A is $\neg$B and B is a sentence of $\mathscr{L}_u$;
(3) A is B$\vee$C and B and C are sentences of $\mathscr{L}_u$.

In other terms, A is a sentence of the language $\mathscr{L}_u$ if it is an elementary sentence of $\mathscr{L}_u$ or the negation of a sentence of $\mathscr{L}_u$ or the disjunction of two sentences of $\mathscr{L}_u$.

Definition 2.3. Let A and B be any two sentences of $\mathscr{L}_u$.

A$\wedge$B stands for $\neg[(\neg A) \vee (\neg B)]$;
'$\wedge$' is short for 'and' and is called *intersection*.

The language $\mathscr{L}_u$ contains parentheses, which are used only as auxiliary signs to avoid ambiguity.

Finally, it must be observed that the axioms introduced in this section serve as structural features of the language $\mathscr{L}_u$, i.e., of a meaningless formal system possessing a determinate structure. Hence, such axioms need no natural or intuitive motivations to be accepted.

3. Basic properties of $\mathscr{L}_u$

We agree to place each elementary sentence of $\mathscr{L}_u$ in one of two mutually exclusive and exhaustive sets K and $-K$. Sentences that are not elementary fall in these sets pursuant to the following conventions:

(1) A sentence having the form A$\vee$B is placed in set $-K$, if both A and B are in $-K$; otherwise, it is placed in K;
(2) A sentence having the form $\neg$A is placed in $-K$, if A is in K; otherwise, it is placed in K.

Definition 3.1. Let α is an element of the set **E** and $\bar{O}$ an observer of $\mathscr{L}_u$, i.e., a member of Ω. We shall say that the elementary sentence ${}_{\bar{O}}KNOW(\alpha)$ of $\mathscr{L}_u$ falls in set K if and only if the element α of **E** also belongs to the patrimony of observer $\bar{O}$.

Definition 3.2. A sentence of $\mathscr{L}_u$ is a *tautology* if, and only if, it falls in set K no matter in which of the two sets K and $-K$ its simple constituents are placed (more details can be found in Nagel and Newman (2001)).

Now, by definitions 3.1 and 3.2, we are equipped to introduce some fundamental properties of $\mathscr{L}_u$.

Proposition 3.1. *If α is an element of* **E**, *then the elementary sentence*

$$_{\Theta}KNOW(\alpha)$$

of $\mathscr{L}_u$ is placed in K if and only if α belongs to the set Y.

Proof. If $_{\Theta}KNOW(\alpha)$ falls in **K**, then, by definition 3.1, α belongs to the set $P(\Theta)$, i.e., to the patrimony of observer Θ. It follows that α falls in Y, since $P(\Theta)$ is by axiom 2.3 a subset of Y.

On the other hand, if $_{\Theta}KNOW(\alpha)$ is in $-K$, then, by definition 3.1, α cannot be in $P(\Theta)$ and so, by axioms 2.2 and 2.3, it must be in set $-Y$. □

More shortly, given a member x of **E**, '$_{\Theta}x$' will be hereafter abbreviation for '$_{\Theta}KNOW(x)$'.

Remark 1. On the basis of interpretation 1 and proposition 3.1, it is thinkable that both the disjunction and the intersection of two elementary sentences of $\mathscr{L}_u$ of type $_{\Theta}\varepsilon$, with ε in the set **E**, are elementary sentences of $\mathscr{L}_u$ of the same type. Indeed, it is known that both the disjunction and the conjunction of two statements (each of which is true or false) are true or false statements.

This consideration motivates the next two axioms of $\mathscr{L}_u$.

Axiom 3.1. Let α_1, α_2 be any two elements of **E**. Then, there is an element α_3 of **E** such that both the sentences of $\mathscr{L}_u$

$$[\neg(_{\Theta}\alpha_1 \vee_{\Theta} \alpha_2)] \vee_{\Theta} \alpha_3, \quad (_{\Theta}\alpha_1 \vee_{\Theta} \alpha_2) \vee (\neg_{\Theta}\alpha_3)$$

are tautologies.

Axiom 3.2. Let α_1, α_2 be any two elements of **E**. Then, there is an element α_4 of **E** such that both the sentences of $\mathscr{L}_u$

$$[\neg(_{\Theta}\alpha_1 \wedge_{\Theta} \alpha_2)] \vee_{\Theta} \alpha_4, \quad (_{\Theta}\alpha_1 \wedge_{\Theta} \alpha_2) \vee (\neg_{\Theta}\alpha_4)$$

are tautologies.

In the proposition below, we show that some statements about the structural proprieties of elementary sentences of $\mathscr{L}_u$ (in particular, the propriety of being a sufficient condition) can be accurately mirrored within the language $\mathscr{L}_u$ itself.
This is the main reason for introduction of the symbolic language $\mathscr{L}_u$.

Proposition 3.2. *Let C is any sentence of $\mathscr{L}_u$ and D any elementary sentence of $\mathscr{L}_u$. Then,*

(1) *there is an elementary sentence D' of $\mathscr{L}_u$ such that D' is in set K if and only if the sentence $(\neg C) \vee D$ of $\mathscr{L}_u$ is a tautology, i.e., if and only if C in K is a sufficient condition for D in K;*
(2) *there is an elementary sentence D" of $\mathscr{L}_u$ such that D" falls in K if and only if $(\neg C) \vee D$ is not a tautology, i.e., if and only if C in K is not a sufficient condition for D in K.*

Proof. Suppose D stands for ${}_{\bar{O}}KNOW(\varepsilon)$, where ε is an element of **E** and $\bar{O}$ is an observer of $\mathscr{L}_u$, i.e., a member of Ω.
Let S be the set which consists of all elements x of **E** such that

$$(\neg C) \vee_{\bar{O}} KNOW(x)$$

is a tautological sentence of $\mathscr{L}_u$.
We indicate the elements of the subset $\{\varepsilon, \sim\varepsilon\}$ of $\mathbf{E}^2$ in the following way:

$$\lambda_1, \quad \lambda_2$$

where λ_1 and λ_2 are the members of the set $\{\varepsilon, \sim\varepsilon\}$ corresponding to the natural numbers (indices) 1 and 2, respectively.
We can suppose that, in this correspondence, the index 1 is associated with the element ε of the set **E** (i.e., it is $\lambda_1 = \varepsilon$), if ε falls in set S; if not, it is linked to the opposite of ε (i.e., it results $\lambda_1 = \sim\varepsilon$).[3]
It follows that λ_1 coincides with ε if and only if the sentence $(\neg C) \vee D$ of $\mathscr{L}_u$ is a tautology or, in other terms, if and only if C in K is a sufficient condition for D in K.
Furthermore, axiom 2.4 yields that there is an observer $\bar{O}'$ of the language $\mathscr{L}_u$ whose patrimony is the set (singleton) $\{\varepsilon\}$.
By assuming that D' and D" are the sentences of $\mathscr{L}_u$

$${}_{\bar{O}'}KNOW(\lambda_1) \quad \text{and} \quad {}_{\bar{O}'}KNOW(\sim\lambda_1),$$

respectively, we complete the proof. □

Remark 2. The possibility of mirroring statements about the formal system $\mathscr{L}_u$ in the system itself is the key to the argument of the paper. In fact, exploiting this form of mapping we can formally translate an informal random event into a logical formula of $\mathscr{L}_u$, which is then used to prove that this event must occur at least once.

Now, we are able to introduce two new connective signs.

Let C, D, D' and D" be the sentences of $\mathscr{L}_u$ used in proposition 3.2.

Definition 3.3. $C \Rightarrow D$ is defined as D';
'$\Rightarrow$' is short of 'implies'.

Definition 3.4. $C \nRightarrow D$ is defined as D";
'$\nRightarrow$' is short of 'does not imply'.

Remark 3. By the above two definitions and proposition 3.2, we obtain the following important facts:

[2]Remember that $\sim\varepsilon$ is by axiom 2.5 the opposite of the individual constant ε of **E**. Hence, $\{\varepsilon, \sim\varepsilon\}$ is a subset of **E**.

[3]E.g., we can exhibit the correspondence caused by application

$$f_S : \{(\varepsilon, \sim\varepsilon)\} \to \{(\varepsilon, \sim\varepsilon), (\sim\varepsilon, \varepsilon)\}$$

defined as follows: $f_S((\varepsilon, \sim\varepsilon)) = (\lambda_1, \lambda_2) = (\varepsilon, \sim\varepsilon)$, if ε is in set S; otherwise,

$$f_S((\varepsilon, \sim\varepsilon)) = (\lambda_1, \lambda_2) = (\sim\varepsilon, \varepsilon),$$

where $(\varepsilon, \sim\varepsilon)$, $(\sim\varepsilon, \varepsilon)$ and (λ_1, λ_2) are three ordered pairs (of members of $\{\varepsilon, \sim\varepsilon\}$). It is thus evident that $\lambda_1 = \varepsilon$, if ε is in set S; if not, $\lambda_1 = \sim\varepsilon$.

(1) the sentence $C \Rightarrow D$ falls in K if and only if C in K is a sufficient condition for D in K. Hence, if the sentences C and $C \Rightarrow D$ are both in K, then the sentence D also is in K;
(2) the sentence $C \not\Rightarrow D$ falls in K if and only if $C \Rightarrow D$ is placed in $-K$.

Using remark 3, we can state the following:

Proposition 3.3. *Let α_1, α_2 be any two elements of $\mathbf{E}$.*
Then,
the sentence of $\mathscr{L}_u$

$$_{\Theta}\alpha_1 \Rightarrow_{\Theta} \alpha_2$$

falls in K if and only if α_1 in Y is a sufficient condition for α_2 in Y.

Proof. It is evident from proposition 3.1 that $_{\Theta}\alpha_1$ in K is a sufficient condition for $_{\Theta}\alpha_2$ in K if and only if α_1 in Y is a sufficient condition for α_2 in Y. On the other hand, because of remark 3, the sentence $_{\Theta}\alpha_1 \Rightarrow_{\Theta} \alpha_2$ of $\mathscr{L}_u$ falls in K if and only if $_{\Theta}\alpha_1$ in K is a sufficient condition for $_{\Theta}\alpha_2$ in K. Thus, we have proved the proposition 3.3. □

4. The random events

Let b_h and g_h be any two individual functions of $\mathscr{L}_u$ whose known terms are b and g, respectively (i.e., two of the signs 'a_1', 'a_2', ..., 'a_n', ...), and h is the variable of $\mathscr{L}_u$ that occurs in any one of them.

Let H is the *K-set* of the individual function b_h, that is, the non-empty set which consists of all elements τ of $\mathbf{T}$ (i.e., of the set of numerals '1', '2', ..., 'n', ...) such that $_{\Theta}b_\tau$ is a sentence of $\mathscr{L}_u$ belonging to the set K.[4]

H is thus a finite or countable infinite set, since it is by definition a subset of a countable infinite set.

We shall call the members of the set H *trials*.

In accordance with proposition 3.2, we proceed by introducing within the language $\mathscr{L}_u$ concepts such as: outcome of a trial, event and random event.

Definition 4.1. Let $\hat{h}$ be any trial (i.e., any member of H).
If the sentence $_{\Theta}g_{\hat{h}}$ of $\mathscr{L}_u$ falls in K, we shall say that g is an outcome of the trial $\hat{h}$ or, in other words, that g has occurred on trial $\hat{h}$ or also that the condition g is realized in trial $\hat{h}$.

Definition 4.2. Let $\hat{h}$ be any trial.
We shall say that the known term g of the individual function g_h is an *event of the trial* $\hat{h}$ (briefly, *event*) if and only if each of the following holds:

i There is an individual function r_h of $\mathscr{L}_u$, with r as its known term and h as its unique variable, such that both the sentences of language $\mathscr{L}_u$

$$_{\Theta}(\sim r_{\hat{h}}) \vee \left\{ [_{\Theta}b_{\hat{h}} \not\Rightarrow _{\Theta}(\sim g_{\hat{h}})] \wedge (_{\Theta}b_{\hat{h}} \not\Rightarrow _{\Theta}g_{\hat{h}}) \right\}, \quad _{\Theta}r_{\hat{h}} \vee \left\{ [_{\Theta}b_{\hat{h}} \Rightarrow _{\Theta}(\sim g_{\hat{h}})] \vee (_{\Theta}b_{\hat{h}} \Rightarrow _{\Theta}g_{\hat{h}}) \right\}$$

are tautologies;

[4] It is easy to show that the set H exists. In fact, because of axiom 2.8, there is at least one numeral τ of $\mathbf{T}$ such that the member b_τ of the set $\mathbf{E}$ also belongs to the set Y and so, by proposition 3.1, such that the sentence $_{\Theta}b_\tau$ of $\mathscr{L}_u$ is in set K.

ii One of two mutually exclusive sentences of $\mathscr{L}_u$[5]

$$_\Theta b_{\hat{h}} \Rightarrow_\Theta r_{\hat{h}}, \quad _\Theta b_{\hat{h}} \Rightarrow_\Theta (\sim r_{\hat{h}})$$

falls in set K.

In words: g is an event of the trial $\hat{h}$ if and only if the condition b reflects all of the necessary and sufficient reasons for g to be one that may or may not appear in trial $\hat{h}$ or one that must occur or can never occur on trial $\hat{h}$.

Definition 4.3. Let $\hat{h}$ be any trial.
We shall say that an event g of the trial $\hat{h}$ is *random with respect to the known term b* (briefly, *random event*) if and only if the sentence of $\mathscr{L}_u$

$$[_\Theta b_{\hat{h}} \not\Rightarrow_\Theta (\sim g_{\hat{h}})] \wedge (_\Theta b_{\hat{h}} \not\Rightarrow_\Theta g_{\hat{h}})$$

falls in K.

In words: an event g of the trial $\hat{h}$ is random with respect to the known term b if and only if, realized the condition b (in $\hat{h}$), it may or may not be an outcome of the trial $\hat{h}$.

Remark 4. Notice that the expression $[_\Theta b_{\hat{h}} \not\Rightarrow_\Theta (\sim g_{\hat{h}})] \wedge (_\Theta b_{\hat{h}} \not\Rightarrow_\Theta g_{\hat{h}})$ is, by proposition 3.3, the translation into the formal language $\mathscr{L}_u$ of the claim '$b_{\hat{h}}$ in Y is a sufficient condition neither for $g_{\hat{h}}$ in Y nor for $\sim g_{\hat{h}}$ in Y'. This, in connection with interpretation 1, leads us to consider such expression, whenever requirements i and ii of definition 4.2 are satisfied simultaneously, as a formal string inside $\mathscr{L}_u$ of the informal notion of random event, which is the starting point of the paper. In other words, we can exhibit the above expression as a transcription into the language $\mathscr{L}_u$ of something characterizable by a statement (here represented by the individual constant '$g_{\hat{h}}$') of which, under certain conditions (in this case, when the condition b is realized in trial $\hat{h}$), we do not know whether it is true or not.

For the sake of simplicity, we shall omit hereafter the expression 'falls in K' in all cases concerning sentences of $\mathscr{L}_u$ whose form comprises at least one of two connective signs '$\Rightarrow$' and '$\not\Rightarrow$'.[6]

Now, we are equipped to prove the following:

Theorem 4.1. Let **I** be the set $\{1,2,\ldots,m\}$. Suppose that b_h, g_{1h}, g_{2h},..., g_{mh} are m+1 individual functions of $\mathscr{L}_u$ whose known terms are b, g_1, g_2,..., g_m, respectively, and h is the variable of $\mathscr{L}_u$ that appears in any one of them. Assume H is the K-set of the individual function b_h. We shall call the members of *H trials*. Suppose that, for every member h of H, g_1, g_2,..., g_m are m events of the trial h, random with respect to the known term b.
Then,
for every element i of the set **I**, the event g_i has occurred in at least one trial.

[5]Namely, if one of these two sentences is in set K the other must be in set $-K$.
[6]In other terms, if C and D are any two sentences of $\mathscr{L}_u$, instead of 'the sentence $C \Rightarrow D$ of $\mathscr{L}_u$ falls in K' we will write '$C \Rightarrow D$'; and instead of '$C \not\Rightarrow D$ falls in K', simply '$C \not\Rightarrow D$'.

Proof. Let ι be any element of **I**.
We indicate by $\bigvee_{(h\in H)} {}_{\Theta}g_{\iota h}$ the disjunction of all sentences ${}_{\Theta}g_{\iota h}$ of $\mathscr{L}_u$ when h varies in set H.

Let q_h be an individual function of $\mathscr{L}_u$ whose known term is q (i.e., one of the signs 'a_1', 'a_2', ..., 'a_n', ...), and h is its sole variable.
Without loss of generality we suppose, according axiom 3.1, that the sentences of $\mathscr{L}_u$

$$ {}_{\Theta}q_{\iota} \vee [\neg(\bigvee_{(h\in H)} {}_{\Theta}g_{\iota h})], \quad (\neg_{\Theta}q_{\iota}) \vee (\bigvee_{(h\in H)} {}_{\Theta}g_{\iota h}) $$

are both tautologies.

It follows that the sentence ${}_{\Theta}q_{\iota}$ of $\mathscr{L}_u$ is in K if and only if the event g_{ι} has occurred in at least one trial.
Also we denote by $\bigwedge_{(h\in H)} {}_{\Theta}b_h$ the intersection of all sentences ${}_{\Theta}b_h$ of $\mathscr{L}_u$ when h varies in set H.
Noting that $\bigwedge_{(h\in H)} {}_{\Theta}b_h$ in set K requires, by definitions 4.2 and 4.3, that each of the random events g_1, g_2,..., g_m may or may not be an outcome of any single trial, we assume for contradiction that there is one element $\hat{\imath}$ of **I** such that $\bigwedge_{(h\in H)} {}_{\Theta}b_h \not\Rightarrow_{\Theta} q_{\hat{\imath}}$.

In addition, we can suppose that each of the following conditions is satisfied:

C 1. *The sentence $\bigwedge_{(h\in H)} {}_{\Theta}b_h$ of the language $\mathscr{L}_u$ falls in set K.*

This follows directly from the fact that H is by hypothesis the K-set of the individual function b_h.

C 2. *There is an element ξ of **E** such that ${}_{\Theta}\xi$ is placed in K if and only if, for every member h of H, $[{}_{\Theta}b_h \not\Rightarrow_{\Theta} (\sim g_{\hat{\imath}h})] \wedge ({}_{\Theta}b_h \not\Rightarrow_{\Theta} g_{\hat{\imath}h})$ is a sentence of $\mathscr{L}_u$ falling in set K.*

With the aim of showing this, let $\check{r}_h$ be an individual function of $\mathscr{L}_u$ with $\check{r}$ as its known term, and h as its sole variable.
Also we write $\bigwedge_{(h\in H)} {}_{\Theta}\check{r}_h$ as short for the intersection of all sentences ${}_{\Theta}\check{r}_h$ of $\mathscr{L}_u$ when h varies in H.
According to definition 4.2 and to the assumption that, for every h in H, $g_{\hat{\imath}}$ is an event of the trial h, we can suppose that the sentence $\bigwedge_{(h\in H)} {}_{\Theta}\check{r}_h$ of $\mathscr{L}_u$ falls in K if and only if, for every h in set H, $[{}_{\Theta}b_h \not\Rightarrow_{\Theta} (\sim g_{\hat{\imath}h})] \wedge ({}_{\Theta}b_h \not\Rightarrow_{\Theta} g_{\hat{\imath}h})$ is a sentence of $\mathscr{L}_u$ falling in set K.
On the other hand, there is, by axiom 3.2, a member ξ of **E** such that the sentences of $\mathscr{L}_u$

$$ [\neg(\bigwedge_{(h\in H)} {}_{\Theta}\check{r}_h)] \vee_{\Theta} \xi, \quad [\bigwedge_{(h\in H)} {}_{\Theta}\check{r}_h] \vee (\neg_{\Theta}\xi) $$

are both tautologies.

C 3. *The elementary sentence ${}_{\Theta}\xi$ of $\mathscr{L}_u$ falls in set K.*

This follows immediately from C 2 and definition 4.3, since, for every member h of H, $g_{\hat{\imath}}$ is by hypothesis an event of the trial h, random with respect to know term b.

C 4. *O is an observer of $\mathscr{L}_u$ whose patrimony, $P(O)$, is the set $P(\Theta)\backslash Q$,*[7] *where $P(\Theta)$ is the patrimony of observer Θ (see axiom 2.3) and Q is the set defined as follows:*

Q contains only the element $q_{\hat{\imath}}$ (i.e., Q is the singleton set $\{q_{\hat{\imath}}\}$), if

(i) ξ in Y is not a sufficient condition for $q_{\hat{\imath}}$ in Y,[8] *and*

(ii) $q_{\hat{\imath}}$ is in set $P(\Theta)$;
otherwise, it has no members.

This is permissible, since there is no restriction on how to choose the patrimony of any observer of $\mathscr{L}_u$.

C 5. ${}_{\Theta}(\sim q_{\hat{\imath}}) \Rightarrow [{}_{\Theta}q_{\hat{\imath}} \Rightarrow_O KNOW(q_{\hat{\imath}})]$.[9]

If ${}_{\Theta}(\sim q_{\hat{\imath}})$ falls in K, then, by proposition 3.1 and axiom 2.5, the element $q_{\hat{\imath}}$ of **E** also belongs to the set $-Y$ and therefore it is not in the patrimony $P(\Theta)$ of observer Θ, by axiom 2.3. If so, the patrimony $P(O)$ of observer O coincides by C 4 with the set $P(\Theta)$.

C 6. *The sentence ${}_{\Theta}q_{\hat{\imath}}$ of $\mathscr{L}_u$ is in set K.*

If $\bigwedge_{(h\in H)} {}_{\Theta}b_h \Rightarrow_{\Theta} (\sim q_{\hat{\imath}})$, then, for some $\hat{h}$ member of H, we have

$$\bigwedge_{(h\in H)} {}_{\Theta}b_h \Rightarrow_{\Theta} (\sim g_{\hat{\imath}\hat{h}}).$$

But, if this were the case, $g_{\hat{\imath}}$ would not be a random event. In fact, we get from definitions 4.2 and 4.3

$${}_{\Theta}b_{\hat{h}} \Rightarrow [{}_{\Theta}b_{\hat{h}} \not\Rightarrow_{\Theta} (\sim g_{\hat{\imath}\hat{h}})].$$

From which, given that $\bigwedge_{(h\in H)} {}_{\Theta}b_h \Rightarrow {}_{\Theta}b_{\hat{h}}$, we easily have

$$\bigwedge_{(h\in H)} {}_{\Theta}b_h \Rightarrow [{}_{\Theta}b_{\hat{h}} \not\Rightarrow_{\Theta} (\sim g_{\hat{\imath}\hat{h}})]$$

and consequently

$$\bigwedge_{(h\in H)} {}_{\Theta}b_h \not\Rightarrow_{\Theta} (\sim g_{\hat{\imath}\hat{h}}).$$

[7]In words: $P(O)$ is the relative complement of the set Q in set $P(\Theta)$, that is, the set of all elements which are in $P(\Theta)$ but not in Q.

[8]We obviously assume that this requirement can possibly be satisfied. In fact, if ξ in Y necessarily were a sufficient condition for $q_{\hat{\imath}}$ in Y, the constant $q_{\hat{\imath}}$ of **E** also would be in set Y, since ξ is by C 3 and proposition 3.1 a member of the set Y. Thus, without further considerations, we should have the demonstration of the theorem 4.1, given that, because of proposition 3.1, $q_{\hat{\imath}}$ is in Y if and only if the sentence ${}_{\Theta}q_{\hat{\imath}}$ of $\mathscr{L}_u$ is in set K and so if and only if the event $g_{\hat{\imath}}$ has occurred in at least one trial.

[9]Since ${}_{\Theta}q_{\hat{\imath}} \Rightarrow_O KNOW(q_{\hat{\imath}})$ is by definition 3.3 and proposition 3.2 an elementary sentence of $\mathscr{L}_u$, the reader will readily note that the expression

$${}_{\Theta}(\sim q_{\hat{\imath}}) \Rightarrow [{}_{\Theta}q_{\hat{\imath}} \Rightarrow_O KNOW(q_{\hat{\imath}})]$$

also is an elementary sentence of $\mathscr{L}_u$.

Hence, we can suppose that the sentence ${}_{\Theta}q_{\hat{\imath}}$ of $\mathscr{L}_u$ falls in set K, since $\bigwedge_{(h\in H)} {}_{\Theta}b_h$ is by C 1 in K.[10]

C 7. *One of two mutually exclusive sentences of* $\mathscr{L}_u$

$$ {}_{\Theta}\xi \Rightarrow ({}_{\Theta}\xi \not\Rightarrow {}_{\Theta}q_{\hat{\imath}}), \quad {}_{\Theta}\xi \Rightarrow ({}_{\Theta}\xi \Rightarrow {}_{\Theta}q_{\hat{\imath}}) $$

falls in K.

Since, for every h in H, $g_{\hat{\imath}}$ is an event of the trial h, if ${}_{\Theta}\xi$ is in K (see C 2), definition 4.2 and C 1 yield

$$ \bigwedge_{(h\in H)} {}_{\Theta}b_h \Rightarrow_{\Theta} \xi. $$

From which, if ${}_{\Theta}\xi$ is in set K, we get

$$ [{}_{\Theta}\xi \wedge (\bigwedge_{(h\in H)} {}_{\Theta}b_h)] \not\Rightarrow {}_{\Theta}q_{\hat{\imath}}, $$

given that by hypothesis $\bigwedge_{(h\in H)} {}_{\Theta}b_h \not\Rightarrow {}_{\Theta}q_{\hat{\imath}}$.
This means that

$$ {}_{\Theta}\xi \Rightarrow ({}_{\Theta}\xi \not\Rightarrow {}_{\Theta}q_{\hat{\imath}}). $$

Hence, we can maintain that C 7 holds.

In these conditions, having regard to remark 3, we can show that:

$$ ({}_{\Theta}\xi \not\Rightarrow {}_{\Theta}q_{\hat{\imath}}) \Rightarrow_O KNOW(q_{\hat{\imath}}). \tag{1} $$

For this purpose, let us consider the sentence of $\mathscr{L}_u$

$$ {}_{\Theta}q_{\hat{\imath}} \Rightarrow_O KNOW(q_{\hat{\imath}}). \tag{2} $$

Since ${}_{\Theta}q_{\hat{\imath}} \Rightarrow [(2) \Rightarrow_O KNOW(q_{\hat{\imath}})]$, we have, by virtue of C 6, that

$$ (2) \Rightarrow_O KNOW(q_{\hat{\imath}}). $$

Furthermore, by C 5, it results

$$ {}_{\Theta}(\sim q_{\hat{\imath}}) \Rightarrow (2). $$

Hence, we get

$$ {}_{\Theta}(\sim q_{\hat{\imath}}) \Rightarrow_O KNOW(q_{\hat{\imath}}) $$

and consequently

$$ [{}_{\Theta}(\sim q_{\hat{\imath}}) \not\Rightarrow_{\Theta} (\sim \xi)] \Rightarrow [{}_{O}KNOW(q_{\hat{\imath}}) \not\Rightarrow_{\Theta} (\sim \xi)], $$

which can be written in the form

$$ ({}_{\Theta}\xi \not\Rightarrow {}_{\Theta}q_{\hat{\imath}}) \Rightarrow [{}_{O}KNOW(q_{\hat{\imath}}) \not\Rightarrow_{\Theta} (\sim \xi)]. \tag{3} $$

Conversely, since by C 6 and definition 3.1 $q_{\hat{\imath}}$ is a member of the patrimony $P(\Theta)$ of observer Θ, to say that ${}_{O}KNOW(q_{\hat{\imath}})$ falls in K is by C 4 to assert that ξ in Y is a sufficient condition for $q_{\hat{\imath}}$ in Y and so that

$$ {}_{\Theta}\xi \Rightarrow {}_{\Theta}q_{\hat{\imath}}, $$

[10]It is worth observing that the sentence ${}_{\Theta}q_{\hat{\imath}}$ of $\mathscr{L}_u$ may fall in K (and so, by definition 3.1, the member $q_{\hat{\imath}}$ of **E** may be in the patrimony $P(\Theta)$ of observer Θ), regardless of whether the statement that ξ in Y is not a sufficient condition for $q_{\hat{\imath}}$ in Y is true or false.

by proposition 3.3.
From the above consideration, it follows that:

(1) ${}_O KNOW(q_{\hat{\imath}}) \Rightarrow ({}_\Theta\xi \Rightarrow {}_\Theta q_{\hat{\imath}})$;
(2) $({}_\Theta\xi \Rightarrow {}_\Theta q_{\hat{\imath}}) \Rightarrow_O KNOW(q_{\hat{\imath}})$;
(3) If ${}_O KNOW(q_{\hat{\imath}})$ is in $-K$ and thus ${}_\Theta\xi \not\Rightarrow {}_\Theta q_{\hat{\imath}}$, then, applying C 7 and C 3, we have

$${}_\Theta\xi \Rightarrow ({}_\Theta\xi \not\Rightarrow {}_\Theta q_{\hat{\imath}})$$

which is written as

$$({}_\Theta\xi \Rightarrow {}_\Theta q_{\hat{\imath}}) \Rightarrow_\Theta (\sim \xi).$$

It is inferred that

$${}_O KNOW(q_{\hat{\imath}}) \vee [{}_O KNOW(q_{\hat{\imath}}) \Rightarrow_\Theta (\sim \xi)]$$

is a tautology and therefore that

$$[{}_O KNOW(q_{\hat{\imath}}) \not\Rightarrow_\Theta (\sim \xi)] \Rightarrow_O KNOW(q_{\hat{\imath}}). \tag{4}$$

Putting (3) and (4) together, we have that (1) holds. However, because of proposition 3.3 and C 4, this is absurd;[11] hence, if ι is any member of the set $\mathbf{I}$, the sentence $\bigwedge_{(h\in H)} {}_\Theta b_h \Rightarrow {}_\Theta q_\iota$ of $\mathscr{L}_u$ must be in set K. Combining this with the statement that $\bigwedge_{(h\in H)} {}_\Theta b_h$ falls in K (see C 1), we obtain the demonstration of the theorem. □

An interesting application of theorem 4.1 is the following one:

Problem 1. A and B play a game in which they alternately toss a pair of dice assumed fair. Suppose that A is the first to toss and that the result for each toss is independent of the result of any others.
The one who is first to get a total of 7 wins the game.

Prove that the statement F 'neither A nor B will win the game' is surely false.

Solution. Notice that the probability calculus[12] has no ready solution to this problem. In fact, we can calculate the probability of a tie (statement F). It is zero (see Spiegel, Schiller, and Srinivasan 2000, for details). However, we cannot definitely say whether F will never be true, because, as is well known, statements having probability zero are neither certainly true nor necessarily false. It would seem that F is possible in some sense, for example, it seems violate no physical or mathematical laws to suppose that F is true. Nevertheless, this is not the case.
With the aim of showing this, we agree to construe the two sets Y and $-Y$, introduced in axiom 2.2, as the sets of true and false statements, respectively, and thus the individual functions of the language $\mathscr{L}_u$ as propositional functions. (See interpretations 1 and 2).
Based on these interpretations, and assuming that the player A has executed at least one toss, we suppose that b_h and g_h are respectively the individual functions of $\mathscr{L}_u$: 'A has executed the h_{th} toss of a pair of fair dice', and 'A got a total of 7 at h_{th} toss', where h is the variable of

[11] Note that, because of proposition 3.3, ${}_\Theta\xi \not\Rightarrow {}_\Theta q_{\hat{\imath}}$ if and only if ξ in Y is not a sufficient condition for $q_{\hat{\imath}}$ in Y. On the other hand, considering what we have shown in notes 8 and 10, it is evident from C 4 that the claim 'ξ in Y is not a sufficient condition for $q_{\hat{\imath}}$ in Y' cannot require the placing of the sentence ${}_O KNOW(q_{\hat{\imath}})$ of $\mathscr{L}_u$ in set K.
[12] Obviously, we here refer to a formulation of calculus of probability in which probabilities are assigned to simple and compound statement form (see Resnik 1987, for details).

$\mathscr{L}_u$ (hence, a numerical variable) that appears in any one of them.
Also suppose s_h is the individual function of $\mathscr{L}_u$ 'The dice, used at hth toss, are loaded neither so that they roll a total of 7 nor so that the sum of 7 does not turn up', where h is its unique variable.
Let b: 'A has executed the toss of a pair of fair dice', s: 'The dice are loaded neither so that they roll a total of 7 nor so that the sum of 7 does not turn up' and g: 'A got a total of 7' are the known terms of the individual functions b_h, s_h and g_h, respectively.
Suppose H is the K-set of the individual function b_h.[13] (H is therefore a finite or countable infinite set of numerals that serve as distinctive tags or labels of each toss).
Let $\hat{h}$ be any trial, i.e., any member of the set H.
Since the probability of getting 7 on a single toss is $1/6$, we have that, if A has executed the $\hat{h}$th toss of a pair of fair dice, the statement 'A got a total of 7 at $\hat{h}$th toss' may or may not be true.
It follows by interpretation 1 that the individual constant $g_{\hat{h}}$ of **E** may or may not be an element of the set Y.
Hence, by virtue of proposition 3.1 and definition 4.1, we can say that, under the condition b (here represented by the claim 'A has executed the toss of a pair of fair dice'), g may or may not be an outcome of the trial $\hat{h}$.
This, formally, can be written as

$$[{}_\Theta b_{\hat{h}} \not\Rightarrow_\Theta (\sim g_{\hat{h}})] \wedge ({}_\Theta b_{\hat{h}} \not\Rightarrow {}_\Theta g_{\hat{h}})$$

(also see definition 4.3).
Furthermore, a little reflection shows that:

(1) The claim $b_{\hat{h}}$ 'the player A has executed the $\hat{h}$th toss of a pair of fair dice' entails the statement $s_{\hat{h}}$ 'the dice, used at $\hat{h}$th toss, are loaded neither so that they roll a total of 7 nor so that the sum of 7 does not turn up'. Hence, we formally have from interpretation 1 and proposition 3.3

$$ {}_\Theta b_{\hat{h}} \Rightarrow {}_\Theta s_{\hat{h}}; \quad \text{(requisite (ii) of definition 4.2);}$$

(2) The sentences of $\mathscr{L}_u$

$${}_\Theta(\sim s_{\hat{h}}) \vee \{[{}_\Theta b_{\hat{h}} \not\Rightarrow_\Theta (\sim g_{\hat{h}})] \wedge ({}_\Theta b_{\hat{h}} \not\Rightarrow {}_\Theta g_{\hat{h}})\} \quad , \quad {}_\Theta s_{\hat{h}} \vee \{[{}_\Theta b_{\hat{h}} \Rightarrow_\Theta (\sim g_{\hat{h}})] \vee ({}_\Theta b_{\hat{h}} \Rightarrow {}_\Theta g_{\hat{h}})\}$$

are both tautologies.[14] (requisite (i) of definition 4.2).

It is inferred that, for every h in set H, g is an event of the trial h, random with respect to b. Then, by theorem 4.1, we can state that the event g has occurred in at least one element of H. That means that the statement 'neither A nor B will win the game' is necessarily false.

[13]We can suppose that this set exists, since by hypothesis the player A has executed at least one toss.
[14]It is sufficient to observe that, if the player A has executed the $\hat{h}$th toss of a pair of dice (claim implicit in the statement $b_{\hat{h}}$), say that the true-value of the statement $g_{\hat{h}}$ is undecidable is equivalent to saying that the dice, used (by A) at $\hat{h}$th toss, are loaded neither so that they roll a total of 7 nor so that the sum of 7 does not turn up (statement $s_{\hat{h}}$). Clearly, this also holds if the player A has executed the $\hat{h}$th toss of a pair of fair dice (statement $b_{\hat{h}}$); hence we can formally write

$$\{{}_\Theta s_{\hat{h}} \Rightarrow [{}_\Theta b_{\hat{h}} \not\Rightarrow_\Theta (\sim g_{\hat{h}})]\} \wedge [{}_\Theta s_{\hat{h}} \Rightarrow ({}_\Theta b_{\hat{h}} \not\Rightarrow {}_\Theta g_{\hat{h}})], \quad \{[{}_\Theta b_{\hat{h}} \not\Rightarrow_\Theta (\sim g_{\hat{h}})] \wedge ({}_\Theta b_{\hat{h}} \not\Rightarrow {}_\Theta g_{\hat{h}})\} \Rightarrow {}_\Theta s_{\hat{h}}$$

From which we easily get the property looked for.

5. Conclusion

We have achieved our goal. In short, we have tried to prove that every random event must occur at least once. This statement is proved in terms of a newly defined symbolic language, taking an informal notion of random event as starting point, and going on to give it a logical formulation. But the repercussion of this result may go well beyond the treatment of a problem which is not solvable using only the probability calculus. Our discussion, in fact, leaves two issues unsettled:

(1) Is there at least one presentation of the probability calculus in which probabilities are assigned to elementary or compound sentences of language $\mathscr{L}_u$?[15]

(2) Can we show that every random event must appear a uniquely determined (and strictly positive) number of times?

It is clear that any affirmative answer to both these questions would be surely helpful in eliminating some difficulties that the researchers have with the project of interpreting the concept of probability, which is one of the most important such foundational problems.

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[15] According to interpretation 1 and proposition 3.1, it seems possible to answer yes to this question, at least as concerns the elementary sentences of $\mathscr{L}_u$ having the form ${}_{\Theta}\varepsilon$ (with ε belonging to $\mathbf{E}$). In fact, as well known, we can built a formulation of probability calculus in which probability is assigned to simple or compound statement forms (see Resnik 1987), characterizing the degree of belief in their truth.

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