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ON DEFINING $\mathscr{S}$-SPACES

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ABSTRACT. The present work is intended to be an introduction to the Superposition Theory of David Carfì. In particular I shall depict the meaning of his brand new theory, on the one hand in an informally fashion and on the other hand by giving a formal approach of the algebraic structure of the theory: the $\mathscr{S}$-linear algebra. This kind of structure underpins the notion of $\mathscr{S}$*-spaces* (or *Carfi*-spaces) by defining both its properties and its nature. Thus I shall define the $\mathscr{S}$-triple as the fundamental *principle* upon which the $\mathscr{S}$-linear algebra is built up.

1. An outline of distribution theory

In a footnote in Dieudonné (1983) is written:

> The role of Schwartz in the theory of distributions is very similar to the one played by Newton and Leibniz in the history of Calculus: contrary to popular belief, they of course did not invent it, for derivation and integration were practiced by men such as Cavalieri, Fermat and Roberval when Newton and Leibniz were mere schoolboys. But they were able to systematize the algorithms and notations of Calculus in such a way that it became the versatile and powerful tool which we know, whereas before them it could be handled via complicated arguments and diagrams.

This note is a very fascinating one, because it summarizes the extent of the work of Schwartz. During the first part of the 20^{th} Century, many mathematicians struggled with the weak solution of a linear partial differential equation, which had been faced by Poincaré in the late 19^{th} Century. Given a differential operator

$$A : f \to \sum_a c_a D^a f$$

that has C^∞ coefficients defined in an open set $\Omega \subset \mathbb{R}^n$, we can define $\langle f, l \rangle = \int_\Omega f(x) l(x) dx$ where $f(x)$ is locally integrable in Ω and l is continuous in (Ω, K) i.e. with compact support. Rather

$$\langle A \cdot f, l \rangle = \langle f, A^t \cdot l \rangle.$$

It is important to define a solution $\langle A \cdot f \rangle = 0 \quad \forall f \in C^\infty$ and thus for $\langle A^t \cdot l \rangle = 0 \ \ \forall l \in (\Omega, K)$. Following Dieudonné (1983), all the locally integrable functions in Ω are called weak solutions for $\langle A \cdot u = 0 \rangle$, even if they are not differentiable. A general solution it was not so simple to find. Schwartz was unaware of the Sobolev's work, albeit was right him who gave a rigorous definition of distributions in a functional point of view in 1936. Schwartz was influenced by Leray, Cartan, De Rham and the difficulties in defining the δ-function. The point it was to find something that works like the same problem faced for the simpler linear differential operator solved by Du Bois-Reymond in 1879, thus any weak solutions $Du = 0$ must be a constant.

A more interesting step was that of finding a *generalized operator* which acts on functions that are not differentiable. In a particular way, Cartan E. in 1922 defined an external derivative $w = Ldy \wedge dz + Gdz \wedge dx + Rdx \wedge dy$ where, although L, G, R were not differentiable, it was be possible to define $dw = Sdx \wedge dy \wedge dz$ given an S continuous. Rather

$$\iiint_U Sdxdydz = \iint_\Sigma (Ldy \wedge dz + Gdz \wedge dx + Rdx \wedge dy)$$

for any open set U and smooth boundary Σ (Dieudonné 1983). As a little generalization of this, Choquet and Deny gave an interesting theorem in defining an integral $\int_E Fd\gamma$ which is made of all mass distributions (E, γ) similar to (E_0, γ_0), where E_0 is a compact support of a mass distribution γ_0. This theorem is very close to the concept of distributions, and to the concept of Sobolev functional. Rather it has been developed the notion of *generalized derivative* for which it was important the definition of *regularization* owed to Leray. He used the convolution between a sequence of C^∞ functions g_n with compact support tending to 0 and a locally integrable function$f$, where $\int g_n = 1$. He states that if f is continuous, $p * f$ is convergent to f in each compact subset and it is a C^∞ function. Given that, there exists a generalized derivative h such that $g_n * h$ is the derivative of $g_n * f$. Nowadays we say that this function is Lebesgue integrable. As I said before, it has been Sobolev the father of the first rigorous definition of such a functionals (distributions), he defined the generalized derivative for every non-differentible function. Sobolev dealt with linear form such as $f \to f^k(x)$ in the space of C^∞ function with compact support $\Omega(K)$. In order to find a solution for the Cauchy's problem for a second order hyperbolic equations with general boundary conditions, he needed to find some particular functionals in Ω (called distributions by Schwartz). Thus, considering the subspace $\Omega(K)$ of Ω consisting of all C^∞ functions with compact support K, it is a Fréchet space given the seminorm

$$g_{m,K}(f) = \sup_{|\alpha| \leq m, x \in K} |D^\alpha(f)|, \tag{1}$$

we call distributions the linear form T on Ω. Sobolev identified the space L^1_{loc} as the space of all locally integrable functions which is identified with a subspace of the space (denoted) $\mathscr{D}'(\Omega)$ of all distributions. It is now possible to define the Dirac function $x \to \delta(x-a)$ as a measure, rather the more δ is shrinked, the more it becomes high so as to retain its mass 1 at a point a. Also, he gave the notion of derivative of its functional and he considered the weak topology for the space $\mathscr{D}'(\Omega)$, defined $(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))$. Given that, he applied the Leray approach to the distributions in order to verify the existence of the limits in the weak topology: a $g_n * T$ is a distribution $f \to T(\tilde{g}_n * f)$ which it is a C^∞ function, and the $g_n * T$ converges weakly to T as $n \to \infty$, because of this limit property they are called *generalized functions*.

Another step in order to enlarge the extent of the theory of these functionals, was to apply them in harmonic analysis. In fact, the Fourier $\mathfrak{F}f$ transorm had meaning only for those functions $f \in \mathbb{R}^n$ belonged to the space $L^1(\mathbb{R})$, then by Plancherel (Dieudonné 1983) is it possible to define an isometry $f \to \mathfrak{F}f$ for the space $L^2(\mathbb{R}^n)$, namely Hilbert spaces onto Hilbert spaces; it could be defined in the space $L^1 \cap L^2$. Weil extended this property to $f \in L^1 \cap L^\infty$. But Schwartz gave a proper solution to this problem by using the Sobolev functionals, in particular by defining the notion of *test functions* $\mathscr{S}(\mathbb{R}^n)$. This is a very important idea, a function which tests another function, i.e. a function by which it is possible to probe into another one. In order to grasp its meaning it is usually explained by the example of temperature. If we deal with point-function such as $f(x) = y$ we shall define a rigid evaluation of a function which boils down to a single point, and a temperature cannot be defined as a point at all. Thus another method of evaluation could be a uniform avarage of a temperature T throughout a set $[a,b]$ by means of the definite integral $\int_a^b f(T)dt$. But it is still a naive definition of temperature. Rather it is not uniforlmy distribuited throughout a set $[a,b]$, it will be a part of the set that is heating up more than another, for example the two extreme points could be colder than its middle points. Thus it is important to consider a weight for a uniform avarage, by the frequency of the test functions it is possible to modify the uniform avarage in order to study where the heat is more concentrated in that function by the integral $\int_a^b f(T)\varphi(T)dt$. We are thus interested in $f(T)$, and in order to know its behaviour we do need *test functions* $\varphi(T)$ by which one can study $f(T)$. We define a test function φ as a function which belongs to $C^\infty(\mathbb{R}^n)$ and it is rapidly decreasing at infinity, thus a compact support. For the generalization in harmonic analysis it is important to say that a Fourier transorm in $\mathscr{S}(\mathbb{R}^n)$ is a bijection with the Fréchet topology

$$t_{k,m}(f) = \sup_{|\alpha| \leq k, x \in \mathbb{R}^n} (1+|x|)^m |D^\alpha f(x)|.$$

Given that $\mathscr{S}(\mathbb{R}^n, K) \subset \mathscr{S}(\mathbb{R}^n)$ their injection is continuous and their union is dense in $\mathscr{S}(\mathbb{R}^n)$. The continuous linear forms on $\mathscr{S}(\mathbb{R}^n)$ are called, by Schwartz, *tempered distributions* $\mathscr{S}'(\mathbb{R}^n)$. In this space the Fourier transform is the transposed automorphism in $\mathscr{S}(\mathbb{R}^n)$, thus

$$\langle \mathfrak{F}T, f \rangle = \langle T, \mathfrak{F}f \rangle.$$

The most important difference between general distributions and tempered distributions is that the latter act on an *open* support test functions. Let me clarify this difference. The space $\mathscr{S}(\mathbb{R}^n)$ is an *open* support test functions space thus is weaker than the *closed* support test function. The former need not to vanish outside the support, they must be rapidly decreasing together by all their derivatives, whereas compact test functions must vanish identically outside of some compact support. I have not stressed this difference before because it could have arisen some confusion in understanding the right meaning of it. Moreover it is important to recall that a general distribution is a continuous mapping from the set of compact test functions into the set of real or complex numbers, whereas a tempered distribution is a continuous mapping from the open support test functions into the set of complex numbers, also it can be defined as the dual $\mathscr{S}'$ of the Schwartz distribution $\mathscr{S}$. The distribution is a functional itself such as the tempered distributions. This space is placed in an intermediate position between the space Υ' of distributions with compact support and the space of all distributions $\mathscr{D}'$. Thus

$$C_k^\infty \to \mathscr{S} \to C^\infty,$$

these are injections and they are continuous linear mapping, moreover they have danse images. By transposing the above sequence we obtain continuous injections (Trèves 1967)

$$\Upsilon' \to \mathscr{S}' \to \mathscr{D}',$$

and it is possible to understand the relations among them, rather $\mathscr{S}'$ contains Υ', and thus the dual of $\mathscr{S}$ is a space of distributions $\mathscr{S}'$. The notion of tempered distribution is a very important one for our aim, rather the superposition integral is a relation between two tempered distributions which gives rise another tempered distribution in a very particular manner, I shall explain this in the next section.

1.1. The derivative of a distribution. It is a very important tool because of its utmost useful property: a derivative of a distribution is a distribution itself, by this property one can state the existence of infinity derivatives. The *trick* is to move the derivation from f, which can be non-differentiable, to φ which it is well known a differentiable one. Now given a distribution T and a test function φ the $\langle T', \varphi\rangle$ is the derivative of T:

$$\langle T', \varphi\rangle = \int_{-\infty}^{+\infty} T'(x)\varphi(x)dx = [T(x)\varphi(x)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} T(x)\varphi'(x)dx, \tag{2}$$

I used the integration by parts and it is something rather important to understand: the $[\cdots]_{-\infty}^{+\infty}$ is 0 because of the properties of φ. Rather a test function vanishes outside its support, and so $\varphi \to \infty$ is obviously equal to 0. Thus our derivative becomes

$$\langle T', \varphi\rangle = -\int_{-\infty}^{+\infty} T(x)\varphi'(x)dx = -\langle T, \varphi'\rangle. \tag{3}$$

It is a quite important property because one is able to find the derivative even for *non-functions* like the Heaviside one (h). Look at the following example

$$\langle h', \varphi\rangle = -\langle h, \varphi'\rangle,$$

thus

$$\begin{aligned} -\langle h, \varphi'\rangle &= -\int_0^{\infty} h(x)\varphi'(x)dx \\ &= -\int_0^{\infty} \varphi'(x)dx \\ &= [\varphi'(x)]_0^{\infty} \\ &= -[\varphi(\infty) - \varphi(0)] \\ &= \varphi(0) \\ &= \langle \delta, \varphi\rangle, \end{aligned} \tag{4}$$

furthermore

$$\langle h', \varphi\rangle = \langle \delta, \varphi\rangle \therefore h' = \delta.$$

I have just obtained the derivative of the Heaviside function, that is not even a function, this is the power of this generalized derivative: the derivative of a distribution.

2. The superposition theory

2.1. An intuitive approach. The importance of the distribution theory is of course known, but during these years, many mathematical methods have been developed for L^p-spaces. A very used space is that of Hilbert and L^2-spaces or Banach spaces (L^1) et cetera. The development of these (and other) spaces and their methods, has been an important field of study in order to enhance many beautiful tools of the functional analysis, but it has been left, surprisingly, the distribution theory in the background. For instance, when one study spectral theory may happen to face a class of eigenfunctions that cannot be integrable in the Hilbert sense, thus the notion of this space it has been extended in order to deal with those functions which are not square-integrable. It is a serius puzzle because it is important in order to define nuclear spaces. This extension has been done by using the Schwartz space $\mathscr{S}$ and it is now called *rigged Hilbert space*. It is called the Gel'fand triple

$$\phi \subset H \subset \phi'$$

where ϕ is the space of test functions, H the Hilbert space and ϕ' the Schwartz space. Given that L^2 is dense in $\mathscr{S}$, it is possible to treat those eigenfunctions that are not lying in an L^2-space, in Schwartz spaces (see Gel'fand and Vilenkin (1964)). Although this extension allows us to use non-L^2 objects as if they were, many other functions and quasi-functions are impossible to deal with, and thus it is important to enlarge the Gel'fand triple on a richer one: The $\mathscr{S}$-triple (Carfì 2006). This expansion is called: Superposition theory and it has been built up by a very clever intuition of Carfì. In this section I shall depict his theory in an intuitive fashion and I shall clarify the meaning of this brand new structure: the *Carfì space* ($\mathtt{C}^{\uparrow}$). It is rather important to understand that a Carfì space is not just an extension of that of Hilbert, but it is a space made of a new algebraic-topological structure that gives to it some very fascinating properties.

I have said that the rigged Hilbert space leans on the Schwartz space in order to enlarge the action of a Hilbert space. It would be thus intuitive to comprehend the notion of Carfì space such as a space by which one is able to deal with math-objects that cannot be applied in a rigged Hilbert sense. I am afraid to say: no, it is not so simple. A Carfì space has a structure which generalizes the notion of continuity and generalizes the notion of scalar product, the $\mathscr{S}$-triple is thus this structure, and it is composed by

$$\left\{ \mathscr{S}'_n, \left(\int_{\mathbb{R}^m} \right)_{m\in\mathbb{N}}, (\cdot|\cdot)_{L_2} \right\}. \tag{5}$$

In (5) the $\mathscr{S}'_n$ is the space of tempered distributions, the $(\int_{\mathbb{R}^m})_{m\in\mathbb{N}}$ denotes the operations of superposition, and the $(\cdot|\cdot)_{L_2}$ is the L_2-product. The reader has not to be worried if he does not understand them at this stage, the meaning of those objects will be clear throughout the following subsections.

The (5) is a generalization of the topological vector space $(\mathscr{S}'_n, +, \cdot)$ and, as a matter of definition (see A), it is continuous in $+$ and $(\cdot)$. Rather $(\int_{\mathbb{R}^m})_{m\in\mathbb{N}}$ is a generalization of $+$, and $(\cdot|\cdot)_{L_2}$ is a generalization of $(\cdot)$. By using the operations of superposition in lieu thereof the simple sum $+$, it is possible to deal with an infinitive *continuous* sum in $\mathbb{R}^m$ and this enlargement encompasses those functionals that are called *smooth*. It is important to understand that the sum of a product $\sum_{i=1}^m a_i v_i$ is, graphically, a point-like sum which gives

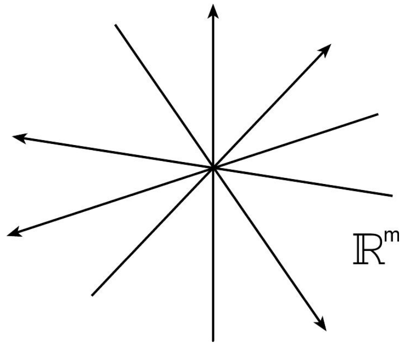

FIGURE 1. Spaces in $\mathbb{R}^m$

us a particular point-value, this has been generalized by Carfì in $\mathbb{R}^m$ (see Fig. 1) by using smooth functionals and defining them through m-directions. The extent of this enlargement becomes plain if the attention is focused on the notion of continuity, but so as to grasp this, it is important to comprehend the concept of L_2-product. Given a complex vector space A, a mapping $\langle \cdot, \cdot \rangle = A \times A \to \mathbb{C}$, where $x, y, z \in A$ and $\alpha, \beta \in \mathbb{C}$, is defined an *inner product* in A if

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle \alpha x + \beta x, z \rangle = \langle \alpha x, z \rangle + \langle \beta y, z \rangle$
- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0 \Rightarrow x = 0$

An inner product involving functions can be thought of as $\langle f, g \rangle = \int_a^b f(x)g(x)dt$ (I omit to list its properties) or, an important one , $\langle f, f \rangle = \int_a^b (f(x))^2 dt < \infty$. The important property belonged to all of the inner products is that are *true*, namely to every function (real numbers, matrices et cetera) corresponds a particular value. For instance look at the following example for $f(x) = 3x - 2$:

$$\begin{aligned} \int_0^1 (f(x))^2 dx &= \int_0^1 (3x-2)(3x-2) \\ &= \int_0^1 9x^2 - 12x + 4 \\ &= [3x^3 - 6x^2 + 4x]_0^1 \\ &= 1 \end{aligned} \tag{6}$$

Given the (6) it is straightforward that it is possible to define the orthogonality so as to build up pointwise-projections. It is a very important property, rather it can be generalized in Carfì spaces by using *non-true* distributions. The meaning of them is quite an important one: a distribution is called *non-true* if it can be involved in an L_2-product, that is to say from a product between a distribution and a family of distributions, it is obtained a *place*-definition of that product rather than a poitwise one. It is crucial in defining Carfì spaces. The figure 2, albeit not so precise, represents an L_2-product, rather there is a vector $\overrightarrow{v}$ times a family v_i and this product gives rise to a global definition φ. The choice of φ is not random, in fact

a test function is a function with some local properties which defines a *place* instead of a *point*.
In this introduction I have defined the core concept of the superposition theory by which it can be possible to build up the notion of Carfì space. It is now plain, even if in an intuitive fashion, that the framework of Carfì spaces involves the space of tempered distributions wherein it is possible to define not just continuous sums, such as topological vector spaces, but smooth sums which tend to infinity. This can be melt into the definition of L_2-product by which one is able to use a particular kind of projection, rather its output is a global definition of the product, i.e. it not possible to know *who* is the i-th coefficent but one is able to know and study its global properties. The world is not in a pointwise connection, as often as not when one studies world events he can only do it in a global view, and thus the study of the world can be done in a more concrete way by using Carfì spaces rather than Hilbert spaces. In the following subsection I shall depict the notion of $\mathscr{S}$-linear algebra, it is the most important object in order to define Carfì spaces.

2.2. Basic concepts on $\mathscr{S}$-linear algebra. The framework underpins the notion of Carfì space is the characterization of the $\mathscr{S}$-linear algebra by which it is possible to define the properties defined for the $\mathscr{S}$-triple. I have stressed the importance of the concept of family, and thus it is fundamental to define that one of interest for our aim. A family $s(\mathbb{R}^m, \mathscr{S}'_n)$ is the space indexed by $\mathbb{R}^m$, i.e. the mappings from $\mathbb{R}^m$ to $\mathscr{S}'_n$, thus if v is one of these families, for $p \in \mathbb{R}^m$, the distribution $v(p)$ is denoted by v_p. The family $s(\mathbb{R}^m, \mathscr{S}'_n)$ is a vector space w.r.t. (with respect to)

$+$: $s(\mathbb{R}^m, \mathscr{S}'_n)^2 \to s(\mathbb{R}^m, \mathscr{S}'_n) : (v,w) \mapsto v+w$ where $v+w : \mathbb{R}^m \to \mathscr{S}'_n : p \mapsto v_p + w_p$;
$\times$: $\mathbb{R} \times (\mathbb{R}^m, \mathscr{S}'_n) \to s(\mathbb{R}^m, \mathscr{S}'_n) : (\lambda, v) \mapsto \lambda v$ where $\lambda v : \mathbb{R}^m \to \mathscr{S}'_n : p \mapsto \lambda v_p$.

Let v be a family in the space S'_n indexed by $\mathbb{R}^m$. The family v is called a *family of class S* ($\mathscr{S}$-family), if for each test function $\phi \in S_n$, the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$ defined by

$$v(\phi)(p) := v_p(\phi)$$

for each index $p \in \mathbb{R}^m$, belongs to the space of S_m. This space is denoted by $S(\mathbb{R}^m, S'_n)$. The operator generated by the family v, is the operator $\hat{v} : \mathscr{S}_n \to \mathscr{S}_m : \phi \mapsto v(\phi)$, sending every test function ϕ of $\mathscr{S}_n$ into its image $v(\phi)$ under the family v (Carfì 2010).
We call the $\mathscr{L}(\mathscr{S}_n, \mathscr{S}_m)$, the set of all the linear and continuous operators among two topological vector spaces $\mathscr{S}_n$ and $\mathscr{S}_m$. Consider a linear operator $A : \mathscr{S}_n \to \mathscr{S}_m$, it is topologically transposable if its adjoint ${}^*A : \mathscr{S}_m^* \to \mathscr{S}_n^*$ defined by ${}^*A(a) = a \circ A$, maps the distribution space $\mathscr{S}'_m \to \mathscr{S}'_n$. It is known that the space $\mathscr{S}$ is a Fréchet space, as we have seen (Sec. **1**), and thus a countable projective limit (intersections) of Banach spaces. The main difference from Sobolev spaces is plain, rather it is, roughly speaking, a Banach space with L^p-norm. The space of tempered distribution is the dual of the Fréchet space $\mathscr{S}$, and it is known that it can be a space rather differet from the Fréchet one, indeed its dual is given by the weak-$\star$topology (i.e. the weakest topology in X' wherein all the linear functions $f \mapsto f(x) \forall x \in X$, are continuous). The family $S(\mathbb{R}^m, S'_n)$ is a subspace of $s(\mathbb{R}^m, +, \cdot)$ and for each v, the operator $\hat{v}$ is linear in $S(\mathbb{R}^m, S'_n)$. We call $\mathscr{S}$-linear combination (or linear *superposition*) of v w.r.t. the system of coefficients a, the distribution

$$\int_{\mathbb{R}^m} av := a \circ \hat{v} : \phi \mapsto a(\hat{v}(\phi)) \tag{7}$$

i.e.

$$\int_{\mathbb{R}^m} av = {}^t(\hat{v})(a).$$

Given (7) we can state that $\hat{v}$ is transposable and weakly continuous, i.e. continuous from $\mathscr{S}_n$ to $\mathscr{S}_m$ w.r.t. the pair of weak topologies $(\sigma(\mathscr{S}_n), \sigma(\mathscr{S}_m))$. Given the definition of $\mathscr{S}$-linear superposition it is important to say that the two vector spaces $\mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$ and $\mathscr{L}(\mathscr{S}_n, \mathscr{S}_m)$ are isomorphic

$$(\cdot)^\wedge : \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n) \to \mathscr{L}(\mathscr{S}_n, \mathscr{S}_m) : v \mapsto \hat{v}. \tag{8}$$

Thus, as a matter of definition, it is continuous in its inverse

$$(\cdot)^\vee : \mathscr{L}(\mathscr{S}_n, \mathscr{S}_m) \to \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n) : A \mapsto A^\vee := (\delta_x \circ A)_{x\in\mathbb{R}^m}. \tag{9}$$

It is worth noting that (8) is the operator generated by an $\mathscr{S}$-family, whereas (9) is the operator which generates an $\mathscr{S}$-family. It is possible to define other kinds of superposition operator, a rather important one is the superposition of a family of numbers (real or complex) w.r.t. a distributional system of coefficients, that is to say: a family of real or complex numbers $x = (x_i)_{i\in\mathbb{R}^m}$ is an $\mathscr{S}$-family if a function $f_x : \mathbb{R}^m \to \mathbb{K}$ defined by $f_x(i) = x_i$ is an $\mathscr{S}$-function $\forall x \in \mathbb{R}^m$. The function$f_x$ is called the test function associated with the family x (Carfì 2007b). Thus, given $a \in \mathscr{S}'_m$, the superposition of the family x w.r.t. a is

$$\int_{\mathbb{R}^m} ax := a(f_x). \tag{10}$$

If we melt the two kinds of superposition we have seen into a new one, we shall obtain a particolar $\mathscr{S}$-linear combination. Thus, we call $\langle\cdot,\cdot\rangle$ the canonical bilinear form on $\mathscr{S}'_n \times \mathscr{S}_n$ where $v \in \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$ and the test function $\phi \in \mathscr{S}_n$, by $\langle v, \phi\rangle$ we denote the family of numbers defined by

$$\langle v, \phi\rangle_i = \langle v_i, \phi\rangle \quad \forall i \in \mathbb{R}^m.$$

Moreover, given $a \in \mathscr{S}'_m$ we obtain a superposition

$$\left\langle \int_{\mathbb{R}^m} av, \phi \right\rangle = \int_{\mathbb{R}^m} a\langle v, \phi\rangle. \tag{11}$$

We have thus two kinds of operator

$$\int_{\mathbb{R}^m} (\cdot,\cdot) : \mathscr{S}'_m \times \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n) \to \mathscr{S}'_n : (a, v) \mapsto \int_{\mathbb{R}^m} av \tag{12}$$

and

$$\int_{\mathbb{R}^m} (\cdot, v) : \mathscr{S}'_m \to \mathscr{S}'_n : a \mapsto \int_{\mathbb{R}^m} av; \tag{13}$$

the superposition operator $\mathscr{S}'_n$ with coefficients-system in $\mathscr{S}'_m$ and the superposition operator in $\mathscr{S}'_n$ associated to an $\mathscr{S}$-family v respectively.

Another interesting type of superposition is the superposition of the family v w.r.t. the regular distribution generated by the $\mathbb{K}$-constant functionals on $\mathbb{R}^m$ of value 1: the distribution $[1_{\mathbb{R}^m}]$, we denote this superposition by

$$\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} [1_{\mathbb{R}^m}]v. \tag{14}$$

This kind of superposition is very important because of the propery of orthonormality **2.2**.

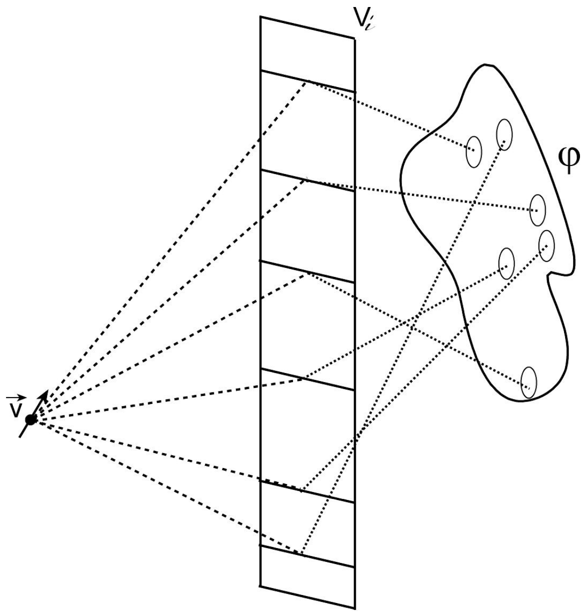

FIGURE 2. L_2-product

The L_2-product. We have seen in Sec. **1** an intuitive introduction to the concept of L_2-product (see Fig. 2). Now it is time to introduce it in a more rigorous manner. We have seen (Sec. **1**) that an L_2-product is a product between a distribution times a family of distributions, by which we obtain a non-locally definite value. This notion stems from the impossibility to compute a product $(\delta_x|\delta_y)$ because of its nonsense by using Hilbert spaces. The very clever intuition of Carfì was to compute a product $(\delta_x|\delta)_{L_2}$ between a distribution and an entire family. Given this intuition it is possible to study the non-locally properties of every function and non-function

$$(\delta_x|\delta) = \delta_x : \varphi(x) : \delta_x(\varphi). \tag{15}$$

It is now straightforward the extent of (15) for its orthonormality implications. We say that $v \in \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$ is δ-orthogonal if there exists a function $f : \mathbb{R}^m \to \mathbb{K}$ such that $(v_p|v) = f\delta_p$ for each n-uple p. If the L_2-product is orthogonal and if $f \in [1_{\mathbb{R}^m}]$, then $(v_p|v) = \delta_p$ is called δ-orthonormal (Carfì 2003). Thus, for what we have seen in (14)

$$\begin{aligned} \int_{\mathbb{R}^m} u\delta &= u \circ \hat{\delta} \\ &= u \circ \mathbb{I}_{\mathscr{S}_n} \\ &= u. \end{aligned} \tag{16}$$

Given this property it can be stated that $\mathscr{S}$-span$(\delta) = \mathscr{S}'_n$ (look at the following paragraph). The $\mathscr{S}$-linear hull. The $\mathscr{S}$-linear hull of an $\mathscr{S}$-family is the set of all the $\mathscr{S}$-linear combinations of the family $v \in \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$ and it is denoted by $\mathscr{S}$-span(v):

$$\mathscr{S}\text{-span}(v) := {}^t(\hat{v})(\mathscr{S}'_m). \tag{17}$$

Moreover (17) denotes

$$\mathscr{S}\text{-span}(v) := \left\{ u \in \mathscr{S}'_n : \exists a \in \mathscr{S}'_m : u = \int_{\mathbb{R}^m} av \right\}.$$

It is now time to give an interesting property of the δ of Dirac: it can be thought of as an extension of the canonical basis e_n in linear algebra. Rather, by δ_n it is possible to span the whole $\mathscr{S}_n$-space (see (16)) and it is possible to define its δ-orthonormal bases. We can say that δ (and the Fourier families) is a system of $\mathscr{S}$-generators for the entire space $\mathscr{S}'_n$. This statement suggests the word $'$generator$'$ to us. Let $v \in \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$, it is a family of $\mathscr{S}$-generators for a *subspace* V of the space $\mathscr{S}'_n$ iff its $\mathscr{S}$-linear hull coincides with the subspace V

$$\mathscr{S}\text{-span} = V.$$

Moreover it is worth noting that the $\mathscr{S}$-span(v) is a subspace of $\mathscr{S}'_n$ and thus it contains all the elements of v, hence

$$\text{span}(v) \subseteq \mathscr{S}\text{-span}(v).$$

Topological comments. We denote by $\beta(\mathscr{S}'_n)$ the strong topology $\beta(\mathscr{S}'_n, \mathscr{S}_m)$, by $\sigma(\mathscr{S}'_n)$ the weak-$\star$topology $\sigma(\mathscr{S}'_n, \mathscr{S}_n)$ and by $\sigma(\mathscr{S}_n)$ the weak topology $\sigma(\mathscr{S}_n, \mathscr{S}'_n)$ (see A.**2**). Given that $\mathscr{S}$ is reflexive, then $\mathscr{S}'$ is semi-reflexive, thus the linear subspaces of $\mathscr{S}'_n$ are colsed in $\beta(\mathscr{S}'_n)$ iff are closed in $\sigma(\mathscr{S}'_n)$. It is interesting to know that the superposition family v is the $\beta(\mathscr{S}'_n)$-limit of sequence of finite linear combinations of the family v. Let a be in the coefficient space $\mathscr{S}'_m$, then the coefficient distribution a is the $\beta(\mathscr{S}'_n)$-limit of a sequence γ of finite combinations of the Dirac family in $\mathscr{S}'_m$. Since $\overline{\text{span}}_{\beta(\mathscr{S}'_n)}(\delta) = \mathscr{S}'_m$, we have

$$\begin{aligned} \int_{\mathbb{R}^m} av &= \int_{\mathbb{R}^m} \left({}^{\beta(\mathscr{S}'_n)} \lim_{k\to\infty} \gamma_k \right) v \\ &= {}^{\beta(\mathscr{S}'_n)} \lim_{k\to\infty} \int_{\mathbb{R}^m} \gamma_k v \end{aligned}$$

by the $(\beta(\mathscr{S}'_m)\beta(\mathscr{S}'_n))$-continuity of the superposition operator ${}^t(\hat{v})$ (Carfì 2010). Rather, given a $v \in \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$ we have that

$$\mathscr{S}\text{-span}(v) \subseteq \overline{\text{span}}_{\beta(\mathscr{S}'_n)}(v) = \overline{\text{span}}_{\sigma(\mathscr{S}'_n)}(v)$$

It is important to introduce the notion of closedeness because if an $\mathscr{S}$-family is $\mathscr{S}$-closed w.r.t. $\sigma(\mathscr{S}'_n)$ and/or $\beta(\mathscr{S}'_n)$ it will encompass very suitable properties. The closedeness is a natural kind of stability fo subsets of the space $\mathscr{S}'_n$ arises with the definition of $\mathscr{S}$-linear combination. Let B be a subset of $\mathscr{S}'_n$, the part B is called $\mathscr{S}$-closed in the space $\mathscr{S}'_n$ iff it contains all the superpositions of the $\mathscr{S}$-families contained in B. In other words we say that B is $\mathscr{S}$-closed iff for each positive integer $k \in \mathbb{N}$, for each $\mathscr{S}$-family $v \in B$ and for each tempered distribution $a \in \mathscr{S}'_n$, the superposition $\int_{\mathbb{R}^m} av$ lies in the set B (Carfì 2010). Now, given a family of $\mathscr{S}$-closed subsets $\Upsilon = (\Upsilon_i)_{i\in I}$ and let v be a family such that $v \in \cap\Upsilon$. The family v is a family in the intersection iff $v_p \in \Upsilon_i$ where $p \in \mathbb{R}^m$ and $i \in I$. Thus, given that Υ_i is $\mathscr{S}$-closed, the superposition $\int_{\mathbb{R}^m} av$ must belong to Υ_i $\forall a \in \mathscr{S}'_m$ and $\forall i \in I$, hence $\int_{\mathbb{R}^m} av \in \cap\Upsilon$ of the family Υ. This is a quite an important statement because gives the property of stability to all of the superpositions. Let B be a subspace of $\mathscr{S}'_n$, the

$\mathscr{S}$*-closed hull* of B is the intersection of all the $\mathscr{S}$-closed subsets containig B. It is denoted by $\mathscr{S}$-cl(B) or simply $\bar{B}^{\mathscr{S}}$. Furthermore we call $\mathscr{S}$-closed *linear* hulls those $\mathscr{S}$-closed hulls w.r.t. the subspaces of B and not only w.r.t. the sets of it, and it is denoted by $\mathscr{S}$-$\overline{\text{span}}(B)$. The relation between these two notions is

$$\mathscr{S}\text{-cl}(B) \subseteq \mathscr{S}\text{-}\overline{\text{span}}(B),$$

the collection of $\mathscr{S}$-closed subsets containing B contains the collection of all $\mathscr{S}$-closed subspaces containing B.
Now, given an $\mathscr{S}$-linear hull $\mathscr{S}$-span of an $\mathscr{S}$-family v, when it will be closed w.r.t. the weak-$\star$topology the superposition operator is a topological homomorphism for the weak-$\star$topologies $\sigma(\mathscr{S}'_m)$ and $\sigma(\mathscr{S}'_n)$, also the operator $\hat{v}$ is a topological homomorphism w.r.t. the pair of weak topologies $(\sigma(\mathscr{S}_n), \sigma(\mathscr{S}_m))$ and for the topological vector space $(\mathscr{S}_n)$ into the space $(\mathscr{S}_m)$ (Carfì 2010). Given the closedeness w.r.t. $\sigma(\mathscr{S}'_n)$, it is plain that the superposition operator is a surjective topological homomorphism for the pair of strong topologies $\beta(\mathscr{S}'_m)$ and $\beta(\mathscr{S}'_n)$ and for the weak-$\star$topologies $\sigma(\mathscr{S}'_m)$ and $\sigma(\mathscr{S}'_n)$. Also, the operator $\hat{v}$ is an injective topological homomorphism for the weak topologies and for the space $(\mathscr{S}_n)$ into the space $(\mathscr{S}_m)$. The following diagram

$$\begin{array}{ccccc} \mathscr{S}'_m & \xrightarrow{\hat{v}^t} & \mathrm{Im}(\hat{v}^t) & \xrightarrow{i} & \mathscr{S}'_n \\ \phi\downarrow & \nearrow_{\bar{v}} & & & \\ \mathscr{S}'_m/\ker_v & & & & \end{array}$$

gives an idea about the relations among the objects I am using and it is important to understand it so as to clarify the notions treated throughout this section. The function $\phi : \mathscr{S}'_m \to \mathscr{S}'_m/\ker_v$ is the *canonical mapping*, it is linear and onto. The $\mathscr{S}'_m/\ker_v$ is thus the quotient space such that for every $x, y \in \mathscr{S}'_m$, the $x - y = 0$ and thus $x - y \in \ker_v$, therefore $\phi(x) = \phi(y)$, but the most important thing is: if $\phi(x) = \phi(y)$ then $\hat{v}^t(x) = \hat{v}^t(y)$. The i is the *natural injection*, that is to say for a $y \in \mathrm{Im}(\hat{v}^t)$, the mapping i assigns the same element y regarded as an element of $\mathscr{S}'_n$. Given the closedeness in $\sigma(\mathscr{S}_n)$ it is plain that $\hat{v}(\mathscr{S}_n)$ is closed in the topological vector space $(\mathscr{S}_m)$. If $\hat{v}$ is homomorphism then $\bar{v}$ will be an isomorphism, $\hat{v}^t$ has a closed $\mathrm{Im}(\hat{v}^t)$ and thus it is a continuous linear mapping from $\mathscr{S}'_m$ to $\mathscr{S}'_n$, therefore it is an homomorphism.
It is now important to give a definition of an $\mathscr{S}$-kernel. Let $v = (v_i)_{i=1}^n$ be a family of linear forms on a vector space V and let h be a linear form vanishing on the kernel of every form v_i of the family. Then, the form h is a linear combination of the family v:

$$\ker_v := \bigcap_{i\in I} \ker_{v_i}. \tag{18}$$

furthermore if W is a subspace of V, the $W^\perp$ denotes the orthogonal of W. Given that h vanishes on the kernel of the family v iff h is a linear combination of the family v, the linear hull of the family v coincides with the orthogonal of its kernel, thus

$$(\ker_v)^\perp = \text{span}(v).$$

Now it can be stated the $\mathscr{S}$-version of the statment: let $v = (v_p)_{p\in\mathbb{R}^m}$ be an $\mathscr{S}$-family of an $\mathscr{S}'_n$-space, then the orthogonal of the kernel of the family coincides with the $\mathscr{S}$-closed

$\mathscr{S}$-linear hull of the family w.r.t. the weak-$\star$topology:

$$(\ker_v)^\perp = \overline{\text{span}}_{\sigma(\mathscr{S}'_n)}(v).$$

If v is topologically exhaustive (i.e. if $\mathscr{S}$-linear hull $\mathscr{S}$-span v is $\sigma(\mathscr{S}'_n)$-closed) we have (Carfì 2010):

$$(\ker_v)^\perp = \mathscr{S}\text{-span}(v).$$

It is justified by

$$\begin{aligned}(\ker_v)^\perp &= (\overline{\text{Im}(A^t)})_{\sigma(\mathscr{S}'_n)} = \\ &= (\overline{\mathscr{S}\text{-span}(v)})_{\sigma(\mathscr{S}'_n)} = \\ &= \overline{\text{span}}_{\sigma(\mathscr{S}'_n)}(v).\end{aligned}$$

$\mathscr{S}$-bases. Let a $v \in \mathscr{S}(\mathbb{R}^m, \mathscr{S}'_n)$, we say that v is $\mathscr{S}$-linear indipendent iff $a \in \mathscr{S}'_m$ and a superposition $\int_{\mathbb{R}^m} av = 0_{\mathscr{S}'_n}$ implies $a = 0_{\mathscr{S}'_m}$

$$\left(a \in \mathscr{S}'_m \wedge \int_{\mathbb{R}^m} av = 0_{\mathscr{S}'_n}\right) \Rightarrow a = 0_{\mathscr{S}'_m}.$$

We say that an exhaustive $\sigma(\mathscr{S}'_n)$-closed $\mathscr{S}$-family v is $\mathscr{S}$-linear indipendent and that the superposition operator is an injective topological homomorphism for the weak-$\star$topology and for the strong topology, whereas $\hat{v}$ is a surjective topological homomorphism for the weak topologies and for the topological vector space $(\mathscr{S}_n)$ onto the space $(\mathscr{S}_m)$. The concept of $\mathscr{S}$-linear independence is fundamental in order to define the $\mathscr{S}$-coordinate operator. Following Carfì (2010), if v is $\mathscr{S}$-linear independent, it is possible to consider an algebraic isomorphism from $\mathscr{S}'_m$ onto the $\mathscr{S}$-linear hull $\mathscr{S}$-span(v) by which one can send every tempered distribution $a \in \mathscr{S}'_m$ to the superposition $\int_{\mathbb{R}^m} av$, that is the restriction of the injection $\int_{\mathbb{R}^m}(\cdot, v)$ to the pair of sets $(\mathscr{S}'_m, \text{span}(v))$. We shall denote the inverse of this isomorphism by the symbol $[\cdot|v]$. Thus

$$[\cdot] :^{\mathscr{S}} \text{span}(v) \to \mathscr{S}'_m \tag{19}$$

is a topological isomorphism w.r.t. the topology induced by the weak-$\star$topology on the $\mathscr{S}$-linear hull $\mathscr{S}$-span(v) iff the v-family is topologically exhaustive. Furthermore, given a $z \in \mathscr{S}$-span(v), the distribution $a \in \mathscr{S}'_m$, such that

$$z = \int_{\mathbb{R}^m} av,$$

is denoted by $[z|v]$ and it is called the system of coordinates of z in v. I can now define the concept of $\mathscr{S}$-bases: given an $\mathscr{S}$-family in $\mathscr{S}'_n$ and let T be a subspace of $\mathscr{S}'_n$. The family v is called $\mathscr{S}$-basis of T if it is $\mathscr{S}$-linear independent and it $\mathscr{S}$-generates V, that is if the superposition operator of the family v is injective and $\mathscr{S}$-span$(v) = T$. Thus there exists a unique $a \in \mathscr{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$, moreover v is an $\mathscr{S}$-basis of $\mathscr{S}_n$ iff ${}^t(v)$ is bijective.
We have seen a particular algebraic structure which is the framework that underpins the notion of Carfì space. Rather this space is characterized by the $\mathscr{S}$-linear algebra and boils down to the notions belonged to the $\mathscr{S}$-triple. Now we are able to comprehend the notion of Carfì space and its importance in order to enlarge those mathematical concepts that were bounded from the L^p spaces to the distribution one. It is thus possible to define non-locally

functions, to define their global properties; there is a new fashion in looking at mathematics: a global one by using Carfì spaces.

Appendix A.

A.**1. Topological Vector Spaces.** Given a space A over $\mathbb{C}$, we have the vector addition and the scalar multiplication

$$+ : A \times A \to A : (x,y) \mapsto x+y$$

$$\times : \mathbb{C} \times A \to A : (\lambda,x) \mapsto \lambda x.$$

In order to define a *topological* vector space it is needed that a topology $\mathfrak{T}$ of A is compatible with the linear structure of A if $+$ and $\times$ are continuous, that is to say A is provided by a topology $\mathfrak{T}$, $A \times A$ with the topology $\mathfrak{T} \times \mathfrak{T}$, $\mathbb{C} \times A$ with the topology $\mathfrak{C} \times \mathfrak{T}$ (Trèves 1967). Thus a topology $\mathfrak{C}$ of λ defines the bases of neighorhoods with open or closed disks provided by a center λ. If $\mathfrak{C}$ is compatible w.r.t. the linear structure of A, then it is a topological vector space, and thus it is *traslation invariant*. Rather, (see Trèves (1967)) the filter of neighborhoods $\mathfrak{F}(x)$ of the point x is the family of set $A+x$ where A varies over $\mathfrak{F}(0)$. This filter of neighborhoods of the origin has to be in a topology $\mathfrak{T}$ compatible w.r.t. the linear structure of A satisfying some properties. Thus the origin belongs to every subset $V \in \mathfrak{F}$, to every V there is a P such that $P+P \subset V$, for every $V \in \mathfrak{F}$ and $\lambda_{\neq 0} \in \mathbb{C}$ then $\lambda V \in \mathfrak{F}$, every subset V is absorbing and balanced.

A.**2. Duality in Topology.** Given a topological vector space A, its dual A' is defined as the vector space of all continuous linear functionals on A, that is to say continuous linear mappings from A to $\mathbb{C}$. The concept of duality is of utmost importance in order to understand the notions of this paper, rather the tempered distributions space is the dual of the space of distributions.
When one talks about duality it is natural to think about orthogonality, informally a duality seems like a rope that from a space goes to another one, the point touched onto the *another one* is, of course, orthogonal w.r.t. the space of origin. This point onto the duality (another one space) is called *polar*, and it is denoted (in the case of the *orogin*-space A) by $\mathring{A}$. Formally, given a subset G of A, the subset of A'

$$\{x' \in A'; \sup_{x \in G} |\langle x', x\rangle| \leq 1\}$$

is defined polar $\mathring{A}$ of A. It is worth noting that given $A \subset U$, the $\mathring{A}$ is a *convex balanced* subset of A', and a rather important property is that of the *cone* of A. In few words, A is a cone when $x \in A$ implies $\lambda x \in A \; \forall \lambda > 0$. In this case $\mathring{A}$ is the set of all continuous linear functions on A which vanish identically in A, thus it is the orthogonal of A and hence $\mathring{A}$ is a linear subspace of A' (Trèves 1967).
The weak topology. Given a family of finite subsets of A, denoted by $\mathfrak{A}$, the corresponding topology is $\sigma(A, A')$. Thus, continuous linear functionals x' on A' converge weakly to 0 if for each $x \in A$, the $\langle x', x\rangle$ converge to zero in $\mathbb{C}$; that is to say a pointwise convergence in A. It is the weakest topology of A that makes all elements of A' continuous.

The weak-$\star$topology. Given $g \in A'$, the seminorm $\kappa_g(x) = |g(x)|$ and $\kappa_x(g) = |g(x)|$, the topology of κ_g is the weak topology for the seminorm $\{\kappa_g | g \in A'\}$ whereas the topology defined by the seminorm $\{\kappa_x | x \in A\}$ generates the topology on A' defined the weak-$\star$topology. Thus we are talking about a topology defined on the dual space.

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