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IMPLIED-IN-PRICES EXPECTATIONS: THEIR ROLE IN ARBITRAGE

SERGEI A. IVANOV *

ABSTRACT. Real prices are created on markets by supply and demand and they do not have to follow some distributions or have some properties, which we often assume. However, prices have to follow some rules in order to make arbitrage impossible. Existence of arbitrage opportunities means existence of inefficiency. Prices always contain expectations about future. Constraints on such expectations and arbitrage mechanisms were investigated with minimum assumptions about price processes (e.g. real prices do not have to be martingales). It was shown that found constraints could be easily failed in some widespread conditions. Fluctuating risk-free interest rates creates excess amount of asset in comparison with case when they are constant. This property allows arbitrage and making risk-free profit. This possibility is hard to use. However, in theory it exists almost on every market. Interest rate is implied in almost every price. The possibility exists where there is uncertainty about future. This leads to assumption that there is very fundamental inefficiency, which potentially is able to change markets dramatically.

1. Introduction

The theory of No Arbitrage plays a serious role in Mathematical Finance. Development of pricing mechanisms (Black and Scholes 1973; Merton 1973), understanding of market efficiency, no arbitrage conditions (Harrison and Kreps 1979; Harrison and Pliska 1981) and many other important themes, which highly influence nowadays markets, are strongly connected to it. However, there are open questions, e.g. Fama (1998) concluded that existing anomalies require new theories of the stock market and we need to continue the search for better models of asset pricing.

Financial instruments are standardized, but market participants are different. They operate in different conditions, have different aims and use different strategies. It is essential that they are different not only because they are differently informed but because they have to operate in fundamentally different environment. It seems to be important how they use securities with respect to their aims and adjacent operations on a market. Interaction and cooperation of such participants, especially from the point of view of game theory, is an important theme. Especially in specific conditions (see Carfì and Musolino 2013; Musolino 2012). It becomes more important when it is connected to countries and their stability (Carfì and Schilirò 2012).

In a modern world we use strategies and securities (e.g. CDOs) that become more and more complex. Moreover, participants are different. They have different numeraires, possibilities and operate in different conditions. Each of them is unique. What if fair price and arbitrage opportunities depend on this uniqueness? This leads to the idea that securities could be analyzed by using the traditional approach of no arbitrage, but with respect to environment.

Arbitrage is potentially possible because there are connections between securities. They are connected because their prices have implied expectations about future that are mutual for different securities. Examples are call and put options, connected through call-put parity, options with different strike prices and others.

We know that derivative's price is discounted expected value of future payoff E_Q under the risk-neutral measure Q (Cox and Ross 1976). Let at a future time T a derivative's payoff is H_T, a random variable on the probability space describing market. The discount factor from the moment when premium is being paid t_0 until expiration time T is $P(t_0,T)$.

Fair value of the derivative at t_0 is

$$H_0 = P(t_0,T) \cdot E_Q(H_T) \tag{1}$$

Expected value of future payoff is mean value of payoff in different possible scenarios multiplied by probabilities of scenarios. In the case of options payoff depends on price of underlying asset. If payoff and premium in Eq. 1 are given in a numeraire that is not interesting to us then we should change it (Jamshidian 1989). We use same derivative with same numeraire but exchange payoff after expiration and premium into different numeraire, i.e. we perform additional operations. Eq. 1 transforms in this case:

$$H_0^N = P^N(t_0,T) \cdot E_{Q^N}(H_T^N) \tag{2}$$

Scenarios are the same but payoff differs:

$$H_T^N = H_T \cdot N_T \tag{3}$$

where N_T is exchange function for payoff from one numeraire to another. Eq. 2 transforms into

$$H_0 \cdot N_0 = P^N(t_0,T) \cdot E_{Q^N}(H_T \cdot N_T) \tag{4}$$

where N_0 is current value of N_T.

Imagine next situation. Asset B costs one unit of asset A. Let $P(t_0,T) = 1$. There are scenarios with equal probabilities: price will be 0.5 or price will be 1.5. We buy one unit of asset B and sell it after price change. We see that today's price is fair because expected payoff is equal to today's price. However, what if we are more interested in asset B as a numeraire? In first case our payoff after exchange is $\frac{0.5-1}{0.5} = -1$, in second case $\frac{1.5-1}{1.5} \approx 0.33$. Consequently, fair price of B should be different. Probabilities, derived from prices, are not equal in the case of asset B as a numeraire. To make today's price of B fair probabilities should be 0.25 and 0.75 for the first and second scenarios correspondingly. We can say that probabilities of scenarios, implied in prices, depend on the environment. Particularly, on what we are going to do with payoff after expiration: leave it in basic numeraire, exchange in somewhat else or maybe reinvest.

2. No arbitrage conditions

If we analyze real market prices we cannot say that they or N_T are martingales or $Q^N$is a risk-neutral measure. However, securities are still connected and there have to be no arbitrage. Implied expectations should reflect this fact.

Transform Eq. 4 for the case of European call option:

$$H_0(K) = P^N(t_0,T) \cdot \int_K^\infty q^N(S_T) \cdot (S_T - K) \cdot \frac{N_T(S_T)}{N_0} dS \tag{5}$$

where K is strike price and $S_T$is price of underlying asset at expiration T.

Then second derivative is

$$\frac{d^2}{dK^2} H_0(K) = P^N(t_0,T) \cdot q^N(K) \cdot \frac{N_T(K)}{N_0} \tag{6}$$

Now consider that there can be different numeraires and N_T. However, H_0 are market prices and do not depend on N_T; $\frac{d^2}{dK^2} H_0(K)$ have to be the same for different N_T. Consequently,

$$q^{N^i}(S_T) \cdot P^{N^i}(t_0,T) \cdot \frac{N_T^i(S_T)}{N_0^i} = q^{N^j}(S_T) \cdot P^{N^j}(t_0,T) \cdot \frac{N_T^j(S_T)}{N_0^j} \tag{7}$$

Or

$$q^{N^i}(S_T) = q^{N^j}(S_T) \cdot \frac{P^{N^j}(t_0,T)}{P^{N^i}(t_0,T)} \cdot \frac{N_T^j(S_T)}{N_0^j} \cdot \frac{N_0^i}{N_T^i(S_T)} \tag{8}$$

The same result could be obtained using payoff $H_T = \delta(x - S_T)$– Dirac delta function.

This equation connects all $q^{N^i}(S_T)$for all numeraires.

So what is $q^{N^i}(S_T)$? It depends on market prices, interest rates and numeraires (exchange functions). It has the meaning of probability density because it reflects implied expectations about future. However, in mathematical sense it doesn't have to be probability density because it was derived from real market prices.

Assume it is still a probability density. Then for every$i$:

$$\int_{-\infty}^{\infty} q^{N^i}(S_T) dS_T = 1 \tag{9}$$

From comparison of Eq. 2 and Eq. 9 follows that Eq. 9 describes the derivative, for which next is true:

(1) Payoff in numeraire i is equal to one, $H_T^{N^i} = H_T \cdot N_T^i = 1$.
(2) Undiscounted expected value of future payoff in numeraire $i$is also equal to one, $H_0^{N^i} = H_0 \cdot N_0^i = 1$.

Eq. 9 describes a security that has certain equal to one payoff in some numeraire independently from underlying asset's price movements. It is like a bond. Premium (without discounting) also have to be equal to one. In fact, this security shows that one dollar tomorrow cost one dollar tomorrow. It is true for every other asset. If it is not then arbitrage is possible. Consequently, $q^{N^i}(S_T)$ have to be probability density function for every N^i.

This security is connected to other more complex securities having same one underlying asset: futures, options and others. If price on some security is changing then implied probability densities $q^{N^i}(S_T)$ are also changing for every possible numeraire. But Eq. 9 has to remain true for every numeraire. Otherwise arbitrage is possible.

Independently from numeraire (even if N^i_T is not martingale) real prices of securities have to reflect this property. There have to no such numeraire that allows making risk-free profit. If such numeraire becomes possible then market becomes inefficient. Every new numeraire adds new Eq. 9 and limits prices of securities.

Actually Eq. 9 is no arbitrage condition. Using Eq. 8 it can be transformed into the next one:

$$\int_{-\infty}^{\infty} q^{N^i}(S_T)\frac{N^i_T(S_T)}{N^j_T(S_T)}dS_T = \frac{P^{N^j}(t_0,T)}{P^{N^i}(t_0,T)}\cdot\frac{N^i_0}{N^j_0} \tag{10}$$

For example, assume that S_T is not expected to be constant and we have three possible numeraires:

$$\begin{array}{l} N^1_T(S_T) = S_T \\ N^2_T = 1 \\ N^3_T(S_T) = \frac{1}{S_T} \end{array} \tag{11}$$

Assume that $P(t_0,T) = 1$. Using Eq. 8 and Eq. 9 we can make next transformations:

$$\int_{-\infty}^{\infty} q^{N^3}(S_T)dS_T = 1 \tag{12}$$

$$\begin{array}{l} \int_{-\infty}^{\infty} q^{N^3}(S_T)dS_T = \int_{-\infty}^{\infty} q^{N^2}(S_T)\cdot\frac{S_T}{S_0}dS_T = \\ = \int_{-\infty}^{\infty} q^{N^2}(S_T)\cdot\frac{S_0}{S_0}dS_T + \int_{-\infty}^{\infty} q^{N^2}(S_T)\cdot\frac{S_T-S_0}{S_0}dS_T = 1 \end{array} \tag{13}$$

$$\int_{-\infty}^{\infty} q^{N^2}(S_T)\cdot\frac{S_T-S_0}{S_0}dS_T = 0 \tag{14}$$

Analogously

$$\int_{-\infty}^{\infty} q^{N^1}(S_T)\cdot\frac{S_T-S_0}{S_0}dS_T = 0 \tag{15}$$

But

$$\begin{array}{l} \int_{-\infty}^{\infty} q^{N^2}(S_T)\cdot\frac{S_T-S_0}{S_0}dS_T = \int_{-\infty}^{\infty} q^{N^1}(S_T)\cdot\frac{S_T}{S_0}\cdot\frac{S_T-S_0}{S_0}dS_T = \\ = \int_{-\infty}^{\infty} q^{N^1}(S_T)\cdot\frac{S_T-S_0}{S_0}dS_T + \int_{-\infty}^{\infty} q^{N^1}(S_T)\cdot\frac{(S_T-S_0)^2}{S_0^2}dS_T = \\ = \int_{-\infty}^{\infty} q^{N^1}(S_T)\cdot\frac{(S_T-S_0)^2}{S_0^2}dS_T \neq 0 \end{array} \tag{16}$$

Consequently, in such system as in Eq. 11 arbitrage is always possible. Is it possible to observe such situation on markets? In fact, interest rates may have such properties. Futures are obligations to buy some amount of asset at expiration. In other words, we buy asset located at some moment of time. To "move" such asset farther to the future we should divide its price by $e^{\int_{t_1}^{t_2} r(t)dt}$. It is the first equation in Eq. 11. To "move" asset from the future closer to initial moment we should multiply by $e^{\int_{t_1}^{t_2} r(t)dt}$. It is the third equation in system (Eq.11). For example, we can exchange between currencies located in a 1 year, 2 years or 3 years (exchange between corresponding futures). If interest rate is independent from time then we get system (Eq.11).

3. The case

If interest rate (exchange rate between futures) is independent from futures, described above situation may arise by itself. In this section we create such situation artificially.

Suppose that there are two securities: C_1 and C_2. Both are traded on a market.C_2 is portfolio that at initial moment consists of $a(0)$ units of C_1.

At moment t manager is able to sell some amount of $C_1$and pay extra dividends. Also manager is able to use dividends from C_1 to buy some amount of C_1 and pay no or little dividends. Expectations about $a(t)$ influence prices of futures on C_2. These prices are fully managed parameters because manager is able to sell all or sell nothing.

Exchange rate between futures on C_2 with expiration time T and futures on $C_2$with another expiration time $T_1 < T$ is:

$$F^t_{C_2}(T_1,T) = \frac{F_{C_2}(T)}{F_{C_2}(T_1)} \tag{17}$$

Exchange rate is a random variable that depends on market situation with C_1 and manager's decisions. Manager can make next property true by managing $a(t)$,$T_1 < T_2 < T_3$:

$$F^t_{C_2}(T_1,T_2) = F^t_{C_2}(T_2,T_3) \tag{18}$$

Suppose there are options, priced in the way of Eq. 4, with next properties:

(1) Underlying asset is $F^t_{C_2}(T_2,T_3)$ – futures contract onC_2 with expiration at T_3 priced in futures with expiration at T_2.
(2) There are three numeraires: futures on$C_2$with expiration at T_1, T_2 or T_3.
(3) Expiration time of options is $T < T_1$.
(4) In the case of futures $P(t_0;T) = 1$ because they have no dividends, and their price increase over time to compensate absence of interest rate.

There are three scenarios for payoff as stated in Eq. 11. Then

$$\begin{aligned} N^1_T &= \tfrac{1}{F^t_{C_2}(T_2,T_1)} = F^t_{C_2}(T_1,T_2) \\ N^2_T &= 1 \\ N^3_T &= \tfrac{1}{F^t_{C_2}(T_2,T_3)} = \tfrac{1}{F^t_{C_2}(T_1,T_2)} \end{aligned} \tag{19}$$

Consequently, arbitrage is possible. It allows making risk-free profit in at least one numeraire. If price of $C_1$is positive then risk-free profit is also positive.

Arbitrage opportunity is inefficiency. People using market inefficiency drive market to efficient state, in which arbitrage is not possible. However, it seems that all we need to create such situation is developed market: market globalization, high liquidity, variety of instruments, small transaction costs and others.

4. Conclusion

It was shown that implied in prices expectations about future depend on numeraire used by certain market participant. Different participants derive different implied expectations depending on what they wish to do with payoff: leave in basic for security numeraire or transform into different numeraire. One dollar of payoff has different significance for different participants. They could use different currencies as a numeraires. Moreover, amount of payoff is connected to situation on market and in the world. This situation could be used differently by different participants and lead to different financial consequences in the future. Fair price and implied in real prices expectations depend on what participant is maximizing, his or her goal. Usually, it is not just payoff minus premium but more complex goals. Ultimately, every participant is individual and it seems market should reflect this.

Consequently, there could be plenty of implied expectations depending on numeraire. In general, picture of future, implied in prices, depend on peculiarity of participant.

These expectations are not probability densities in mathematical sense because they are being derived from real market prices. However, if expectations are not probability densities then arbitrage is possible. It is possible to combine common securities and create a portfolio that costs less than certain payoff in some numeraire (buy one dollar for less than one dollar). Consequently, this requirement is no arbitrage condition. There have to be no such numeraire that makes expectations to be not a probability density.

Interest rates allow creation of such numeraire. It seems that there is very fundamental inefficiency in interest rates. Risk-free interest rate is a property of creation process: it shows how fast amount of asset is increasing in time. However, if interest rate fluctuates then amount of asset increases faster. The more fluctuations are, the more difference is.

It is possible not only on a money market. In global, interest rates are everywhere. Price of every asset depends on time when this asset is delivered. There are also behavioral aspects (A. G. S. Ventre and V. Ventre 2012), which could be seen in a new way using presented in this paper point of view. In fact every asset could be borrowed or lent. Commonly, price depends on period of time and reflects interest rate of an asset. All is needed to create such numeraire is developed market. Examined in this paper case shows that everything that has market price also has not constant interest rate. It allows making an assumption that found inefficiency is not only connected to interest rates. It lies deeper. If it is true then almost every market is inefficient and allows making risk-free profit.

This all may lead to increase of cooperative behavior between participants on financial markets and usage of consequent theories (Carfì and Ricciardello 2009; Carfì and Schilirò 2011). Using of arbitrage opportunities by participants drive market to the efficient state. What efficient market should be in our case? Maybe conceptions and consequent models should be more complex? It makes sense if we remember that market's goal is to give buyer and seller best conditions for exchanging of goods. For example, global financial markets cannot do this perfectly because one price cannot reflect individuality. We can make an assumption that price and value should be more individual. Maybe price is fair only for buyer and seller but not for any third party? Sounds pretty obvious. It should be noted that markets on which complex goods and services are traded with individual prices exist.

While markets are inefficient it is possible to use this fact. Inefficiency is fundamental and so can be profit.

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* Acribia Holdings
Saint Petersburg, Russian Federation

Email: ivanov.sci@gmail.com

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