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NEW QUALITATIVE FEATURES FOR THE 2D DYNAMICAL SYSTEM ASSOCIATED TO A MIXING FLOW MODEL

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ABSTRACT. The aim of this paper is to approach the 2D mixing flow model from a new qualitative standpoint. The feedback linearization method is applied for a slight perturbation of this model, in order to analyze the form of the inverse system and the parameter influence.

1. Introduction: mixing flow mathematical context

The mixing flow theory appears in an area with far from completely solved problems: the flow kinematics. Its methods and techniques developed the significant relation between turbulence and chaos. The turbulence is an important feature of dynamic systems with few degrees of freedom, the so-called *far-from-equilibrium systems*. These are widespread between the models of excitable media, and a recent goal is to find a consistent and coherent theory to establish that a mixing model in excitable media leads to a far-from-equilibrium model.

After a hundred years of stability study, the problems of flow kinematics are far from being completely solved. Since the beginnings, considering the stability of laminar flows with infinitesimal turbulences was a fruitful investigation method. The non-linearity could operate in the sense of stabilizing the flow, and then the basic flow is replaced with a new stable flow, which is considered a secondary flow. This secondary flow could be further replaced by a tertiary flow, and so on. In fact, it is about a *bifurcations sequence*, and the Couette flow could be the best example in this sense. This context becomes more difficult if the non-linearity is in the sense of increasing of the growing rate of linear unstable modes. In fact, we are talking about *strong turbulence problems*, an area which still needs a lot of analysis.

Generally, the statistical idea of a flow is represented by a map:

$$\mathbf{x} = \Phi_t(\mathbf{X}),\ \mathbf{X} = \Phi_{t=0}(\mathbf{X}) \tag{1}$$

We say that $\mathbf{X}$ is *mapped in* $\mathbf{x}$ after a time t. In continuum mechanics, the relation (1) is named *flow*, and it is a diffeomorphism of class C^k. Moreover, (1) must satisfy the relation

$$J = \det(D(\Phi_t(\mathbf{X}))) = \det\left(\frac{\partial x_i}{\partial X_j}\right) \tag{2}$$

where D denotes the derivation with respect to the reference configuration, in this case $\mathbf{X}$. The relation (2) implies two particles, X_1 and X_2, which occupy the same position $\mathbf{x}$ at a moment. With respect to $\mathbf{X}$, the basic measure of deformation, the *deformation gradient* $\mathbf{F}$, is defined by

$$\mathbf{F} = (\nabla_{\mathbf{X}}\Phi_t(\mathbf{X}))^T,\ F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right) \tag{3}$$

where $\nabla_{\mathbf{X}}$ denotes differentiation with respect to $\mathbf{X}$. According to (3), $\mathbf{F}$ is non singular. The basic measure for the deformation with respect to $\mathbf{x}$ is the *velocity gradient*.

After defining the basic deformation of a material filament and the corresponding relation for the area of an infinitesimal material surface, we can define the basic deformation measures: the *length deformation* λ and *surface deformation* η, with the relations (Ottino 1989):

$$\lambda = (\mathbf{C} : \mathbf{MM})^{1/2},\ \eta = (\det\mathbf{F}) \cdot (\mathbf{C}^{-1} : \mathbf{NN})^{1/2} \tag{4}$$

in which $\mathbf{C}$ $(= \mathbf{F}^T \cdot \mathbf{F})$ is the *Cauchy-Green deformation tensor*, and the vectors $\mathbf{M}$ and $\mathbf{N}$ are respectively the orientation versors in length and surface, defined by

$$\mathbf{M} = \frac{d\mathbf{X}}{d|\mathbf{X}|},\ \mathbf{N} = \frac{d\mathbf{A}}{d|\mathbf{A}|} \tag{5}$$

In practice the scalar form of (4), namely

$$\lambda^2 = C_{ij} \cdot M_i \cdot N_j,\ \eta^2 = (\det\mathbf{F}) \cdot C_{ij}^{-1} \cdot M_i \cdot N_j \tag{6}$$

is often used.

In the above context, we say that the flow $\mathbf{x} = \Phi_t(\mathbf{X})$ has a *good mixing* if the mean values $D(ln\lambda)/Dt$ and $D(ln\eta)/Dt$ are not decreasing to zero, for any initial position $\mathbf{P}$ and any initial orientations $\mathbf{M}$ and $\mathbf{N}$.

From both the analytical and computational standpoint, the following relations are basic in the flow kinematics: the *deformation efficiency in length*, $e_\lambda = e_\lambda(\mathbf{X}, \mathbf{M}, t)$ of the material element $d\mathbf{X}$, is defined as:

$$e_\lambda = \frac{D(ln\lambda)/Dt}{(\mathbf{D} : \mathbf{D})^{1/2}} \leq 1 \tag{7}$$

where $\mathbf{D}$ is the deformation tensor (Ottino 1989), and, similarly, the *deformation efficiency in surface*, $e_\eta = e_\eta(\mathbf{X}, \mathbf{N}, t)$ of the area element $d\mathbf{A}$. In the case of an isochoric flow (the Jacobian is equal to 1), we have:

$$e_\eta = \frac{D(ln\eta)/Dt}{(\mathbf{D} : \mathbf{D})^{1/2}} \leq 1 \tag{8}$$

The deformation tensor $\mathbf{F}$ and the associated tensors $\mathbf{C}$, $\mathbf{C}^{-1}$, form the fundamental quantities for the analysis of deformation of infinitesimal elements. In most cases, the flow $\mathbf{x} = \Phi_t(\mathbf{X})$ is unknown and has to be obtained by integration from the Eulerian velocity field. If this can be done analytically, then $\mathbf{F}$ can be obtained by differentiation of the flow with respect to the material coordinates $\mathbf{X}$. The flows of interest belong to two classes: i) flows with a special form of ∇v and ii) flows with a special form of $\mathbf{F}$. The second class is of very large interest, as it contains the so-called Constant Stretch History Motion $-$ CSHM flows.

2. Recent results and methods

Studying a mixing for a flow implies the analysis of successive stretching and folding phenomena for its particles, together with the influence of parameters and initial conditions. It can concern simple mixing phenomena, or the phenomenon of a mixing of a biological material in a host fluid. In the previous works it was studied the mixing phenomenon produced when a biological material is vortexed in a host fluid. A first aim was to study the deformation efficiency in length and surface for the mixing flow model. The mathematical model used as starting point in the analysis is basically the widespread isochoric two-dimensional flow, namely (Ottino 1989):

$$\left.\begin{array}{rcl} \dot{x}_1 & = & G \cdot x_2 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 \end{array}\right\} \tag{9}$$

in which $-1 < K < 1$ and $G > 0$.

In the 3D case, the associated mathematical model was constructed according to the experiments realized for a vortex phenomenon. This was realized by simply adding the vortex velocity as third component (Ionescu 2002):

$$\left.\begin{array}{rcl} \dot{x}_1 & = & G \cdot x_2 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 \\ \dot{x}_3 & = & c \end{array}\right\} \tag{10}$$

where c is a constant.

The study of the 3D non-periodic models exhibited a quite complicated behaviour. In agreement with experiments, some significant events - the so-called *rare events* were involved. The variation of parameters had a great influence on the length and surface deformations. The experiments were realized with a special vortex installation, a well-known aquatic algae as biologic material was used, and the water as basic fluid (Ionescu 2002).

Quite complex relations for e_λ and e_η were obtained. The analysis of the mathematical model contained an analytical and a computational / simulation stage. Procedures of the Maple software for discrete time were used. The events studied were very few, about 60 both for 2D and 3D cases. They were statistically collected and interpreted by Ionescu (2002).

This stage of the analysis produced a panel of random events for the mixing flow mathematical model. Although it has a not very complicated mathematical form, it is going to turbulence for certain values of the parameters. A very important point is that when adding similar terms to the model, in the 2D case, the model turns its behaviour into a far-from-equilibrium one.

The analysis recently has been continued with more computational simulations, for the 2D model, both in the periodic and the non-periodic case, and for the 3D model too. A lot of comparisons between the periodic and the non-periodic case, 2D and 3D cases were realized. At the same time, the computational appliances were varied. If initially the model was studied from the standpoint of mixing efficiency, in the works that come after new appliances of the Maple 11 software were tested (Abell and Braselton 2005; Ionescu and Coman 2011) in order to collect more statistical data for the turbulent mixing model behaviour. Also, some interesting versions of the mixing dynamical system were analyzed, perturbing the model with a logistic-type term (Ionescu 2012). In the analysis, the same set of parameters values were taken into account, for a better accuracy of the comparative analysis. The phase-portrait analysis offered new features concerning the influence of parameters on the model behaviour.

3. Feedback linearization for the 2D mixing flow

The aim of this paper is to analyze the mixing model dynamical system from a new analytical standpoint, the feedback linearization method. This approach is based on concepts from non-linear systems theory and contains two fundamental non-linear controller design techniques: input-output linearization and state-space linearization (Isidori 1989; Henson and Seborg 1997). The resulting controller includes the inverse of the dynamic model of the process, provided that such an inverse exists. Thus, the method is applicable to a broad class of non-linear control problems.

The feedback method is applied generally to differential systems of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot u \tag{11}$$

where $\mathbf{f}, \mathbf{g} : \mathbf{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ and $u \in \mathbb{R}$.

We look for a diffeomorphism $\mathbf{T} : \mathbf{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ which defines a coordinate transformation

$$\mathbf{z} = \mathbf{T}(\mathbf{x}) \tag{12}$$

in order to find for the system (11) a state space realization of the form

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}v \tag{13}$$

The method was presented in detail by Isidori (1989). In this approach, u has the role of control, and the relation with v (the new control) is given by

$$u = \Phi(\mathbf{x}) + \omega^{-1}(\mathbf{x}) \cdot v \tag{14}$$

where Φ and ω are scalar functions obtained by the imposed relations on the vector functions $\mathbf{f}$, $\mathbf{g}$ and the transformation $\mathbf{T}$.

The transformation $\mathbf{T}$ has to be obtained in special conditions for the partial derivatives of $\mathbf{f}$, $\mathbf{g}$ (Isidori 1989). Also, $\mathbf{A}$ and $\mathbf{B}$ are in the controllable form ($n \times n$, $n \times 1$ respectively):

$$\mathbf{A}_C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B}_C = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{15}$$

Synthesizing, we can say that, given the non-linear system (11), the problem of feedback linearization consists in finding, if possible, a coordinate transformation of the form (12), and a static feedback control law of the form

$$u = \alpha(\mathbf{x}) + \beta(\mathbf{x}) \cdot v \tag{16}$$

where v is the new control and $\beta(\mathbf{x})$ is assumed to be non-zero for all $\mathbf{x}$, such that the composed dynamics of the new system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \alpha(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \beta(\mathbf{x}) \cdot v, \qquad \mathbf{x} \in \mathbb{R}^n \tag{17}$$

expressed in the new coordinates $\mathbf{z}$ is the controllable system

$$\left.\begin{array}{rcl}\dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ \dots & & \\ \dot{z}_{n-1} & = & z_n \\ \dot{z}_n & = & v\end{array}\right\} \tag{18}$$

Taking into account that we are analyzing a far-from-equilibrium model, we shall apply the feedback linearization method first for a slightly perturbed mixing flow model. Thus, let us consider the following perturbed version for the dynamical system (9), analyzed also by Ionescu and Coman (2011):

$$\left.\begin{array}{rcl}\dot{x}_1 & = & G\cdot x_2 + x_1 \\ \dot{x}_2 & = & K\cdot G\cdot x_1 - x_2\end{array}\right\} \tag{19}$$

with $-1 < K < 1$ and $G > 0$. We are in the case $n = 2$, and we search for a transformation

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} T_1(\mathbf{x}) \\ T_2(\mathbf{x}) \end{pmatrix} \tag{20}$$

in order to transform this system. According to the above statements, let us put the system (19) in the vector form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} Gx_2 \\ KGx_1 \end{pmatrix} + u\cdot \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \tag{21}$$

In this form we consider the vectors $\mathbf{f}$ and $\mathbf{g}$ defined by

$$\mathbf{f} = \begin{pmatrix} Gx_2 \\ KGx_1 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \tag{22}$$

The transformation $\mathbf{T}(\mathbf{x})$ is found, after some calculus, as follows:

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} x_1\cdot x_2 \\ KGx_1^2 + Gx_2^2 \end{pmatrix}, \quad \mathbf{x} = (x_1, x_2) \tag{23}$$

Furthermore, the functions ω and Φ are found as:

$$\omega(\mathbf{x}) = 2KGx_1^2 - 2Gx_2^2, \quad \Phi(\mathbf{x}) = \frac{2KGx_1x_2}{Kx_1^2 - x_2^2} \tag{24}$$

The inverse model (13) can be found easily, according to the method of Ionescu and Munteanu (2015), as follows:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}v = \mathbf{A}\mathbf{z} + \mathbf{B}\cdot\bar{\omega}(\mathbf{z})\left(u - \bar{\Phi}(\mathbf{z})\right), \quad \mathbf{z} = (z_1, z_2) \tag{25}$$

with $\bar{\omega}(\mathbf{z}) = \omega\left(\mathbf{T}^{-1}(\mathbf{z})\right)$, $\bar{\Phi}(\mathbf{z}) = \Phi\left(\mathbf{T}^{-1}(\mathbf{z})\right)$. Thus, taking the controllers $\mathbf{A} = \mathbf{A}_C$ and $\mathbf{B} = \mathbf{B}_C$ as

$$\mathbf{A}_C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B}_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{26}$$

the inverse system becomes, in the coordinate $\mathbf{z} = (z_1, z_2)$:

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \omega\cdot(u - \Phi)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{27}$$

After some calculus, we find that the dynamical system associated to the perturbed 2D mixing flow has the following inverse:

$$\left.\begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & 2G\cdot(Kz_1^2 - z_2^2)\cdot u - 4KG^2 z_1 z_2 \end{array}\right\} \qquad (28)$$

4. Concluding remarks

The dynamical system associated to the 2D mixing flow model admits an inverse system in a controllable form. The system (28) proves that the form (18) of the inverse model can be reached. The transformation **T** does exist if certain conditions are fulfilled for the functions **f** and **g** of the system. In fact, the form (18) is the form that is reached by the feedback linearization method, provided that **T** exists, and passing through a relation like (25). Ionescu and Munteanu (2015) found a similar form for the inverse of the Lotka-Volterra model.

The new controllable form of the dynamical system has a significant change in the parameter distribution. We can state that this form is similar with that of the second order non-linear oscillator with polynomial non-linearities. Therefore, a next aim could be the analysis of the stability and the non-linear dynamics of the inverse model from this standpoint.

The approach of the feedback linearization method brings new data in the information panel for the mixing flow dynamical system. A further aim is to test the existence of an inverse system for a perturbed version of the 2D dynamical system with a logistic-type term. A version of this type was analyzed by Ionescu and Coman (2011). The results will be used for further analytical and computational analysis of other versions of this model type, including possible generalisations and control theory approach.

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