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AN EXISTENCE AND QUALITATIVE RESULT FOR DISCONTINUOUS IMPLICIT DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $T > 0$ and $Y \subseteq \mathbf{R}^n$. Given a function $f : [0,T] \times \mathbf{R}^n \times Y \to \mathbf{R}$, we consider the Cauchy problem $f(t,u,u') = 0$ in $[0,T]$, $u(0) = \xi$. We prove an existence and qualitative result for the generalized solutions of the above problem. In particular, our result does not require the continuity of f with respect to the first two variables. As a matter of fact, a function $f(t,x,y)$ satisfying our assumptions could be discontinuous (with respect to x) even at all points $x \in \mathbf{R}^n$. We also study the dependence of the solution set $\mathscr{S}_T(\xi)$ from the initial point $\xi \in \mathbf{R}^n$. In particular, we prove that, under our assumptions, the multifunction $\mathscr{S}_T$ admits a multivalued selection Φ which is upper semicontinuous with nonempty compact acyclic values.

1. Introduction

Let $n \in \mathbf{N}$, $T > 0$, $Y \subseteq \mathbf{R}^n$, and let $f : [0,T] \times \mathbf{R}^n \times Y \to \mathbf{R}$ be a given function. Let $\xi \in \mathbf{R}^n$. As known, a generalized solution of the Cauchy problem

$$\begin{cases} f(t,u,u') = 0 & \text{in } [0,T], \\ u(0) = \xi \end{cases} \tag{1}$$

is an absolutely continuous $u : [0,T] \to \mathbf{R}^n$ such that $u(0) = \xi$,

$$u'(t) \in Y \quad \text{and} \quad f(t,u(t),u'(t)) = 0 \quad \text{for a.e.} \quad t \in [0,T].$$

It is known that, under the mere continuity of f, problem (1) may admit no solutions. For instance, Herzog and Lemmert (private communication, see Di Bella 2002) proved that for every $\lambda \in \mathbf{R}$ the scalar problem

$$\begin{cases} u' = 2\pi(1-2t)(1+u^2) + \lambda & \text{in } [0,2], \\ u(0) = 0 \end{cases} \tag{2}$$

has no solutions (of course, a generalized solution of problem (2) is also a classical solution).

The aim of this note is to prove an existence and qualitative result for the generalized solutions of problem (1), without assuming the continuity of the function $f(t,x,y)$ with respect to the variables t and x. In particular, a function $f : [0,T] \times \mathbf{R}^n \times Y \to \mathbf{R}$ satisfying our assumptions can be discontinuous (with respect to the second variable) even at all points $x \in \mathbf{R}^n$.

Up to our knowledge, there are not many contributions for the implicit problem (1) associated with a discontinuous function f. We have to mention that the existence of explicit Cauchy problem (that is, the case $f(x,u,u') = u' - g(x,u)$) associated with discontinuous functions has been widely studied (see, for instance, Filippov 1964; Matrosov 1967; Cambini and Querci 1969; Pucci 1971; Giuntini and Pianigiani 1974; Sentis 1978; Binding 1979). As regards the implicit case, some existence results can be found in the paper by Ricceri (1985).

Our result will be stated and proved in Section 3, while in Section 2 we shall fix some notations and recall some results which will be fundamental in the sequel.

2. Preliminaries

Firstly, for every $k \in \mathbf{N}$, we denote by m_k the k-dimensional Lebesgue measure in $\mathbf{R}^k$. For every $i \in \{0,,\dots,n\}$, we denote by P_i the ith projection from $\mathbf{R} \times \mathbf{R}^n$ to $\mathbf{R}$, that is we put

$$P_i(t,x) = \begin{cases} t & \text{if } i = 0, \\ x_i & \text{if } i \in \{1,\dots,n\}, \end{cases}$$

for every $(t,x) = (t,x_1,\dots,x_n) \in \mathbf{R} \times \mathbf{R}^n$. In the following, we shall denote by $\mathscr{F}$ the family of all subsets $U \subseteq \mathbf{R} \times \mathbf{R}^n$ such that there exist sets $V_0, V_1, \dots, V_n \subseteq \mathbf{R} \times \mathbf{R}^n$, with $m_1(P_i(V_i)) = 0$ for all $i = 0,1\dots,n$, such that $U = \bigcup_{i=0}^n V_i$. Of course, any set $U \in \mathscr{F}$ satisfies $m_{n+1}(U) = 0$.

Let us denote by $\mathscr{G}_n$ the family of all subsets $A \subseteq \mathbf{R}^n$ such that, for all $i = 1,\dots,n$, the supremum and the infimum of the projection of $\overline{\text{conv}}\,(A)$ on the ith axis are both positive or both negative ("$\overline{\text{conv}}$" standing for "closed convex hull").

As usual, we denote by $C([a,b];\mathbf{R}^n)$ (respectively, $AC([a,b];\mathbf{R}^n)$) the space of all continuous (respectively, absolutely continuous) functions from $[a,b]$ to $\mathbf{R}^n$. The space $C([a,b];\mathbf{R}^n)$ will be considered with the distance of uniform convergence. Following Aubin and Cellina (1984), we put

$$\mathscr{A}([0,T],\mathbf{R}^n) := \big\{u \in C([0,T],\mathbf{R}^n) :\ u' \in L^\infty([0,T],\mathbf{R}^n)\big\},$$

where the derivative is taken in the sense of distributions. The space $\mathscr{A}([0,T],\mathbf{R}^n)$ is considered with the initial topology which makes the map $u \to (u,u')$ continuous from $\mathscr{A}([0,T],\mathbf{R}^n)$ to $C([0,T],\mathbf{R}^n) \times L^\infty([0,T],\mathbf{R}^n)$, where $C([0,T],\mathbf{R}^n)$ is considered with its strong topology, and $L^\infty([0,T],\mathbf{R}^n)$ with its weak-star topology. For every $\xi \in \mathbf{R}^n$, we put

$$\mathscr{S}_T(\xi) = \{u \in AC([0,T],\mathbf{R}^n) : u \text{ is a generalized solution of problem (1)}\}.$$

Let $G : [0,T] \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be a given multifunction. As known, a generalized solution of the differential inclusion $u' \in G(t,u)$ in $[0,T]$ is an absolutely continuous $u : [0,T] \to \mathbf{R}^n$ such that $u'(t) \in G(t,u(t))$ for a.e. $t \in [0,T]$. For every $\xi \in \mathbf{R}^n$, we put

$$\mathscr{T}_T^G(\xi) := \{u \in AC([0,T],\mathbf{R}^n) : u'(t) \in G(t,u(t)) \text{ a.e. in } [0,T], u(0) = \xi\},$$

$$\mathscr{A}_T^G(\xi) := \{u(T) : u \in \mathscr{T}_T^G(\xi)\}.$$

In what follows, we shall denote by $B(x,r)$ (resp., $\overline{B}(x,r)$) the open (resp., closed) ball of center $x \in \mathbf{R}^n$ with radius $r > 0$, with respect to the Euclidean norm $\|\cdot\|_n$ of $\mathbf{R}^n$. Moreover, we shall denote by $\mathscr{L}([a,b])$ the family of all Lebesgue measurable subsets of the interval $[a,b]$.

Finally, we recall (see Willard 2004, Definition 27.4) that a topological space X is said to be locally connected at $x \in X$ if for every open neighborhood V of x there exists a connected open set $U \subseteq X$ such that $x \in U \subseteq V$. The space X is said to be locally connected if it is locally connected at each $x \in X$. We also recall that connectedness and local connectedness are not related to one another.

For the basic definitions and facts about multifunctions, we refer to Klein and Thompson (1984) and Aubin and Frankowska (1990). For the reader's convenience, we now recall some results that will be useful in the sequel.

Proposition 2.1. (Cubiotti and Yao 2015, Proposition 2.6). *Let $\psi : [a,b] \times \mathbf{R}^n \to \mathbf{R}^k$ be a given function, $E \subseteq \mathbf{R}^n$ a Lebesgue measurable set, with $m_n(E) = 0$, and let D be a countable dense subset of $\mathbf{R}^n$, with $D \cap E = \emptyset$. Assume that:*

(i) *for all $t \in [a,b]$, the function $\psi(t,\cdot)$ is bounded;*
(ii) *for all $x \in D$, the function $\psi(\cdot,x)$ is $\mathscr{L}([a,b])$-measurable.*

Let $G : [a,b] \times \mathbf{R}^n \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(t,x) \in [a,b] \times \mathbf{R}^n$,

$$G(t,x) := \bigcap_{m \in \mathbf{N}} \overline{\mathrm{conv}} \overline{\left(\bigcup_{\substack{y \in D \\ \|y-x\|_n \leq \frac{1}{m}}} \{\psi(t,y)\} \right)}.$$

Then, one has:

(a) *G has nonempty closed convex values;*
(b) *for all $x \in \mathbf{R}^n$, the multifunction $G(\cdot,x)$ is $\mathscr{L}([a,b])$-measurable;*
(c) *for all $t \in [a,b]$, the multifunction $G(t,\cdot)$ has closed graph;*
(d) *if $t \in [a,b]$, and $\psi(t,\cdot)|_{\mathbf{R}^n \setminus E}$ is continuous at $x \in \mathbf{R}^n \setminus E$, then one has*

$$G(t,x) = \{\psi(t,x)\}.$$

The following result summarizes several results proved in (Aubin and Cellina 1984, pp. 103–109).

Theorem 2.2. *Let $x^* \in \mathbf{R}^n$, and let $\Omega \subseteq \mathbf{R} \times \mathbf{R}^n$ be an open set, such that $(0,x^*) \in \Omega$. Let $G : \Omega \to 2^{\mathbf{R}^n}$ be an upper semicontinuous multifunction, with nonempty compact convex values. Assume that there exist $M > 0$, $b > 0$, $T > 0$ such that*

$$Q := [0,T] \times \overline{B}(x^*, b + MT) \subseteq \Omega \quad \text{and} \quad G(Q) \subseteq \overline{B}(0_{\mathbf{R}^n}, M).$$

Then, one has:

(i) *For every $\xi \in B(x^*,b)$, the solution set $\mathscr{T}_T^G(\xi)$ is nonempty. Moreover, the multifunction $\xi \to \mathscr{T}_T^G(\xi)$ is upper semicontinuous from $B(x^*,b)$ to $\mathscr{A}([0,T],\mathbf{R}^n)$, with nonempty compact acyclic values;*
(ii) *The multifunction $\xi \to \mathscr{A}_T^G(\xi)$ is upper semicontinuous from $B(x^*,b)$ to $\mathbf{R}^n$, with nonempty compact connected values;*

3. The result

The following is our result.

Theorem 3.1. *Let $Y \in \mathscr{G}_n$ be a compact, connected and locally connected set. Let $f : [0,T] \times \mathbf{R}^n \times Y \to \mathbf{R}$ be a given function, D_1, D_2 two dense subsets of Y. Let $S := [0,T] \times \mathbf{R}^n$. Assume that there exists $U \subseteq S$, with $U \in \mathscr{F}$, such that:*

(i) *for every $y \in D_1$, the function $f(\cdot,\cdot,y)|_{S\setminus U}$ is lower semicontinuous;*

(ii) *for every $y \in D_2$, the function $f(\cdot,\cdot,y)|_{S\setminus U}$ is upper semicontinuous;*

(iii) *for every $(t,x) \in S \setminus U$, the function $f(t,x,\cdot)$ is continuous in Y, $0 \in \operatorname{int}_{\mathbf{R}}(f(t,x,Y))$ and*
$$\operatorname{int}_Y(\{y \in Y :\ f(t,x,y) = 0\}) = \emptyset.$$

Then, for every $\xi \in \mathbf{R}^n$, the solution set $\mathscr{S}_T(\xi)$ of problem (1) is nonempty. Moreover, there exists an upper semicontinuous multifunction
$$\Phi : \mathbf{R}^n \to 2^{\mathscr{A}([0,T];\mathbf{R}^n)}$$
with nonempty compact acyclic values, such that:

(1) $\Phi(\xi) \subseteq \mathscr{S}_T(\xi)$ *for all* $\xi \in \mathbf{R}^n$*;*

(2) *the multifunction*
$$\xi \in \mathbf{R}^n \to \{u(T) : u \in \Phi(\xi)\}$$
is upper semicontinuous with nonempty connected and compact values;

(3) *the multifunction*
$$\xi \in \mathbf{R}^n \to \{u' \in L^\infty([0,T],\mathbf{R}^n) :\ u \in \Phi(\xi)\}$$
is upper semicontinuous (with compact values) from $\mathbf{R}^n$ to $L^\infty([0,T],\mathbf{R}^n)$, endowed with its weak-star topology;

(4) *for every $\xi \in \mathbf{R}^n$ and every $u \in \Phi(\xi)$, one has that $(t,u(t)) \in S \setminus U$ for a.e. $t \in [0,T]$.*

Proof. Let $M > 0$ be such that $Y \subseteq \overline{B}(0_{\mathbf{R}^n}, M)$. Let $V_0, V_1, \dots, V_n \subseteq \mathbf{R} \times \mathbf{R}^n$, with $m_1(P_i(V_i)) = 0$ for all $i = 0, 1 \dots, n$, be such that
$$U = \bigcup_{i=0}^{n} V_i.$$
For every $(t,x) \in S$, put
$$I(t,x) = \{y \in Y : f(t,x,y) = 0\},$$
$$H(t,x) = \{y \in Y : y \text{ is a local extremum for } f(t,x,\cdot)\},$$
$$\Psi(t,x) = I(t,x) \setminus H(t,x).$$
By Theorem 2.2 proved by Ricceri (1982), the multifunction $\Psi|_{S\setminus U}$ is lower semicontinuous with nonempty closed (in Y, hence in $\mathbf{R}^n$) values. By Lemma 2.4 proved by Cubiotti and Yao (2014), there exist sets $Q_0 \subseteq [0,T]$, $Q_1, \dots, Q_n \subseteq \mathbf{R}$, with
$$Q_i \in \mathscr{B}(\mathbf{R}) \quad \text{and} \quad m_1(Q_i) = 0 \quad \forall i = 0, \dots, n,$$

and a function $\psi : S \setminus U \to Y$ such that:

(a) $\psi(t,x) \in \Psi(t,x)$ for every $(t,x) \in S \setminus U$;

(b) ψ is continuous at each point

$$(t,x) \in \Big[([0,T] \setminus Q_0) \times (\mathbf{R} \setminus Q_1) \times \cdots \times (\mathbf{R} \setminus Q_n)\Big] \cap (S \setminus U) =$$

$$= S \setminus \Big[U \cup \bigcup_{i=0}^{n} P_i^{-1}(Q_i)\Big] .$$

Of course, the set

$$U^* := U \cup \bigcup_{i=0}^{n} (P_i^{-1}(Q_i) \cap S)$$

belongs to $\mathscr{F}$. Now, put

$$U_1^* := U^* \cup (\{0,T\} \times \mathbf{R}^n) .$$

Of course, $U_1^* \in \mathscr{F}$. Fix $y^* \in \psi(S \setminus U)$, and let $\psi^* : \mathbf{R} \times \mathbf{R}^n \to Y$ be defined by

$$\psi^*(t,x) = \begin{cases} \psi(t,x) & \text{if } (t,x) \in S \setminus U_1^* \\ y^* & \text{otherwise.} \end{cases}$$

We now observe that

$$\psi^*|_{(\mathbf{R}\times\mathbf{R}^n)\setminus U_1^*}$$

is continuous. To see this, fix $(t,x) \in (\mathbf{R} \times \mathbf{R}^n) \setminus U_1^*$. Firstly, assume that $(t,x) \notin S$. Since the set $W := (\mathbf{R} \times \mathbf{R}^n) \setminus S$ is open in $\mathbf{R} \times \mathbf{R}^n$, and

$$W \subseteq (\mathbf{R} \times \mathbf{R}^n) \setminus U_1^*,$$

the set W is open in $(\mathbf{R} \times \mathbf{R}^n) \setminus U_1^*$. Since ψ^* is constant over W, we get the continuity of $\psi^*|_{(\mathbf{R}\times\mathbf{R}^n)\setminus U_1^*}$ at (t,x). Conversely, assume that $(t,x) \in S$. Hence, in particular, we have

$$(t,x) \in (]0,T[\times\mathbf{R}^n) \setminus U_1^* = (]0,T[\times\mathbf{R}^n) \setminus U^*.$$

Now, observe that

$$\psi^*|_{(]0,T[\times\mathbf{R}^n)\setminus U_1^*} = \psi|_{(]0,T[\times\mathbf{R}^n)\setminus U_1^*} = \psi|_{(]0,T[\times\mathbf{R}^n)\setminus U^*}.$$

Since $\psi|_{(]0,T[\times\mathbf{R}^n)\setminus U^*}$ is continuous and the set

$$(]0,T[\times\mathbf{R}^n) \setminus U^* = (]0,T[\times\mathbf{R}^n) \setminus U_1^*$$

is open in $(\mathbf{R} \times \mathbf{R}^n) \setminus U_1^*$, we get the continuity of ψ^* at (t,x), as desired.

Let $D \subseteq \mathbf{R} \times \mathbf{R}^n$ be a countable set, dense in $\mathbf{R} \times \mathbf{R}^n$, such that $D \cap U_1^* = \emptyset$. Of course, such a set D exists since $m_{n+1}(U_1^*) = 0$. Let

$$G : \mathbf{R} \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$$

be the multifunction defined by putting, for each $(t,x) \in \mathbf{R} \times \mathbf{R}^n$,

$$G(t,x) := \bigcap_{m\in\mathbf{N}} \overline{\mathrm{conv}} \overline{\Big(\bigcup_{\substack{(\lambda,y)\in D \\ \|(\lambda,y)-(t,x)\|_{n+1} \le \frac{1}{m}}} \{\psi^*(\lambda,y)\}\Big)}.$$

By Proposition **2.1**, one has:

(a)$'$ G has nonempty closed convex values;

(b)′ the multifunction G has closed graph;
(c)′ for each $(t,x) \in (\mathbf{R}\times\mathbf{R}^n)\setminus U_1^*$, one has

$$G(t,x) = \{\psi^*(t,x)\}.$$

Moreover, $G(\mathbf{R}\times\mathbf{R}^n) \subseteq \overline{\mathrm{conv}}\,(Y)$, hence

$$G(\mathbf{R}\times\mathbf{R}^n) \in \mathscr{G}_n \tag{3}$$

and

$$G(\mathbf{R}\times\mathbf{R}^n) \subseteq \overline{B}(0,M).$$

Now, fix $\xi^* \in \mathbf{R}^n$, and choose $b > 0$. Put $\Omega := \mathbf{R}\times\mathbf{R}^n$. By Theorem **2.2**, the multifunction

$$\xi \in B(\xi^*,b) \to \mathscr{T}_T^G(\xi)$$

is upper semicontinuous from $B(\xi^*,b)$ to $\mathscr{A}([0,T],\mathbf{R}^n)$, with nonempty compact acyclic values, and the multifunction

$$\xi \in B(\xi^*,b) \to \mathscr{A}_T^G(\xi)$$

is upper semicontinuous with nonempty compact connected values. By the arbitrariness of ξ^*, it follows at once that the multifunction

$$\xi \in \mathbf{R}^n \to \mathscr{T}_T^G(\xi)$$

is upper semicontinuous from $\mathbf{R}^n$ to $\mathscr{A}([0,T],\mathbf{R}^n)$, with nonempty compact acyclic values, and the multifunction

$$\xi \in \mathbf{R}^n \to \mathscr{A}_T^G(\xi)$$

is upper semicontinuous in $\mathbf{R}^n$ with nonempty compact connected values. By the continuity of the function

$$u \in \mathscr{A}([0,T],\mathbf{R}^n) \to u' \in L^\infty([0,T],\mathbf{R}^n)$$

(where $L^\infty([0,T],\mathbf{R}^n)$ is considered with its weak-star topology), it follows that the multifunction

$$\xi \in \mathbf{R}^n \to \{u' \in L^\infty([0,T],\mathbf{R}^n) :\ u \in \mathscr{T}_T^G(\xi)\}$$

is upper semicontinuous in $\mathbf{R}^n$ with nonempty compact values.

Now, let $u \in AC([0,T],\mathbf{R}^n)$ be a generalized solution of the differential inclusion $u' \in G(t,u)$ in $[0,T]$, and let $K_0 \subseteq [0,T]$, with $m_1(K_0) = 0$, be such that

$$u'(t) \in G(t,u(t)) \quad \text{for all} \quad t \in [0,T]\setminus K_0. \tag{4}$$

Fix $i \in \{1,\dots,n\}$, and let us denote by $u_i : [0,T] \to \mathbf{R}$ the ith component of the function u. By (3) and (4), the function u_i' has constant sign in $[0,T]\setminus K_0$. Suppose

$$u_i'(t) > 0 \quad \text{for all} \quad t \in [0,T]\setminus K_0$$

(if conversely, $u_i'(t) < 0$ for all $t \in [0,T]\setminus K_0$, then the argument is analogous). Thus, the function u_i is stricly increasing in $[0,T]$. By Theorem 2 proved by Villani (1984), the function u_i^{-1} is absolutely continuous. Put

$$H_i := u_i^{-1}\Big([u_i(0),u_i(T)] \cap (Q_i \cup P_i(V_i))\Big).$$

By Theorem 18.25 proved in (Hewitt and Stromberg 1965), it follows that

$$m_1(H_i) = 0.$$

Put

$$K_1 = K_0 \cup \{0,T\} \cup P_0(V_0) \cup Q_0 \cup \bigcup_{i=1}^{n} H_i.$$

Since $V_0 \subseteq S$, by the above construction we get $K_1 \subseteq [0,T]$ and $m_1(K_1) = 0$. Now, fix $t \in [0,T] \setminus K_1$.

Since $t \notin \bigcup_{i=1}^{n} H_i$, we have that

$$u_i(t) \notin Q_i \cup P_i(V_i) \quad \text{for all} \quad i = 1, \ldots, n. \tag{5}$$

Taking into account that $t \notin P_0(V_0)$, we get $(t,u(t)) \in S \setminus U$. Since $t \notin Q_0$, by (5) we also get $(t,u(t)) \notin U^*$. Finally, since $t \in]0,T[$, we get $(t,u(t)) \in S \setminus U_1^*$. Taking into account that $t \notin K_0$, by (4) and (c)$'$ we have

$$u'(t) \in G(t,u(t)) = \{\psi^*(t,u(t))\} = \{\psi(t,u(t))\} \subseteq \Psi(t,u(t)),$$

hence

$$f(t,u(t),u'(t)) = 0.$$

Consequently, the function u is a generalized solution of the equation $f(t,u,u')$ in $[0,T]$, and $(t,u(t)) \in S \setminus U$ for a.e. $t \in [0,T]$. Therefore, in particular, we have $\mathscr{T}_T^G(\xi) \subseteq \mathscr{S}_T(\xi)$ for all $\xi \in \mathbf{R}^n$. At this point, it suffices to take $\Phi = \mathscr{T}_T^G$, and this completes the proof. □

Remark 3.2. Theorem **3.1** does not hold without the assumption $Y \in \mathscr{G}_n$. To see this, consider the case $n = 1$, $Y = [-1,1]$, $f : [0,T] \times \mathbf{R} \times Y \to \mathbf{R}$ defined by

$$f(t,x,y) = \begin{cases} y & \text{if } x \neq 0, \\ y-1 & \text{if } x = 0. \end{cases}$$

As shown by Ricceri (1985, Example 1), in this case the Cauchy problem

$$\begin{cases} f(x,u,u') = 0 & \text{in } [0,T], \\ u(0) = 0 \end{cases}$$

has no generalized solution for any $T > 0$. It is immediate to check that in this case all the assumptions of Theorem **3.1** (with the exception of $Y \in \mathscr{G}_1$) are satisfied by choosing $U = [0,T] \times \{0\}$. However, in this case we have $\mathscr{S}_T(0) = \emptyset$ for every $T > 0$.

Remark 3.3. As remarked in the Introduction, a function $f : [0,T] \times \mathbf{R}^n \times Y \to \mathbf{R}$ can satisfy the assumptions of Theorem **3.1** even if it is discontinuous (with respect to the second variable) at all points $x \in \mathbf{R}^n$. To see this, let $T > 0$, and take $n = 1$, $Y = [1,4]$, and $f : [0,T] \times \mathbf{R} \times Y \to \mathbf{R}$ defined by

$$f(t,x,y) = \begin{cases} y+1 & \text{if } t \in \mathbf{Q} \text{ or } x \in \mathbf{Q}, \\ y-2 & \text{otherwise.} \end{cases}$$

It is immediate to check that all the assumptions of Theorem **3.1** are satisfied, with $D_1 = D_2 = Y$ and

$$U = \big(([0,T] \cap \mathbf{Q}) \times \mathbf{R}\big) \cup \big([0,T] \times \mathbf{Q}\big).$$

However, for every $y \in Y$ the function $f(\cdot,\cdot,y)$ is discontinuous at all points $(t,x) \in [0,T] \times \mathbf{R}$. Moreover, for every $t \in [0,T] \setminus \mathbf{Q}$ and every $y \in Y$ the function $f(t,\cdot,y)$ is discontinuous at all points $x \in \mathbf{R}$.

As a matter of fact, the function f in the statement of Theorem **3.1** could be even defined only over the set $(S \setminus U) \times Y$, since its behaviour over the set $U \times Y$ plays no role. This is the main peculiarity of Theorem **3.1**, since the existence results in the literature (up to our knowledge) usually require the function $f(t,\cdot,y)$ to be defined either on the whole space $\mathbf{R}^n$, or on a ball, or on a closed set with empty interior ((see, for instance, Webb and Welsh 1989; Ricceri 1991; Heikkilä *et al.* 1996; Heikkilä and Lakshmikantham 1996; Carl and Heikkilä 1998; Pouso 2001; Cid 2003; Cid *et al.* 2006, and references therein).

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