DOI: 10.1478/AAPP.972A1

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. 2, A1 (2019)

ON THE RATIONAL FUNCTION FIELD OF REAL CURVES WITHOUT REAL POINTS

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(communicated by Gaetana Restuccia)

ABSTRACT. Let F be the field of rational functions of a real algebraic curve without real points. It is well-known that in F we can express -1 as a sum of two squares. We show that -1 is also a sum of four fourth powers, six sixth powers, and so on.

1. Introduction

A field F is said to be *quasi algebraically closed* if any homogeneous polynomial

$$f(X_0,\ldots,X_n) \in F[X_0,\ldots,X_n]$$

in which the number $n+1$ of indeterminates is larger than the degree, has a nontrivial zero in F^{n+1}. Well-known examples of such fields are finite fields, the field $\mathbb{C}(t)$ of complex rational functions in one variable, as well as all their algebraic extensions (see Greenberg 1969; Shatz 1972). In particular, the field $\mathbb{C}(\mathscr{C})$ of rational functions over any irreducible complex curve $\mathscr{C}$ is quasi algebraically closed.

Let now $\mathscr{C}$ be a quasi-projective curve defined over the real field $\mathbb{R}$, reduced and absolutely irreducible. Suppose that $\mathscr{C}$ has no real points or only a finite number of them. It seems plausible that the field $F = \mathbb{R}(\mathscr{C})$ of rational functions on $\mathscr{C}$ could be quasi algebraically closed. The purpose of this note is to confirm a very special case of this conjecture. We will show that any polynomial of the form

$$X_0^n + X_1^n + \cdots + X_n^n \tag{1}$$

has a nontrivial zero in F^{n+1}. If n is odd this assertion is obvious. If n is even it can be rephrased by saying that in the field F we can write -1 as a sum of two squares, four fourth powers, six sixth powers, and so on.

2. The result

Theorem 1. *Let $\mathbb{R}$ be the real field and $\mathscr{C}$ a quasi-projective curve over $\mathbb{R}$, reduced and absolutely irreducible. Let $F = \mathbb{R}(\mathscr{C})$ be the field of rational functions over $\mathscr{C}$. Assume that $\mathscr{C}$ has no real points or only a finite number of them. Then, for every positive natural number n, the equation*

$$1 + X_1^n + \cdots + X_n^n = 0$$

has a solution in F^n.

For $n = 2$ this result is due to Witt (1934). Witt's result has been reproved in different ways and extended in several directons. In particular Pfister (1967) proved that, in the rational function field of a real d-dimensional variety without real points, -1 is a sum of 2^d squares. Thus we may assume that F contains two functions u and v such that

$$u^2 + v^2 + 1 = 0.$$

This means that F contains the real algebra

$$R = \mathbb{R}[X,Y]/(X^2 + Y^2 + 1) = \mathbb{R}[x,y]$$

and therefore it suffices to prove the theorem for the field of fractions of R, that is the field of rational functions over the imaginary conic:

$$F = \mathbb{R}(x,y),\ x^2 + y^2 = -1.$$

Extending scalars from $\mathbb{R}$ to $\mathbb{C}$ we can write $(x+iy)(x-iy) = -1$ and, setting $t = x+iy$, we obtain $x - iy = -t^{-1}$. Denoting by σ the involution on $\mathbb{C}[t,t^{-1}]$ which is the complex conjugation on the constants and maps t to $\sigma(t) = -t^{-1}$, we can identify R with the subring

$$\mathbb{C}[t,t^{-1}]^{\sigma} \subset \mathbb{C}[t,t^{-1}]$$

consisting of all complex Laurent polynomials fixed by σ. In particular, every element of $\mathbb{C}[t,t^{-1}]$ of the form

$$Z(x,\varphi) = -e^{-i\varphi}t^{-1} + x + e^{i\varphi}t,\ x,\varphi \in \mathbb{R}$$

is in R. We shall prove that, for certain real values of x and $\varphi_0, \ldots, \varphi_n$,

$$\big(Z(x,\varphi_0), \ldots, Z(x,\varphi_n)\big)$$

is a solution of (1). We define

$$P^{[n]}(x,t) = \left(x + t - \frac{1}{t}\right)^n = \sum_{\nu=-n}^{n} P_\nu^{[n]}(x)t^\nu \qquad (2)$$

Expanding the n-th power we obtain

$$P_\nu^{[n]}(x) = \sum_{j\geq 0}(-1)^j \binom{n}{\nu+2j}\binom{\nu+2j}{j} x^{n-\nu-2j}. \qquad (3)$$

We want to show that, for any $n > 0$, $P_0^{[n]}(x)$ has a real root, but it is easier to prove a little more, namely that for any n and any ν all the roots of $P_\nu^{[n]}(x)$ are real.

From $P^{[n]}(x,t) = P^{[n]}(x,-1/t)$ we get $P_\nu^{[n]}(x) = (-1)^\nu P_{-\nu}^{[n]}(x)$, hence we may assume that $0 \leq \nu \leq n$. Using (3) we can check that, for a fixed $n \geq 0$, the polynomials $P_\nu^{[n]}(x)$ are determined by $P_n^{[n]}(x) = 1$, $P_{n-1}^{[n]}(x) = nx$, and, for $\nu = n-2, n-3, \ldots, 0$, by the descending recursion formula

(4) $$P_\nu^{[n]}(x) = \frac{\nu+1}{n-\nu} x P_{\nu+1}^{[n]}(x) - \frac{n+\nu+2}{n-\nu} P_{\nu+2}^{[n]}(x).$$

If, for instance, $n = 4$ the $P_\nu^{[4]}$ look as follows:

$$P_0^{[4]} = x^4 - 12x^2 + 6$$
$$P_1^{[4]} = 4x^3 - 12x$$
$$P_2^{[4]} = 6x^2 - 4$$
$$P_3^{[4]} = 4x$$
$$P_4^{[4]} = 1$$

Lemma 2. *For $0 \le \nu \le n$ the polynomials $P_\nu^{[n]}$ have the following properties:*

- *Every $P_\nu^{[n]}(x)$ is of degree $n-\nu$, has positive leading coefficient and, as a function of x, has the same parity as $n-\nu$.*
- *If $n-\nu$ is even, then $P_\nu^{[n]}(0)$ is different from zero and its sign is $(-1)^{(n-\nu)/2}$.*
- *If $n-\nu$ is odd, then $\frac{dP_\nu^{[n]}}{dx}(0)$ is different from zero and its sign is $(-1)^{(n-\nu-1)/2}$.*

Proof. For any value of n the three assertions can be proved by descending induction on ν using the recursion formula (4) or directly using (3). □

Lemma 3. *For $0 \le \nu \le n$ the polynomial $P_\nu^{[n]}(x)$ has $n-\nu$ real simple roots. Furthermore, if $\nu < n$ and $s_1 < s_2 < \cdots < s_{n-\nu}$ are the roots of $P_\nu^{[n]}(x)$, then the roots $r_1 < r_2 < \cdots < r_{n-\nu-1}$ of $P_{\nu+1}^{[n]}(x)$ are intercalated between those of $P_\nu^{[n]}(x)$:*

(5) $$s_1 < r_1 < s_2 < \cdots < s_{n-\nu-1} < r_{n-\nu-1} < s_{n-\nu}.$$

Proof. For $\nu = n-1$, $n-2$ or $n-3$ these assertions can be immediately verified. Thus we may suppose that $\nu \le n-3$ and argue by descending induction on ν. To simplify notation we set

$$h = P_\nu^{[n]},\ g = P_{\nu+1}^{[n]},\ f = P_{\nu+2}^{[n]},\ a = \frac{\nu+1}{n-\nu},\ b = \frac{n+\nu+2}{n-\nu}$$

so that (4) becomes

(6) $$h = axg - bf,\ a > 0,\ b > 0.$$

We start from the assumption that the roots of f and g are all real and simple and that the roots r_i of f are intercalated between the roots s_j of g. We prove that the situation remains the same if we replace the pair (f,g) with (g,h).

Let $s < s'$ be two consecutive roots of g and let r be the root of f in the open interval (s,s'). Since r is a simple root we have $f(s)f(s') < 0$. Equation (6) then implies that $h(s)h(s') < 0$, hence h has a zero between s and s'. This shows that between the smallest and the largest root of g there exist at least as many roots of h as the degree of g diminished by 1. To complete the proof it now suffices to check that $h(x)$ vanishes for some x smaller than all the roots of g and for some x larger than all these roots. From this it will follow that the number of real roots of h is equal to the degree of h, hence all its roots are simple. Obviously the roots of g will be intercalated between the roots of h.

Let us first assume that the degree of f is even. In this case, by Lemma 2, $g(x)$ is an odd function of x with a simple zero at $x=0$: $g(0)=0$ and $g'(0)\neq 0$. The degree of h is even, hence, again by Lemma 2, the sign of $h(0)$ is the opposite of the sign of $g'(0)$. Suppose for instance that $g'(0)>0$, so that $h(0)<0$. For x sufficiently large $g(x)$ is positive, hence the number of strictly positive roots of g is even. The sign of h changes every time we pass from a root of g to the next, hence if s is the largest of these roots, $h(s)$ is negative. Since $h(x)$ is positive for x sufficiently large, $h(x)$ must vanish for some x larger than s. By Lemma 2 $h(-x)=h(x)$, and so $h(x)$ also vanishes for some x smaller than any root of g.

If the degree of f is even but $g'(0)<0$ then Lemma 2 implies that $h(0)>0$ and that the number of strictly positive roots of g is odd. If s is the largest root of g we again find $h(s)<0$, whence the existence of a root of h larger than s. By symmetry, h also has a root smaller than any root of g. The case in which f has odd degree is entirely similar. □

We now prove Theorem 1. We choose $n+1$ elements $Z_k\in R$ of the form

$$Z_k = x + te^{2\pi ik/(n+1)} - t^{-1}e^{-2\pi ik/(n+1)},\ x\in\mathbb{R},\ k=0,1,\dots n.$$

It follows from

$$Z_k^n = P^{[n]}(x, te^{2\pi ik/(n+1)}) = \sum_{\nu=-n}^{n} P_\nu^{[n]}(x)t^\nu e^{2\pi ik\nu/(n+1)}$$

that

$$\sum_{k=0}^{n} Z_k^n = \sum_{\nu=-n}^{n} P_\nu^{[n]}(x)t^\nu \sum_{k=0}^{n} e^{2\pi ik\nu/(n+1)}.$$

For $0<\nu\leq n$ the sum

$$\sum_{k=0}^{n} e^{2\pi ik\nu/(n+1)}$$

vanishes, hence

$$Z_0^n + Z_1^n + \cdots + Z_n^n = (n+1)P_0^{[n]}(x).$$

It suffices now to replace x with a root of $P_0^{[n]}$. □

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Manuscript received 20 February 2019; communicated 6 June 2019; published online 6 July 2019