A REMARK ON PROPER SEQUENCES OF MODULES*

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ABSTRACT. A bound for the depth of a quotient of the symmetric algebra, $S(E)$, of a finitely generated module $E$, over a C.M. ring by an ideal of $S(E)$ generated by a subsequence of $x_1, \ldots, x_n$ is obtained in the case when $E$ satisfies the sliding depth condition, with maximal irrelevant ideal generated by a proper sequence $x_1, \ldots, x_n$ in $E$.

Introduction

Let $R$ be a commutative noetherian C.M. local ring of dimension $d$, and let $E$ be a finitely generated $R$-module of rank $e$.

We denote with $\text{Sym}_R(E)$ or with $S(E)$, the symmetric algebra of $E$ over $R$, that is the graded algebra over $R$

$$S(E) = \bigoplus_{t \geq 0} \text{Sym}_t(E),$$

with $S_+ = \mathbf{x} = (x_1, \ldots, x_n)$ the graded maximal irrelevant ideal of $S(E)$ and with $S = R[T_1, \ldots, T_n]$ the polynomial ring in $n$ variables.

When $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a proper sequence in $E$, $\mathbf{x}_i = \{x_1, \ldots, x_i\}$ is a subsequence of $\mathbf{x}$, $i = 1, \ldots, n$, the complex of homology modules of the Koszul complex on the elements $\mathbf{x}_i$ is exact and gives us informations on the quotient ring $S(E)/(x_{i+1}, \ldots, x_n)$.

It is possible to obtain bounds on the depth of $S(E)/(x_{i+1}, \ldots, x_n)$ knowing bounds on the depth of the $d$th components of the homology modules of the Koszul complex. For example, the condition $SD_k$ on modules is able to give us such conditions, introduced in [5].

In this article we obtain a bound for the depth of the quotient ring $S(E)$ by ideals generated by proper sequences of 1-forms of the maximal irrelevant ideal of $S(E)$, under weaker hypothesis given in theorem 2 of [9], where we used approximation complex $Z(E)$ of the module $E$.

We obtain the following:
Theorem 1. Let \((R, m)\) be a C.M. local ring of dimension \(d\). Let \(E\) a f.g. \(R\)-module that satisfies \(SD_k\) and \(x_1, \ldots, x_n\) a proper sequence of \(E\).

Then \(\text{depth} S(E)/(x_{n-i+1}, n) \geq d - i + k, i = 0, \ldots, n\).

Our result generalizes to the case of a module the result in [6], where the problem is studied in the case of an ideal \(I \subset R\) generated by proper sequences, obtaining bounds on the depth of the quotient of the ring \(R\) by \(I\).

1. Preliminaries

Let \(R\) be a commutative noetherian C.M. local ring of dimension \(d\), and let \(E\) be a finitely generated \(R\)-module of rank \(e\).

We denote with \(Sym_R(E)\) or with \(S(E)\), the symmetric algebra of \(E\) over \(R\), that is the graded algebra over \(R\):

\[ S(E) = \bigoplus_{t \geq 0} Sym_t(E) \]

and with \(S_+\) the maximal irrelevant ideal of \(S(E)\)

\[ S_+ = \bigoplus_{t > 0} Sym_t(E). \]

Let \(S_+ = (x_1, \ldots, x_n) = (x)\). We can consider the Koszul complex on the generating set \(\{x_1, \ldots, x_n\}\) of \(S_+\), that is a graded complex.

In particular in degree \(t > 0\) we have

\[ 0 \to \bigwedge^n R^n \otimes S_t-n(E) \xrightarrow{d_0} \bigwedge^{n-1} R^n \otimes S_{t-n+1}(E) \xrightarrow{d_1} \cdots \]

\[ \cdots \bigwedge^2 R^n \otimes S_{t-2}(E) \xrightarrow{d_2} R^n \otimes S_{t-1}(E) \xrightarrow{d_3} S_{t-1}(E) \to 0 \]

with differential \(d_t\)

\[ d_t(e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes f(x)) = \sum_{j=1}^{p} (-1)^{p-j} e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_p} \otimes x_{i_j} f(x) . \]

We denote with \(H_i(x; S(E))\), \(j \geq i\), the \(j\)-th graded component of the Koszul homology \(H_i(x; S(E))\).

It results:

\[ H_i(x; S(E)) = \bigoplus_{j \geq i} H_i(x; S(E))_j \]

Now, is possible to define the complex:

\[ 0 \to H_n(x; S(E))_n \otimes S[-n] \to \cdots \to H_1(x; S(E))_1 \otimes S[-1] \to S \]

and in general the complexes associated to the subsequences \(x_i = \{x_1, \ldots, x_i\}\)

\[ 0 \to H_i(x_i; S(E))_i \otimes S[-i] \to \cdots \to H_1(x_i; S(E))_1 \otimes S[-1] \to S \]
Definition 1. [1] Let \( R \) be a graded ring. A graded ideal \( \mathfrak{m} \) of \( R \) is called \( \ast \)maximal, if every graded ideal that contains \( \mathfrak{m} \) properly, equals \( R \). The ring \( R \) is called \( \ast \)local, if it has a unique \( \ast \)maximal ideal \( \mathfrak{m} \). A \( \ast \)local ring \( R \) with \( \ast \)maximal ideal \( \mathfrak{m} \) will be denoted by \( (R, \mathfrak{m}) \).

Example 1. Let \( R = \oplus_{i \geq 0} R_i \) be a graded ring for which \( R_0 \) is a local ring with maximal ideal \( \mathfrak{m}_0 \), and call \( R_\mathfrak{m} \) the ideal \( \oplus_{i \geq 0} R_i \). Then \( R \) is a \( \ast \)local ring with \( \ast \)maximal ideal \( \mathfrak{m} = \mathfrak{m}_0 \oplus R_\mathfrak{m} \).

Remark 1. Let \( M, N \) graded \( R \)-modules. We can denote by \( \text{Hom}_i(M, N) \) the module of homogeneous homomorphisms of degree \( i \).

We can define \( \ast \text{Hom}(M, N) = \bigoplus \text{Hom}_i(M, N) \) as the submodule of homogeneous homomorphisms of \( \text{Hom}(M, N) \).

It is known that \( \ast \text{Hom}(M, N) = \text{Hom}(M, N) \) when \( M \) is finitely generated ([1], chapter 1.5).

The same remark is possible to give for the \( i \)-th right derived functor \( \ast \text{Ext}^i \) of \( \ast \text{Hom}(\_\_ N) \), that is \( \ast \text{Ext}^i(M, N) = \text{Ext}^i(M, N) \), when \( M \) is finitely generated ([1], chapter 1.5).

Definition 2. Let \( M \) be a graded module on a \( \ast \)local ring \( (R, \mathfrak{m}) \). A sequence \( x = x_1, \ldots, x_n \) of graded elements in \( R \) is an \( \ast \)regular sequence if the following conditions are satisfied:

1) \( x_i \) is an \( M/(x_1, \ldots, x_{i-1})M \)-regular element for \( i = 1, \ldots, n \);
2) \((x)M \neq M\).

Now is possible to define the depth for a \( \ast \)local ring

Definition 3. Let \( M \) be a graded module on a \( \ast \)local ring \( (R, \mathfrak{m}) \). The depth of \( M \) w.r.t. \( \mathfrak{m} \), denoted with

\[ \text{depth}_\mathfrak{m} M \]

is the length of the maximal \( \ast \)regular sequence of graded elements contained in \( \mathfrak{m} \), or equivalently

\[ \min\{i : \ast \text{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\} \].

Lemma 1. Let \((R, \mathfrak{m})\) be a \( \ast \)local ring and

\[ 0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0 \]

an exact sequence of graded and finitely generated \( R \)-modules. Then

1) \( \text{depth}_\mathfrak{m} M \geq \min\{\text{depth}_\mathfrak{m} U, \text{depth}_\mathfrak{m} N\} \);
2) \( \text{depth}_\mathfrak{m} U \geq \min\{\text{depth}_\mathfrak{m} M, \text{depth}_\mathfrak{m} N + 1\} \);
3) \( \text{depth}_\mathfrak{m} N \geq \min\{\text{depth}_\mathfrak{m} U - 1, \text{depth}_\mathfrak{m} M\} \).

Proof: The assertion is proved using proposition 1.2.9 of [1] and remark 1.

Definition 4. [10] Let \( x = x_1, \ldots, x_n \) be a sequence of 1-form generating the maximal irrelevant ideal \( S_+ \) of \( S(E) \). Then \( x \) is called a proper sequence in \( E \) if:

\[ x_{i+1} Z_j(x_1, \ldots, x_i; S(E))j/B_j(x_1, \ldots, x_i; S(E))j+1 = 0 \quad 0 \leq i \leq n-1, j > 0. \]
where $Z_j(x_1, \ldots, x_j; S(E))$ and $B_j(x_1, \ldots, x_j; S(E))$ are respectively the $j$-th graded component of the cycles and boundaries of the Koszul complex.

Now we recall the sliding depth condition $SD_k$ over finitely generated modules (see [5]).

**Definition 5.** Let $(R, m_0)$ be a C.M. local ring of dimension $d$. Let $E$ be a finitely generated $R$-module, $S_k = (x_1, \ldots, x_n) = (x)$ the maximal irrelevant ideal of the symmetric algebra $S_R(E)$.

We say that $E$ satisfies the sliding depth condition $SD_k$, with $k$ integer, if $\forall i \geq 0$:

$$\text{depth}_{m_0} H_i (x, S(E))_i \geq d - n + i + k, \quad 0 \leq i \leq n - k.$$ 

If $k = \text{rank}(E)$ we shall say that $E$ satisfies the sliding depth condition $SD$.

**Remark 2.** In particular if $E$ satisfies the sliding depth condition $SD_k$, with $S_k = (x_1, \ldots, x_n) = (x)$ the maximal irrelevant ideal of the symmetric algebra $S_R(E)$, then for every subsequence $x_j = \{x_1, \ldots, x_j\}$ of $x$ we have

$$\text{depth}_{m_0} H_i(x_j, S(E))_i \geq d - j + i + k, \quad 0 \leq i \leq n - k.$$

2. Main result

Let $(R, m_0)$ be a local ring and $E$ a f.g. $R$-module. In this section we look at the symmetric algebra $S_R(E)$ of a module $E$, and a proper sequence in $E$, $x = \{x_1, \ldots, x_n\}$, generating the maximal irrelevant ideal $S_k$, and we call $S = R[T_1, \ldots, T_n]$ the polynomial ring with coefficients in $R$ with the natural grading.

**Lemma 2.** Let $(R, m_0)$ be a C.M. local ring of dimension $d$. Let $x = \{x_1, \ldots, x_n\}$ a proper sequence of the module $E$.

We call $x_i = \{x_1, \ldots, x_i\}$ and $x_{i,n} = \{x_i, x_{i+1}, \ldots, x_n\}$ subsequence of $x$, $H_j(x_i) = H_j(x_i; S(E))$. The following sequences are exact:

1) $0 \rightarrow H_{j+1}(x_{i+1}) \cap S[-j - 1] \rightarrow H_{j+1}(x_{i+1}) \cap S[-j - 1] \rightarrow H_{j+1}(x_{i+1}) \cap S[-j - 1] \rightarrow 0$

2) $0 \rightarrow Q^{(1)} \rightarrow S(E)/(x_{i+1,n}) \rightarrow S(E)/(x_{i,n}) \rightarrow 0$, with $Q^{(1)} = (x_{i,n})/(x_{i+1,n})$

3) $0 \rightarrow M^{(1)} \rightarrow S(E)/(x_{i+1,n}) \rightarrow S(E)/(x_{i,n}) \rightarrow 0$, with $M^{(1)} = ((x_{i+1,n} : x_i)/(x_{i+1,n})$

4) $0 \rightarrow H_1(x_{i+1}) \cap S[-1] \rightarrow H_1(x_{i+1}) \cap S[-1] \rightarrow M^{(1)} \rightarrow 0$, where $M^{(1)}$ is seen as an $S$-module.

**Proof:** Let $x = \{x_1, \ldots, x_n\}$ a proper sequence of $E$. Then $\forall j > 1$

$0 \rightarrow H_j(x_{i+1}) \cap S[-j] \rightarrow H_j(x_{i+1}) \cap S[-j] \rightarrow H_{j-1}(x_j) \cap S[-j] \rightarrow 0$

the sequences of $S$-modules are exact.

The tail of this homology sequence is

$0 \rightarrow H_1(x_{i+1}) \cap S[-1] \rightarrow H_1(x_{i+1}) \cap S[-1] \rightarrow S(E)/(x_{i+1,n}) \rightarrow S(E)/(x_{i+1,n}) \rightarrow 0$. 


We can observe that kernel of the homomorphism
\[ S(E)/(x_{i+1,n}) \rightarrow S(E)/(x_{i,n}) \rightarrow 0 \]
is \( Q^{(i)} = (x_{i,n})/(x_{i+1,n}) \).

Now, considering the morphism
\[ S(E)/(x_{i+1,n}) \xrightarrow{\varphi} Q^{(i)} \rightarrow 0, \]
we obtain the kernel that is \( M^{(i)} = ((x_{i+1,n}) : x_i)/(x_{i+1,n}) \).

And so we can break the long sequence (*) into the shorter sequence 2), 3) and 4).

Let \( T = \{ T_1, \ldots, T_n \} \), from now on, the depth of each \( S \)-modules, and therefore also \( S(E) \)-modules, is calculated with respect to the \( \mathfrak{m} \) maximal ideal
\[ m = m_0 \oplus (T). \]

**Remark 3.** Let \( m = m_0 \oplus (T) \) and \( m' = m_0 \oplus S_+ \). Then
\[ \text{depth}_m S(E) = \text{depth}_{m'} S(E), \]
where depth w.r.t. \( m \) is calculated considering \( S(E) \) as an \( S \)-module, while depth w.r.t. \( m' \) is calculated considering \( S(E) \) as an \( S(E) \)-module.

**Remark 4.** Let \( (R, m_0) \) be local ring and \( E \) a f.g. \( R \)-module.

If \( x = \{ x_1, \ldots, x_n \} \) is a proper sequence of \( E \), \( x_i = \{ x_1, \ldots, x_i \} \) a subsequence of \( x \), then the following complexes of \( S \)-modules are exact:

- \[ 0 \rightarrow H_1(x_i, S(E))_1 \otimes S[-1] \rightarrow \cdots \rightarrow H_n(x, S(E))_n \otimes S[-n] \rightarrow S \rightarrow S(E); \]
- \[ 0 \rightarrow H_i(x_i, S(E))_1 \otimes S[-i] \rightarrow \cdots \rightarrow H_1(x_i, S(E))_1 \otimes S[-1] \rightarrow S \rightarrow S(E)/(x_{i+1}, \ldots, x_n). \]

**Theorem 1.** Let \( (R, m_0) \) be a C.M. local ring of dimension \( d \) and \( E \) a f.g. \( R \)-module. Suppose that:

1) \( E \) satisfies \( SD_k \);
2) \( x_1, \ldots, x_n \) is a proper sequence of \( E \).

Then \( \text{depth}_m S(E)/(x_{n-i+1,n}) \geq d - i + k \), \( i = 0, \ldots, n \).

**Proof:**
We suppose that the assertion is true for \( j = n - i + 1 \) and by contradiction let
\[ \text{depth}_m S(E)/(x_{j-1,n}) = l < d - i + k - 1 \quad (*) \]
Consider the exact sequence
\[ 0 \rightarrow H_1(x_{j-1}; S(E))_1 \otimes S[-1] \rightarrow H_1(x_j; S(E))_1 \otimes S[-1] \rightarrow M^{(j-1)} \rightarrow 0 \]

By lemma 1, we have
\[ \text{depth}_m M^{(j-1)} \geq \min \{ \text{depth}_m H_1(x_{j-1}; S(E))_1 \otimes S[-1] - 1, \text{depth}_m H_1(x_j; S(E))_1 \otimes S[-1] \} \]
\[ = d + k - j + 1 + n > l \]
From the sequence of \( S(E) \)-modules (and therefore \( S \)-modules)

\[
0 \to M^{(j-1)} \to S(E)/(x_{j,n}) \to Q^{(j-1)} \to 0
\]

we obtain the long exact sequence:

\[
\cdots \to \text{Ext}^{j-1}(k, M_{j-1}) \to \text{Ext}^{j}(k, S(E)/(x_{j,n})) \to \text{Ext}^{j+1}(k, M^{(j-1)}) \to \cdots
\]

where \( k \cong R/m_0 \).

But \( \text{Ext}^{j-1}(k, M^{(j-1)}) = 0 \), since \( \text{depth} M^{(j-1)} > l \), it follows that the map

\[
\alpha : \text{Ext}^{j}(k, S(E)/(x_{j,n})) \to \text{Ext}^{j}(k, Q^{(j-1)})
\]

is injective.

In the same way, from the exact sequence of \( S(E) \)-modules

\[
0 \to Q^{(j-1)} \to S(E)/(x_{j,n}) \to S(E)/(x_{j-1,n}) \to 0
\]

we obtain

\[
\cdots \to \text{Ext}^{j-1}(k, S(E)/(x_{j-1,n})) \to \text{Ext}^{j}(k, Q^{(j-1)}) \to \text{Ext}^{j+1}(k, S(E)/(x_{j-1,n})) \to \cdots
\]

But \( \text{Ext}^{j-1}(k, S(E)/(x_{j-1,n})) = 0 \) by hypothesis (*), so the map

\[
\beta : \text{Ext}^{j}(k, Q^{(j-1)}) \to \text{Ext}^{j}(k, S(E)/(x_{j,n}))
\]

is injective, too. Therefore the composite \( \beta \alpha \) is injective.

But this gives us a contradiction, since \( \beta \alpha \) is induced by the multiplication by \( x_{j-1} \), and it is the null mapping, since \( \{ x_1, \ldots, x_n \} \) is a proper sequence.

**Example 2.** Let \( R = k[x_1, \ldots, x_d] \), \( I_1 = (f_1) \) and \( I_2 = (f_2, f_3) \) ideals of \( R \) with \( f_1, f_2, f_3 \) monomials and \( E = I_1 \oplus I_2 \).

Since \( E \) has rank 2

\[
H_1(x, S(E)) = 0
\]

for \( i > 1 \) (see [5]).

If \( i = 1 \), \( H_1(x, S(E)) \) is the first syzygy module of \( E \) and by easy calculations we have

\[
H_1(x, S(E)) = Rf,
\]

\( f = (0, f_2/GCD[f_2, f_3], f_2/GCD[f_2, f_3]) \in R^3 \), and in particular \( Rf \cong R \). Therefore \( E \) satisfies \( SD_2 \).

The sequence \( f_1, f_2, f_3 \) is a strong \( s \)-sequence in the sense of [4], so \( y_1, y_2, y_3 \in S(E) \cong R[y_1, y_2, y_3] \cong R[y_1, y_2, y_3]/J \) is a \( d \)-sequence (see [4]) and this implies that the sequence is a proper sequence, too.

We have that

1. \( \text{depth}_m S(E) \geq d + 2 \);
2. \( \text{depth}_m S(E)/(y_3) \geq d - 1 + 2 \);
3. \( \text{depth}_m S(E)/(y_2, y_3) \geq d - 2 + 2 \);
4. \( \text{depth}_m S(E)/(y_1, y_2, y_3) \geq d - 3 + 2 \).

then the assertion of theorem 1 is satisfied.
Remark 5. The computation of the depth was performed using CoCoA (see [2]) a computer algebra system entirely devoted to computing in polynomial rings. Other examples have been verified.

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