ACTIONS OF FORMAL GROUPS 
ON SPECIAL QUOTIENTS OF ALGEBRAS 

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ABSTRACT. Let $k$ be a field of characteristic $p > 0$ and let $F$ be a one dimensional commutative formal group over $k$. The endomorphisms of a $k$-algebra $A$ that defines an action of $F$ on $A$ when $A$ is isomorphic to the quotient $B/pB$, with $B$ torsion free $Z$-algebra, are studied. 

1. Introduction 

If $F = F(X,Y)$ is a one dimensional commutative formal group over a field $k$ and $A$ is a $k$-algebra, an action $D$ of $F$ on $A$ is a sequence of additive endomorphisms $\{D_i\}_{i \in \mathbb{N}}$ of $A$ such that $D_0 = id_A$ and $\sum_{i,j} D_i \circ D_j(a)X^iY^j = \sum_t D_t(a)F(X,Y)^t$, for every $a \in A$. 

$D_1$ is always a derivation while the $D_i$’s, for $i > 1$, are only additive endomorphisms such that 

$$D_n(ab) = \sum_{i+j=n} D_i(a)D_j(a), \text{ for every } n.$$ 

If $F = F_a = X + Y$, an action of $F$ on a $k$-algebra is a strongly integrable differentiation in the sense of H.Matsumura [1, 2]. In particular if $\text{char}(k) = 0$, every endomorphism $D_i$ can be expressed in terms of $D_1$, for every $i$. In fact it is $D_i = \frac{1}{i!} D_1$, for every $i$. 

If $\text{char}(k) = p > 0$ and $F = F_a$, if one considers the endomorphisms $D_1, D_p, \ldots, D_{p^i}, \ldots$, there is the nice formula [3]: 

$$D_n = \frac{D_1^{n_0}D_p^{n_1} \cdots D_{p^r}^{n_r}}{n_0!n_1! \cdots n_r!}, \text{ for every } n > 0,$$ 

while if $F = F_m = X + Y + XY$, it is 

$$D_n = \frac{(D_1)_{n_0}(D_p)_{n_1} \cdots (D_{p^r})_{n_r}}{n_0!n_1! \cdots n_r!}, \text{ for every } n > 0$$ 

where $(D_{p^i})_m = D_{p^i}(D_{p^i} - 1) \cdots (D_{p^i} - m + 1) = \prod_{t=0}^{m-1} (D_{p^i} - t)$ and $n = n_0 + n_1 p + \cdots + n_r p^r$, $0 \leq n_i < p$, is the $p$-adic expansion of $n$ [4]. 

The problem is the following:
Let \( \text{char}(k) = p > 0 \) and let \( F \) be a one dimensional commutative formal group over \( k \). Is it possible to express every \( D_n \) in terms of \( D_1, D_p, \ldots, D_{p^i}, \ldots \), as when \( F = F_a \) and \( F = F_m \)?

We give a positive answer when \( F \) acts on a \( k \)-algebra \( A \) which is a quotient of a torsion free \( \mathbb{Z} \)-algebra \( B \), modulo the ideal generated by the prime number \( p \), \( p \in B \), that is \( A \cong B/pB \).

Precisely we prove the following result (Theorem 3.1):

Let \( k \) be a separably closed field of characteristic \( p > 0 \), \( F \) a one dimensional commutative formal group over \( k \) of height greater than 2, and let \( D : A \rightarrow A[[X]] \) be an action of \( F \) on a \( k \)-algebra \( A \) that is isomorphic to \( B/pB \), where \( B \) is a \( \mathbb{Z} \)-algebra and \( B \hookrightarrow B \otimes \mathbb{Z} \mathbb{Q} \), with \( \mathbb{Q} \) the rational number field.

Then we have

\[
D_i \circ D_{p^j} = \left( i + p^j \right) D_{i+p^j},
\]

for \( j = 0, 1 \) and for every \( i \in \mathbb{N} \).

From this expression, it will be possible to express every \( D_n \) in terms of \( D_1, D_p, \ldots, D_{p^i}, \ldots \) (Corollary 3.3). Our result uses in a crucial way the structure theorem of one dimensional formal group over a separably closed field ([5], Chap. III, §2, Theorem 2).

2. Preliminaries

All rings are assumed to be commutative with a unit element. A local ring is assumed to be noetherian.

We recall that a one dimensional commutative formal group \( F \) over a ring \( k \) is a formal series \( F(X,Y) \in k[[X,Y]] \) such that

\[
i) \quad F(X,0) = X, \quad F(0,Y) = Y, \\
ii) \quad F(F(X,Y),Z) = F(X,F(Y,Z)), \\
iii) \quad F(X,Y) = F(Y,X)
\]

For simplicity a one dimensional commutative formal group over a ring \( k \) will be called a formal group over \( k \).

The most known formal groups are the additive formal group \( F_a = X + Y \) and the multiplicative formal group \( F_m = X + Y + XY \).

An action of the formal group \( F \) on a \( k \)-algebra \( A \) is a morphism of \( k \)-algebras \( D : A \rightarrow A[[X]] \) such that if \( D(a) = \sum_i D_i(a)X^i, a \in A \), then

\[
D_0 = \text{id}_A \quad \text{and} \quad \sum_{i,j} D_i \circ D_{j}(a)X^iY^j = \sum_{t} D_t(a)F(X,Y)^t,
\]

or every \( a \in A \).

If \( F \) and \( G \) are formal groups over \( k \), then a homomorphism \( f : F \rightarrow G \) is a power series \( f(X) \in k[[X]] \) such that \( f(0) = 0 \) and \( f(F(X,Y)) = G(f(X),f(Y)) \).

A homomorphism \( f \) is said to be an isomorphism if \( f'(0) \) is an invertible element in \( k \) \( (f'(X) = \partial f/\partial X) \).
Moreover for any formal group $F$ there exists a unique formal power series $i(X) \in k[[X]]$ such that $i(0) = 0$ and $F(X, i(X)) = 0 = F(i(X), X)$.

Now, for later use, let us recall the notion of height of a formal group. Let $F = F(X, Y)$ be a formal group over a ring $k$. As $F(X, Y) = F(Y, X)$, the induction formula: $[1]_F(X) = X$, $[m]_F(X) = F([m - 1]_F(X), X)$, $m \in \mathbb{N}$, determine a sequence of endomorphisms of the group $F$. If $pR = 0$, then ([5], Chap. III, §3, Theorem 2) each homomorphism $f : G \to G'$ of formal groups over $k$ can be uniquely written in the form $f(X) = f_1(X^{p^h})$, where $f_1(X) \in k[[X]]$, $f'_1(0) \neq 0$, and $h \in N \cup \infty (h = \infty$, if $f = 0)$. The number $h$ is called the height of $f$.

Now the height of a formal group $F$ over a field $k$ of characteristic $p > 0$ is defined to be the height of the endomorphism $[p]_F(X)$. We denote it by $Ht(F)$.

It is easy to see that $Ht(F) \geq 1$ for any $F$ and that $Ht(F_0) = \infty$, $Ht(F_m) = 1$. Moreover if $Ht(F) = Ht(F')$ then $F \simeq F'$.

Let $k$ be a separably closed field of characteristic $p > 0$, we want to study the action of a formal group $F$ over $k$ on a restricted class $A$ of $k$-algebras. More precisely we will study $k$-algebras such that $A \cong B/pB$, where $B$ is a $Z$-algebra and $B \hookrightarrow B \otimes Q$, $Z$ is the ring of integers and $Q$ is the field of rationals.

We recall the following:

**Theorem 2.1.** ([6], Lemme 19.7.1.3) Let $(A, m, K)$ be a local $Z$-algebra with $\text{char}(K) = p > 0$ and let $B_0$ be a $K$-algebra which is a regular complete local ring. Then there exists a topological local $A$-algebra $B$ with respect to the topology given by the maximal ideal and such that

i) $B$ is a complete ring which is a flat $A$-module

ii) $B_0$ is $K$-isomorphic to $B \otimes A K = B/mB$.

The previous theorem is due to Grothendieck. It is interesting to look at it because thanks to it we can have example of the algebras $A$ considered during the paper.

**Example 2.2.** Let $Z_p = (Z, pZ)^\wedge$ be the local complete ring of the $p$-adic integers, with $p \in Z$, $p$ a prime number. $Z_p$ is a topological ring with respect to the $pZ_p$-adic topology.

Set $B = Z_p[[X]]$, $B$ is a complete local ring whose maximal ideal is $(pZ_p, X)B$ and $B$ is a topological ring with respect to the $(pZ_p, X)B$-adic topology. Since the standard injection $Z_p \hookrightarrow Z_p[[X]]$ is a local continuous ring homomorphism, it follows that $B$ is a topological $Z_p$-algebra which is a flat $Z_p$-module.

Moreover put $B_0 = F_p[[X]]$, where $F_p = Z_p/pZ_p$ is the prime field, which is a perfect field, it is

$B_0 = Z_p/pZ_p[[X]] \cong Z_p/pZ_p \otimes_{Z_p} Z_p[[X]].$

Since $B = Z_p[[X]]$ is a flat $Z_p$-module, from the exactness of the sequence of the $Z_p$-modules:

$0 \to pZ_p \xrightarrow{j} Z_p \xrightarrow{\pi} Z_p/pZ_p \to 0,$

where $j$ is the injection and $\pi$ is the standard epimorphism, the exactness of the following sequence of $Z_p$-modules follows:

$0 \to pZ_p \otimes_{Z_p} Z_p[[X]] \xrightarrow{j \otimes 1_{Z_p[[X]]}} Z_p \otimes_{Z_p} Z_p[[X]] \xrightarrow{\pi \otimes 1_{Z_p[[X]]}} Z_p/pZ_p \otimes_{Z_p} Z_p[[X]] \to 0,$
$(\otimes = \otimes_{p})$. Hence:

$$Z_{p}/pZ_{p} \otimes_{p} Z_{p}[X] \simeq \frac{Z_{p} \otimes_{p} Z_{p}[X]}{pZ_{p} \otimes_{p} Z_{p}[X]} \simeq Z_{p}[X]/pZ_{p}[X]$$

and so $B_{0} \simeq Z_{p}[X]/pZ_{p}[X] = B/pB$.

Finally $Z_{p}$ is a $Z$-flat module, $Z_{p}[X]$ is a flat $Z_{p}$-module and $B$ is a flat $Z$-module too.

3. Formal group actions on special algebras

Our main result is the following:

**Theorem 3.1.** Let $k$ be a separably closed field of characteristic $p > 0$, $F$ a formal group over $k$, and $D : A \to A[[X]]$ be an action of $F$ on a $k$-algebra $A$.

Suppose $A \simeq B/pB$, where $B$ is a $Z$-algebra such that $B \hookrightarrow B \otimes Z Q$. Then

i) if $Ht(F) \geq 2$

$$D_{i} \circ D_{1} = (i + 1)D_{i+1},$$

for every $i \in N$;

ii) if $Ht(F) > 2$

$$D_{i} \circ D_{p} = \binom{i + p}{i} D_{i+p},$$

for every $i \in N$.

**Proof.** When $F$ is a formal group of height $h \geq 2$, $F$ may be replaced by a formal group $\tilde{F}_{h}$ constructed as follows.

Consider the following power series from $Q[[X, Y]]$:

$$f_{h}(X) = X + \sum_{r=1}^{\infty} p^{-r} X^{p^{r}h} (f_{\infty}(X) = X).$$

If $f_{h}^{-1}$ is the inverse homomorphism determined by $f_{h}$, we put

$$F_{h}(X, Y) = f_{h}^{-1}(f_{h}(X) + f_{h}(Y)).$$

Then $F_{h} = F_{h}(X, Y)$ is a formal group over $Z$ and

$$[p]F_{h}(X) \equiv X^{p^{h}} \pmod{pZ[[X]]}$$

($X^{p^{\infty}} = 0$) ([5], Chap. I, §3.2). Hence we can define $\tilde{F}_{h}$ as the formal group over $k \supset Z/pZ$ obtained by reducing all the coefficients of $F_{h}$ modulo $p$. It follows that $Ht(\tilde{F}_{h}) = h = Ht(F)$ and so we can assume that the formal group $F$ is equal to $\tilde{F}_{h}$ when $Ht(F) \geq 2$ ([5], Chap. III, §3, Theorem 2).

Now we can prove the theorem.

Consider the formal group $F_{h}(X, Y)$ over $Z$ and an action $D$ of this group on the $Z$-algebra $B$, we have

$$\sum_{i, r \geq 0} D_{i} \circ D_{r}(a)X^{i}Y^{r} = \sum_{s \geq 0} D_{s}(a)F_{h}(X, Y)^{s}$$

for all $a \in A$.

i): the proof is similar to that of ([7], Lemma 4.1). But we include it for completeness.
Differentiating both sides of (1) with respect to $Y$ and putting $Y = 0$ one obtains:

$$\sum_i D_i \circ D_1(a)X^i = \sum_s sD_s(a)F_h(X, 0)^{s-1} \frac{\partial F_h(X, 0)}{\partial Y} = \sum_s sD_s(a) \frac{\partial F_h(X, 0)}{\partial Y} X^{s-1}$$

From the equality $f_h(F_h(X, Y)) = f_h(X) + f_h(Y)$, by differentiating with respect to $Y$, we have:

$$f'_h(X) \frac{\partial F_h}{\partial Y} = 1.$$

Hence

$$f'_h(X) \frac{\partial F_h}{\partial Y} = 1,$$

where $\tilde{f}_h(X)$ is obtained by reducing all the coefficients of $f'_h(X)$ modulo $p$. Since

$$f'_h(X) = 1 + \sum_{r=1}^{\infty} p^{r(h-1)} X^{prh-1}$$

and $h \geq 2$, $\tilde{f}_h(X) = 1$. Finally $\frac{\partial F_h}{\partial Y} = 1$ and we get the stated result.

ii): Differentiating both sides of (1) $p$-times with respect to $Y$, one obtains:

$$\sum_{i,r} r(r-1) \ldots (r-p+1)D_i \circ D_r(a)X'^r Y^{r-p} = \sum_s s(s-1) \ldots (s-p+1)D_s(a)F_h(X, Y)^{s-p} \left( \frac{\partial F_h(X, Y)}{\partial Y'} \right)^p$$

$$+ \text{terms with the factor} \quad \frac{\partial^t F_h(X, Y)}{\partial Y'^t}, \quad t \geq 2.$$

From the equality $f_h(F_h(X, Y)) = f_h(X) + f_h(Y)$, by differentiating with respect to $Y$, we have:

$$f'_h(F_h(X, Y)) \frac{\partial F_h}{\partial Y} = f'_h(Y).$$

Moreover

$$f'_h(X) = 1 + \sum_{r=1}^{\infty} p^{r(h-1)} X^{prh-1}.$$

Since $h > 2$, $h - 1 > 1$ and $r(h-1) > 1$, we have

$$f''_h(X) = \sum_{r=1}^{\infty} p^{r(h-1)} (p^{rh} - 1)X^{prh-2}$$

and going on we obtain:

$$f^{(p)}_h(X) = \sum_{r=1}^{\infty} p^{r(h-1)} (p^{rh} - 1) \ldots (p^{rh} - p + 1)X^{prh-p},$$

but $h > 2$ and so $f''_h(0) = \cdots = f^{(p)}_h(0) = 0.$
If we calculate (3) for $Y = 0$, we have

\[ f_h'(X) \frac{\partial F_h(X, 0)}{\partial Y} = 1. \]  

By reducing now (4) mod $p$, $\frac{\partial \tilde{F}_h(X, 0)}{\partial \tilde{Y}} = 1, \tilde{f}_h(X) = 1$.

Differentiating (3) with respect to $Y$ we obtain

\[ f_h''(F_h(X, Y)) \left( \frac{\partial F_h(X, Y)}{\partial Y} \right)^2 + f_h'(F_h(X, Y)) \frac{\partial^2 F_h(X, Y)}{\partial Y^2} = f_h''(Y). \]

If we calculate (5) for $Y = 0$, 
\[ f_h''(X) \left( \frac{\partial F_h(X, 0)}{\partial Y} \right)^2 + f_h'(X) \frac{\partial^2 F_h(X, 0)}{\partial Y^2} = 0, \]

and so

\[ f_h'(X) \frac{\partial^2 F_h(X, 0)}{\partial Y^2} = -f_h''(X) \left( \frac{\partial F_h(X, 0)}{\partial Y} \right)^2. \]

Finally

\[ \frac{\partial^2 F_h(X, 0)}{\partial Y^2} = -f_h'(X)^{-1} f_h''(X) \left( \frac{\partial F_h(X, 0)}{\partial Y} \right)^2. \]

Claim. $f_h''(X)$ has $p^2$ as a factor for $h > 2$.
In fact, since $h > 2, r(h - 1) > 1$ and from
\[ f_h''(X) = \sum_{r=1}^{\infty} p^{r(h-1)} (p^{r-1} - 1) X^{p^{r-2}}, \]
we get the assertion.

By reducing (6) mod $p$, we have $\frac{\partial^2 \tilde{F}_h(X, 0)}{\partial \tilde{Y}^2} = 0$, since $f_h''(X)$ contains $p^2$ as a factor.

In general, from (7), we obtain that $\frac{\partial^t \tilde{F}_h(X, 0)}{\partial \tilde{Y}^t}$ contains $p^2$ as a factor, for $t \geq 2$, and that
\[ \frac{\partial^t \tilde{F}_h(X, 0)}{\partial \tilde{Y}^t} = 0, \text{ for } t \geq 2. \]

Consider now (1). For $Y = 0$, we obtain
\[ \sum_i (p(p - 1) \ldots 1) D_i \circ D_p(a) X^i = \]
\[ = \sum_{s \geq 0} s(s - 1)(s - 2) \ldots (s - p + 1) D_s(a) X^{s-p} \left( \frac{\partial F_h(X, 0)}{\partial Y} \right)^p + \]
\[ + \text{terms with the factor } p^2. \]
\[
\sum_i p! D_i \circ D_p(a) X^i = \sum_{s \geq p} \binom{s}{p} D_s(a) X^{s-p} \left( \frac{\partial F_h(X,0)}{\partial Y} \right)^p + \text{ terms with the factor } p^2.
\]

Since we can divide by \( p \) (\( p \) is not a 0-divisor in \( B \)), we have
\[
\sum_i (p-1)! D_i \circ D_p(a) X^i = \sum_{s \geq p} \binom{s}{p} D_s(a) X^{s-p} \left( \frac{\partial F_h(X,0)}{\partial Y} \right)^p + \text{ terms with the factor } p.
\]

By reducing mod \( p \), we have:
\[
\sum_{i \geq 0} D_i \circ D_p(a) X^i = \sum_{s \geq p} \binom{s}{p} D_s(a) X^{s-p}.
\]

Hence
\[
D_i \circ D_p = \binom{i + p}{i} D_{i+p},
\]
for every \( i \in \mathbb{N} \).

**Remark 3.2.** Observe that \( \binom{i + p}{i} \neq \binom{i}{i} \) in characteristic \( p > 0 \).

**Corollary 3.3.** Under the same hypotheses of Theorem 3.1, let \( D : A \to A[[X]] \) be an action of \( F \) on the \( k \)-algebras \( A \). If \( D(a) = \sum_{i \geq 0} D_i(a) X^i \), put
\[
\delta_0 = D_1, \delta_1 = D_p, \delta_2 = D_{p^2}, \ldots, \delta_i = D_{p^i}, \ldots
\]
we have

a) if \( Ht(F) > 2 \)

i) \( \delta_i \delta_j = \delta_j \delta_i \) for \( i = 0, 1 \) and \( j \geq 0 \),

ii) \( \delta_i^p = 0 \) for \( i = 0, 1 \),

iii) for every \( m > 0 \)
\[
D_m = \frac{\delta_0^{m_0} \delta_1^{m_1} \cdots \delta_r^{m_r}}{m_0! m_1! \cdots m_r!},
\]
where \( m = m_0 + m_1 p + \cdots + m_r p^r \), \( 0 \leq m_i < p \), is the \( p \)-adic expansion of \( m \).

b) if \( Ht(F) = 1 \)

i') \( \delta_i \delta_j = \delta_j \delta_i \) for all \( i, j \geq 0 \),

ii') \( \delta_i^p = \delta_i \) for all \( i \),

iii') for every \( m > 0 \)
\[
D_m = \frac{(\delta_0 m_0) (\delta_1 m_1) \cdots (\delta_r m_r)}{m_0! m_1! \cdots m_r!},
\]
where \( (\delta_i)_m = \delta_i (\delta_{i-1}) \cdots (\delta_{i-m+1}) = \prod_{k=0}^{m-1} (\delta_i - k) \).
Proof. If we suppose that $k$ is separably closed, the equality $Ht(F) = Ht(F')$ imply that $F$ is isomorphic to $F'$. More precisely, for any $h \in N$, there exists a formal group $G$ (that is unique up to isomorphisms) such that $Ht(G) = h$.

Then a) follows from Theorem 3.1 and b) from [8].

Remark 3.4. Let $k$ be a field and let $H$ be a finite dimensional Hopf algebra over $k$ with comultiplication $\Delta : H \to H \otimes H$ and antipode $S : H \to H$, and counity $\epsilon : H \to k$.

A coaction of $H$ on a $k$-algebra is a morphism of algebras $D : A \to A \otimes H$ such that $(1 \otimes \epsilon)D \simeq 1$ and $(1 \otimes \Delta) = (D \otimes 1)D$.

From now on let $k$ be a field of characteristic $p > 0$.

We consider the Hopf algebra which “lives” on the coalgebras $C_n = (\sum_{s=0}^{p^n-1} ke_s, \Delta, \epsilon)$, $n=0, 1, \ldots$, where $\Delta(e_s) = \sum_{i+j=s} e_i \otimes e_j$ and $\epsilon(e_s) = \delta_{s,0}$ (Kroneker delta). More precisely, we say that a Hopf algebra $H$ “lives” on $C_n$ if $H$, as a coalgebra, is equal to $C_n$.

For example $H_n(F_n) = (C_n, M : C_n \otimes C_n \to C_n, S : C_n \to C_n, \eta : k \to C_n)$, $n=0, 1, \ldots$, where multiplication $M$ is given by $M(e_i \otimes e_j) = \binom{i+j}{i} e_{i+j}$ if $i + j < p^n$ and 0, otherwise, antipode $S$ is determined by the equalities $\sum_{i+j=s} e_i S(e_j) = \delta_{s,0}$, and the structural map $\eta$ is defined by $\eta(t) = \epsilon(t)$.

If we fix a natural number $n$ we can consider the Hopf algebra $H$ which “lives” on the algebra $H_n = k[X]/(X^{p^n})$. We say also that $H$ is a Hopf algebra structure on $H_n$.

A coaction of such a Hopf algebra $H$ on a $k$-algebra $A$ is a morphism of $k$-algebras such that if $D(a) = \sum_i D_i(a) \otimes x^i, a \in A$, with $x = X + (X^{p^n})$ and $D_i : A \to A$ addive endomorphisms, then

$$D_0 = 1, \quad \text{and} \quad D_s(ab) = \sum_{i+j=s} D_i(a) D_j(b) \text{ for all } 0 \leq s < p^n,$$

with $a, b \in A$ i.e. $\{D_i : 0 \leq i < p^n\}$ is an higher derivation of order $p^n$ in $A$.

For a given formal group $F$ over the field $k$ and a natural $n$ we define the Hopf algebra structure $H_n(F)$ on the algebra $H_n = k[X]/(X^{p^n})$ as follows:

a) comultiplication $\Delta : H_n \to H_n \otimes H_n \simeq k[X, Y]/(X^{p^n}, Y^{p^n})$ defined by $\Delta(x) = F(x, y)$, where $x = X + (X^{p^n}, Y^{p^n})$ and $y = Y + (X^{p^n}, Y^{p^n})$, and we identify $x = X + (X^{p^n})$ with $x = X + (X^{p^n}, Y^{p^n})$.

b) antipode $S : H_n \to H_n$ given by $S(x) = i(x)$

c) counity $\epsilon : H_n \to k$ defined by $\epsilon(x) = 0$.

We can easily verify that if $F$ is a formal group over $k$ and $A$ a $k$-algebra a coaction of $H_n(F)$ on $A$ is a morphism of algebras $D : A \to A \otimes H_n(F)$ such that $D_0 = 1$ and

$$\sum_{0 \leq i,j < p^n} D_i D_j(a) \otimes x^i y^j = \sum_{0 \leq s < p^n} D_s(a) \otimes F(x, y)^s$$

for all $a \in A$.

By using the same techniques of Theorem 3.1, the following result holds:

Let $k$ be a separably closed field of characteristic $p > 0$, $F$ a formal group over $k$, and $D : A \to A \otimes H_n(F)$ be a coaction of $H_n(F)$ on a $k$-algebra $A$.

Suppose

i) $A \simeq B/pB$, with $B$ $\mathbb{Z}$-algebra such that $B \to B \otimes \mathbb{Q}$ is injective
ii) $Ht(F) > 2$

Then
\[ D_i \circ D_p = \left( i + \frac{p^i}{i} \right) D_{i+p^i}, \]

for $j = 0, 1$ and $0 \leq i < p^n$.

In fact put $H_n(F) = H_n$. It is easy to verify that for any action $D : A \to A[[X]]$, $D(a) = \sum_{i \geq 0} D_i(a) X^i$ of a formal group $F(X, Y)$, the application:

\[ D^{(n)} : A \to A \otimes H_n, \quad \text{with} \quad D^{(n)}(a) = \sum_{0 \leq i < p^n} D_i(a) \otimes x^i, \]

$x = X + (X^{p^n})$, $\Delta(x) = F(x, y)$, is a coaction of the Hopf algebra $H_n$ on $A$.

Hence
\[ \sum_{0 \leq i < p^n} D_i \circ D_p(a) \otimes x^i = \sum_{s \geq p} \binom{s}{p} D_s(a) \otimes x^{s-p} \]

and so
\[ D_i \circ D_p = \left( i + \frac{p^i}{i} \right) D_{i+p}, \]

for every $i$, $0 \leq i < p^n$.

The case $j = 0$ follows from ([7], Lemma 4.1).

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