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ASYMPTOTIC STUDY OF THE RAYLEIGH EQUATION

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ABSTRACT. This paper provides an asymptotic treatment of the Cauchy problem for the Rayleigh equation $\mu\ddot{y} + \frac{\dot{y}^3}{3} + ay - \dot{y} = 0$ modelling a harmonic oscillator. The method of boundary layer functions and our version of the Tikhonov theorem are used. The models of asymptotic approximations are derived.

1. Mathematical problem

Consider the Cauchy problem $y(0) = y^0$, $\dot{y}(0) = z^0$ for the perturbed Rayleigh nonlinear ordinary differential equation (ode) $\mu\ddot{y} + \frac{\dot{y}^3}{3} + ay - \dot{y} = 0$, where $a, \mu \in \mathbb{R}$ are parameters, $\mu > 0$ is small, and $y : \mathbb{R} \to \mathbb{R}$, $y = y(t)$ is the unknown function [1]. The dot over the quantities stands the derivative $\frac{d}{dt}$, where t is the independent variable (time). This equation reads, equivalently, in the form

$$\left\{ \begin{array}{rcl} \mu\dot{z} & = & z - ay - \frac{1}{3}z^3, \\ \dot{y} & = & z, \end{array} \right. \tag{1}$$

where, $y, z : \mathbb{R} \to \mathbb{R}$, $Z = Z(t)$, $y = y(t)$ are the new unknown functions.

The initial conditions become

$$z(0) = z^0, \;\; y(0) = y^0. \tag{2}$$

Due to the smoothness of the right-hand sides in (1), the solution of (1), (2) exists and is unique.

The generalized Van der Pal oscillators are governed by the following system of ordinary differential equations (sode)

$$\left\{ \begin{array}{rcl} \mu\frac{dz}{dt} & = & F(z, y, t), \\ \frac{dy}{dt} & = & f(z, y, t), \end{array} \right. \tag{3}$$

The Rayleigh equations (1) are particular cases of (3), corresponding to $F(z, y, t) := z - ay - \frac{z^3}{3}$, $f(z, y, t) := z$.

From the asymptotic point of view (1), (2) is a singularly perturbed problem of boundary layer type [2]. The existence, uniqueness and asymptotic representation of its solutions are stated by the Tikhonov theorem [3], while terms of higher order of asymptotic approximation are shown to satisfy the corresponding models of asymptotic approximation for (1), (2) given by our improved version.

The method based of the Tikhonov theorem is referred to as the method of the boundary layer functions and it yields the uniformly valid asymptotic expansion for the solution in the form of a composite power series of μ. More precisely, it can be easily shown that the conditions of our theorem are fulfilled.

The existence of the mathematical boundary layer makes necessary to re-scale the time t by introducing a new time $z = \frac{t}{\mu}$. Then the unknown functions of the concrete Cauchy problem (1), (2) are looked for in the form of composite series as $\mu \to 0$

$$z = \overline{z}(t,\mu) + \Pi z(\tau,\mu), \;\; y = \overline{y}(t,\mu) + \Pi y(\tau,\mu), \tag{4}$$

where

$$\begin{cases} \overline{z}(t,\mu) &= \sum\limits_{k=0}^{\infty} \mu^k \overline{z}_k(t), \text{ where } \overline{z}_k(t) = \sum\limits_{j=0}^{\infty} z_{kj} t^j, \; z_{kj} = \frac{\overline{z}_k^{(j)}(0)}{j!}, \\ \Pi z(\tau,\mu) &= \sum\limits_{k=0}^{\infty} \mu^k \Pi_k z(\tau), \text{ where } \Pi_k z(\tau) = \sum\limits_{j=0}^{\infty} (\Pi_k z)_j \tau^j, \\ \overline{y}(t,\mu) &= \sum\limits_{k=0}^{\infty} \mu^k \overline{y}_k(t), \text{ where } \overline{y}_k(t) = \sum\limits_{j=0}^{\infty} y_{kj} t^j, \; y_{kj} = \frac{\overline{y}_k^{(j)}(0)}{j!}, \\ \Pi y(z,\mu) &= \sum\limits_{k=0}^{\infty} \mu^k \Pi_k y(\tau), \text{ where } \Pi_k y(\tau) = \sum\limits_{j=0}^{\infty} (\Pi_k y)_j \tau^j. \end{cases} \tag{5}$$

Writing F and f in the form

$$F = \overline{F}(t,\mu) + \Pi F(\tau,\mu), \;\; f = \overline{f}(t,\mu) + \Pi f(\tau,\mu)$$

where $\Pi F = F - \overline{F}$, $\overline{F}(t,\mu) = F(\overline{y}(t,\mu), \overline{z}(t,\mu), t)$, $\overline{f} = f(\overline{y}(t,\mu), \overline{z}(t,\mu), t)$, $\Pi f = f - \overline{f}$ are similar composite power series in μ.

Taking into account (4) in (3) and the equality $\frac{d}{dt} g(t,\tau) = \frac{\partial y}{\partial t} + \frac{1}{\mu}\frac{\partial y}{\partial z}$, we obtain

$$\begin{cases} \mu \frac{\partial \overline{z}}{\partial t}(t,\mu) + \frac{\partial \Pi z}{\partial \tau}(\tau,\mu) &= \overline{F}(t,\mu) + \Pi F(\tau,\mu) \\ \mu \frac{\partial \overline{y}}{\partial t}(t,\mu) + \frac{\partial \Pi y}{\partial \tau}(\tau,\mu) &= \mu[\overline{f}(t,\mu) + \Pi f(\tau,\mu)]. \end{cases} \tag{6}$$

Introducing the expressions for z, y, F and f in (6) and (2) and matching the obtained series, the models of asymptotic approximations are obtained, whence the asymptotic expansion of the solution.

2. Expansions for F and f

In the following we deal with the problem (1), (2). For F and f we have the following expressions

$$\begin{aligned} \overline{F}(t,\mu) &= \sum_{k=0}^{\infty} \mu^k \overline{z}_k(t) - a \sum_{k=0}^{\infty} \mu^k \overline{y}_k(t) - \frac{1}{3} \left(\sum_{i=0}^{\infty} \mu^i \overline{z}_i(t) \right) \cdot \left(\sum_{j=0}^{\infty} \mu^j \overline{z}_j(t) \right) \cdot \left(\sum_{l=0}^{\infty} \mu^l \overline{z}_l(t) \right) \\ &= \sum_{k=0}^{\infty} \mu^k \left[\overline{z}_k(t) - a \overline{y}_k(t) \right] - \frac{1}{3} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \mu^{i+j+l} \overline{z}_i(t) \cdot \overline{z}_j(t) \cdot \overline{z}_l(t) \\ &= \sum_{k=0}^{\infty} \mu^k \left[\overline{z}_k(t) - a \overline{y}_k(t) \right] - \frac{1}{3} \sum_{i=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{k-j} \mu^k \overline{z}_{k-j-l}(t) \cdot \overline{z}_j(t) \cdot \overline{z}_l(t) \\ &= \sum_{k=0}^{\infty} \mu^k \left[\overline{z}_k(t) - a \overline{y}_k(t) - \frac{1}{3} \sum_{j=0}^{k} \sum_{l=0}^{k-j} \overline{z}_{k-j-l}(t) \cdot \overline{z}_j(t) \cdot \overline{z}_l(t) \right]. \end{aligned} \tag{7}$$

Apart from the supplementary use of the double Taylor series expansion, we reduced the product of three infinite series to a product of an infinite series by two finite series. This reduction will be performed in the following too.

We can write

$$
\begin{aligned}
\Pi F(\tau,\mu) &= \sum_{k=0}^{\infty}\mu^k\Pi_k z(\tau) - a\sum_{k=0}^{\infty}\mu^k\Pi_k y(\tau) - \frac{1}{3}\left(\sum_{i=0}^{\infty}\mu^i\Pi_i z(\tau)z_i(\tau)\right)\cdot \\
&\cdot \left\{\left[\sum_{j=0}^{\infty}\mu^j\overline{z}_j(\tau\mu) + \sum_{j=0}^{\infty}\mu^j\Pi_j z(\tau)\right]\cdot\left[\sum_{l=0}^{\infty}\mu^l\overline{z}_l(\tau\mu) + \sum_{l=0}^{\infty}\mu^j\Pi_l z(\tau)\right] + \right. \\
&+ \left[\sum_{j=0}^{\infty}\mu^j\overline{z}_j(\tau\mu) + \sum_{j=0}^{\infty}\mu^j\Pi_j z(\tau)\right]\cdot\left[\sum_{l=0}^{\infty}\mu^l\overline{z}_l(\tau\mu)\right] + \\
&+ \left.\left[\sum_{j=0}^{\infty}\mu^j\overline{z}_j(\tau\mu)\right]\cdot\left[\sum_{l=0}^{\infty}\mu^l\overline{z}_l(\tau\mu)\right]\right\} = \\
&= \sum_{k=0}^{\infty}\mu^k\left[\Pi_k z(\tau) - a\Pi_k y(\tau)\right] - \frac{1}{3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\mu^{i+j+l}\Pi_i z(\tau)\cdot \\
&\cdot\{[\overline{z}_j(\tau\mu)+\Pi_j z(\tau)]\times[\overline{z}_l(\tau\mu)+\Pi_l z(\tau)] + [\overline{z}_j(\tau\mu)+\Pi_j z(\tau)]\cdot\overline{z}_l(\tau\mu) + \overline{z}_j(\tau\mu)\cdot\overline{z}_l(\tau\mu)\}
\end{aligned} \tag{8}
$$

Since

$$[\overline{z}_j(\tau\mu)+\Pi_j z(\tau)]\times[\overline{z}_l(\tau\mu)+\Pi_l z(\tau)] + [\overline{z}_j(\tau\mu)+\Pi_l z(\tau)]\cdot\overline{z}_l(\tau\mu) + \overline{z}_j(\tau\mu)\cdot\overline{z}_l(\tau\mu) =$$

$$= \left[\Pi_j z(\tau) + \sum_{m=0}^{\infty} z_{jm}(\tau\mu)^m\right]\cdot\left[\Pi_l z(\tau) + \sum_{n=0}^{\infty} z_{ln}(\tau\mu)^n\right] +$$

$$+\left[\Pi_j z(\tau) + \sum_{m=0}^{\infty} z_{jm}(\tau\mu)^m\right]\cdot\left[\sum_{n=0}^{\infty} z_{ln}(\tau\mu)^n\right] + \left[\sum_{m=0}^{\infty} z_{jm}(\tau\mu)^m\right]\cdot\left[\sum_{n=0}^{\infty} z_{ln}(\tau\mu)^n\right] =$$

$$= \Pi_j z(\tau)\cdot\Pi_l z(\tau) + \sum_{n=0}^{\infty}\left[\Pi_j z(\tau)\cdot z_{ln}\cdot\tau^n\right]\mu^n + \sum_{m=0}^{\infty}\left[\Pi_l z(\tau)\cdot z_{jm}\cdot\tau^m\right]\mu^m +$$

$$+\sum_{n=0}^{\infty}\left[\Pi_j z(\tau)\cdot z_{ln}\cdot\tau^n\right]\mu^n + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_{jm}\cdot z_{ln}\cdot\tau^{m+n}\mu^{m+n} +$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_{jm}\cdot z_{ln}\cdot\tau^{m+n}\mu^{m+n} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_{jm}\cdot z_{ln}\cdot\tau^{m+n}\mu^{m+n} = \tag{9}$$

$$= \Pi_j z(\tau)\cdot\Pi_l z(\tau) + \sum_{s=0}^{\infty}\mu^s\cdot\tau^s\left[2\Pi_j z(\tau)\cdot z_{ls} + \Pi_l z(\tau)\cdot z_{js}\right] + 3\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mu^{m+n}\tau^{m+n}\cdot z_{jm}\cdot z_{ln},$$

we have

$$
\begin{aligned}
\Pi F(\tau,\mu) &= \sum_{k=0}^{\infty}\mu^k\left[\Pi_k z(\tau)-a\Pi_k y(\tau)\right]-\overbrace{\frac{1}{3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\mu^{i+j+l}}^{\text{(we denote } i+j+l:=k)}\Pi_i z(\tau)\cdot\Pi_j z(\tau)\cdot\Pi_l z(\tau)-\\
&- \overbrace{\frac{1}{3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\mu^{i+j+l+s}}^{\text{(we denote } i+j+l+s:=k)}\,\tau^s\Pi_i z(\tau)\times\left[2\Pi_j z(\tau)\cdot z_{ls}+\Pi_l z(\tau)\cdot z_{js}\right]-\\
&- \overbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mu^{i+j+l+m+n}}^{\text{(we denote } i+j+l+m+n:=k)}\,\tau^{m+n}\Pi_i z(\tau)\cdot z_{jm}\cdot z_{ln} \qquad (10)\\
&= \sum_{k=0}^{\infty}\mu^k[\Pi_k z(\tau)-a\Pi_k y(\tau)]-\frac{1}{3}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k-j}\mu^k\Pi_{k-j-l}\cdot z(\tau)\Pi_j z(\tau)\cdot\Pi_l z(\tau)-\\
&- \frac{1}{3}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k-j}\sum_{s=0}^{k-j-l}\mu^k\tau^s\cdot\Pi_{k-j-l-s}\cdot z(\tau)\times\left[2\Pi_j z(\tau)\cdot z_{ls}+\Pi_l z(\tau)\cdot z_{js}\right]-\\
&- \frac{1}{3}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k-j}\sum_{m=0}^{k-j-l}\sum_{n=0}^{k-j-l-m}\mu^k\tau^{m+n}\cdot\Pi_{k-j-l-m-n}\cdot z(\tau)\cdot z_{jm}\cdot z_{ln}
\end{aligned}
$$

In the same way

$$f(z,y,t):=z=\sum_{k=0}^{\infty}\mu^k\overline{z}_k(t)+\sum_{k=0}^{\infty}\mu^k\Pi_k z(\tau).$$

In this way, the Cauchy problem (2) for (6) becomes

$$
(11)\quad
\begin{cases}
\mu\sum\limits_{k=0}^{\infty}\mu^k\frac{d\overline{z}_k}{dt}(t)+\sum\limits_{k=0}^{\infty}\mu^k\frac{d\Pi_k z}{d\tau}(\tau)=\\
\quad=\sum\limits_{k=0}^{\infty}\mu^k\left[\overline{z}_k(t)-a\overline{y}_k(t)-\frac{1}{3}\sum\limits_{j=0}^{k}\sum\limits_{l=0}^{k-j}\overline{z}_{k-j-l}(t)\cdot\overline{z}_j(t)\cdot\overline{z}_l(t)\right]+\\
\quad+\sum\limits_{k=0}^{\infty}\mu^k[\Pi_k z(\tau)-a\Pi_k y(\tau)]-\frac{1}{3}\sum\limits_{k=0}^{\infty}\sum\limits_{j=0}^{k}\sum\limits_{l=0}^{k-j}\mu^k\Pi_{k-j-l}z(\tau)\Pi_j z(\tau)\cdot\Pi_l z(\tau)-\\
\quad-\frac{1}{3}\sum\limits_{k=0}^{\infty}\sum\limits_{j=0}^{k}\sum\limits_{l=0}^{k-j}\sum\limits_{s=0}^{k-j-l}\mu^k\tau^s\cdot\Pi_{k-j-l-s}\cdot z(\tau)\times\left[2\Pi_j z(\tau)\cdot z_{ls}+\Pi_l z(\tau)\cdot z_{js}\right]-\\
\quad\sum\limits_{k=0}^{\infty}\sum\limits_{j=0}^{k}\sum\limits_{l=0}^{k-j}\sum\limits_{m=0}^{k-j-l}\sum\limits_{n=0}^{k-j-l-m}\mu^k\tau^{m+n}\cdot\Pi_{k-j-l-m-n}z(\tau)\cdot z_{jm}\cdot z_{ln},\\
\mu\sum\limits_{k=0}^{\infty}\mu^k\frac{d\overline{y}_k}{dt}(t)+\sum\limits_{k=0}^{\infty}\mu^k\frac{d\Pi_k y}{d\tau}(\tau)=\mu(\overline{z}_0(t)+\Pi_0 z(\tau))+\\
\quad+\mu\sum\limits_{k=1}^{\infty}\mu^k\overline{z}(t)+\mu\sum\limits_{k=1}^{\infty}\mu^k\Pi_k z(\tau),\\
\sum\limits_{k=0}^{\infty}\mu^k\overline{z}_k(0)+\sum\limits_{k=0}^{\infty}\mu^k\Pi_k z(0)=z^0,\\
\sum\limits_{k=0}^{\infty}\mu^k\overline{y}_k(0)+\sum\limits_{k=0}^{\infty}\mu^k\Pi_k y(0)=y^0.
\end{cases}
$$

3. The models of asymptotic approximation in the generalized Tikhonov approach

Matching the double series in μ and t or in μ and τ in (11) we obtain the following problem of the zeroth asymptotic approximation as $\mu \to 0$, corresponding to the coefficients of μ^0

$$(12)\qquad \begin{cases} \frac{d\Pi_0 z}{d\tau}(\tau) & = & \left[\overline{z}_0(t) - a\overline{y}_0(t) - \frac{1}{3}(\overline{z}_0(t))^3\right] + \\ & & +[\Pi_0 z(\tau) - a\Pi_0 y(\tau)] - \frac{1}{3}(\Pi_0 z(\tau))^3 - \\ & & -\frac{3}{3}(\Pi_0 z(\tau))^2 \cdot z_{00} - \Pi_0 z(\tau) \cdot z_{00}^2, \\ \frac{d\Pi_0 z}{d\tau}(\tau) & = & 0, \\ \overline{z}_0(0) + \Pi_0 z(0)) & = & z^0, \\ \overline{y}_0(0) + \Pi_0 y(0)) & = & y^0. \end{cases}$$

For the coefficients of μ^{k+1}, $k \geq 0$ we obtain the following problem of the $k+1$ th asymptotic approximation as $\mu \to 0$

$$(13)\qquad \begin{cases} \frac{d\overline{z}_k}{dt}(t) + \frac{d\Pi_{k+1} z}{d\tau}(\tau) = [\overline{z}_{k+1}(t) - a\overline{y}_{k+1}(t) - \\ \quad \frac{1}{3}\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j} \overline{z}_{k+l-j-l}(t)\cdot\overline{z}_j(t)\cdot\overline{z}_l(t)] + \\ \quad +[\Pi_{k+1}z(\tau) - a\Pi_{k+1}y(\tau)] - \\ \quad -\frac{1}{3}\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j} \Pi_{k+l-j-l}z(\tau)\cdot\Pi_j z(\tau)\cdot\Pi_l z(\tau) - \\ \quad -\frac{1}{3}\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j}\sum_{s=0}^{k+1-j-l} \tau^s \cdot \Pi_{k+1-j-l-s}z(\tau) \times \\ \quad \times[2\Pi_j z(\tau)\cdot z_{ls} + \Pi_l z(\tau)\cdot z_{js}] - \\ \quad -\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j}\sum_{m=0}^{k+1-j-l}\sum_{n=0}^{k+1-j-l-m} \tau^{m+n}\Pi_{k+1-j-l-m-n}z(\tau)\cdot z_{jm}\cdot z_{ln}, \\ \frac{d\overline{y}_k}{dt}(t) + \frac{d\Pi_{k+1} y}{d\tau}(\tau) = \overline{z}_k(t) + \Pi_k z(\tau), \\ \overline{z}_{k+1}(0) + \Pi_{k+1}z(0)) = 0, \\ \overline{y}_{k+1}(0) + \Pi_{k+1}y(0))0 = 0. \end{cases}$$

In these equations we have variables separate in t and τ, whence

- for μ^0

$$(14)\qquad S_0 \begin{cases} 0 & = & \overline{z}_0(t) - a\overline{y}_0(t) - \frac{1}{3}\overline{z}_0^3(t), \\ \frac{d\Pi_0 z}{d\tau}(\tau) & = & \Pi_0 z(\tau) - a\Pi_0 y(\tau)] - \frac{1}{3}(\Pi_0 z(\tau))^3 - \\ & & -(\Pi_0 z(\tau))^2 \cdot z_{00} - \Pi_0 z(\tau)\cdot z_{00}^2, \\ \frac{d\Pi_0 z}{d\tau}(\tau) & = & 0, \\ \overline{z}_0 + \Pi_0 z(0)) & = & z^0, \\ \overline{y}_0 + \Pi_0 y(0)) & = & y^0, \end{cases}$$

- for μ^1

$$
(15)\qquad S_1 \begin{cases}
\frac{d\overline{z}_0}{dt}(t) = [\overline{z}_1(t) - a\overline{y}_1(t) - \frac{1}{3}\sum_{j=0}^{1}\sum_{l=0}^{1-j}\overline{z}_{1-j-l}(t)\cdot\overline{z}_j(t)\cdot\overline{z}_l(t), \\
\frac{d\Pi_1 z}{d\tau}(\tau) = \Pi_1 z(\tau) - a\Pi_1 y(\tau)]- \\
\quad -\frac{1}{3}\sum_{j=0}^{1}\sum_{l=0}^{1-j}\Pi_{1-j-l}z(\tau)\cdot\Pi_j z(\tau)\cdot\Pi_l z(\tau)- \\
\quad -\frac{1}{3}\sum_{j=0}^{1}\sum_{l=0}^{1-j}\sum_{s=0}^{1-j-l}\tau^s\cdot\Pi_{1-j-l-s}z(\tau)\times \\
\quad \times[2\Pi_j z(\tau)\cdot z_{ls} + \Pi_l z(\tau)\cdot z_{js}]- \\
\quad -\sum_{j=0}^{1}\sum_{l=0}^{1-j}\sum_{m=0}^{1-j-l}\sum_{n=0}^{1-j-l-m}\tau^{m+n}\Pi_{1-j-l-m-n}z(\tau)\cdot z_{jm}\cdot z_{ln}, \\
\frac{d\overline{y}_0}{dt}(t) = \overline{z}_0(t), \\
\frac{d\Pi_1 y}{d\tau}(\tau) = \Pi_0 z(\tau), \\
\overline{z}_1(0) + \Pi_1 z(0)) = 0, \\
\overline{y}_1(0) + \Pi_1 y(0)) = 0.
\end{cases}
$$

- for $\mu^{k+1},\ k \geq 1$

$$
(16)\qquad S_{k+1} \begin{cases}
\frac{d\overline{z}_k}{dt}(t) = \overline{z}_{k+1}(t) - a\overline{y}_{k+1}(t)- \\
\quad -\frac{1}{3}\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j}\overline{z}_{k+l-j-l}(t)\cdot\overline{z}_j(t)\cdot\overline{z}_l(t), \\
\frac{d\Pi_{k+1} z}{d\tau}(\tau) = \Pi_{k+1} z(\tau) - a\Pi_{k+1} y(\tau) \\
\quad -\frac{1}{3}\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j}\Pi_{k+1-j-l}z(\tau)\cdot\Pi_j z(\tau)\cdot\Pi_l z(\tau)- \\
\quad -\frac{1}{3}\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j}\sum_{s=0}^{k+1-j-l}\tau^s\cdot\Pi_{k+1-j-l-s}z(\tau)\times \\
\quad \times[2\Pi_j z(\tau)\cdot z_{ls} + \Pi_l z(\tau)\cdot z_{js}]- \\
\quad -\sum_{j=0}^{k+1}\sum_{l=0}^{k+1-j}\sum_{m=0}^{k+1-j-l}\sum_{n=0}^{k+1-j-l-m}\tau^{m+n}\Pi_{k+1-j-l-m-n}z(\tau)\times \\
\quad \times z_{jm}\cdot z_{ln}, \\
\frac{d\overline{y}_k}{dt}(t) = \overline{z}_k(t), \\
\frac{d\Pi_{k+1} y}{d\tau}(\tau) = \Pi_k z(\tau), \\
\overline{z}_{k+1}(0) + \Pi_{k+1} z(0) = 0, \\
\overline{y}_{k+1}(0) + \Pi_{k+1} y(0) = 0.
\end{cases}
$$

In this way, we extended the results of Tikhonov and made them algorithmic. For instance, it can be proved that the model S_0 can be obtained from the Tikhonov's model in μ^0 of where only the first term in the power series of $\overline{z}_0(0)$ is kept. The other part of $\overline{z}_0(0)$ occurs in other models.

These models are algorithmic, therefore they are more suitable to analytical as well as numerical computation than the Tikhonov's models. Their solution will be provided elsewhere.

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