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SUPERPOSITIONS IN DISTRIBUTIONS SPACES

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ABSTRACT. We present a deep topological study of the operation of superposition, it is applied in several papers to phisics and economics and the study is just motivated by the increasing use of superpositions in physical and economical theories. Several characterizations in topological terms are presented, with some properties useful in the applications.

1. Preliminaries and notations on tempered distributions

In this paper we shall use some notations. The letters n, m, k are natural numbers, $\mathbb{N}(\leq k)$ is the set of positive integer lower than or equal to k; μ_n is the Lebesgue measure on $\mathbb{R}^n$; $\mathbb{I}_{(\mathbb{R},\mathbb{C})}$ is the canonical immersion of $\mathbb{R}$ in $\mathbb{C}$; if X is a non-empty set, $\mathbb{I}_X$ is the identity map on X. If X and Y are two topological vector spaces on $\mathbb{K}$, $\mathrm{Hom}(X, Y)$ is the set of all the linear operators from X to Y, $\mathcal{L}(X, Y)$ is the set of all the linear and continuous operators from X to Y, $X^* := \mathrm{Hom}(X, \mathbb{K})$ is the algebraic dual of X and $X' := \mathcal{L}(X, \mathbb{K})$ is the topological dual of X. With $\mathcal{S}_n := \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ we shall denote the $(n, \mathbb{K})$-*Schwartz space*, that is to say, the set of all the smooth functions (i.e., of class C^∞) of $\mathbb{R}^n$ in $\mathbb{K}$ *rapidly decreasing at infinity with all their derivatives* (the functions and all their derivatives tend to 0 at $\mp\infty$ faster than the inverse of any polynomial); $\mathcal{S}_{(n)}$ is the *standard Schwartz topology* on $\mathcal{S}_n$, and $(\mathcal{S}_n)$ is the topological vector space on $\mathcal{S}_n$ with its standard topology; the topology $\mathcal{S}_{(n)}$ is induced by a metric, in fact ($\mathcal{S}_n$ is closed under differentiation and multiplication by polynomials) by the family of seminorms (p_k) on $\mathcal{S}_n$ defined by

$$p_k(f) = \sup_{x\in\mathbb{R}^n} \max\left\{\left|x^\beta D^\alpha f(x)\right| : \alpha, \beta \in \mathbb{N}_0^n : 0 \leq |\alpha|, |\beta| \leq k\right\},$$

for every non-negative integer k; each p_k is a norm on $\mathcal{S}_n$, and $p_k(f) \leq p_{k+1}(f)$ for all $f \in \mathcal{S}_n$, the pair $\left(\mathcal{S}_n, (p_k)_{k\in\mathbb{N}_0}\right)$ is a countably complete normed space and consequentely $(\mathcal{S}_n)$ is a Frèchet space (for the necessary preliminaries of Functional Analysis see [1],..., [6]); $\mathcal{S}'_n := \mathcal{S}'(\mathbb{R}^n, \mathbb{K})$ is the space of tempered distributions from $\mathbb{R}^n$ to $\mathbb{K}$, that is, the topological dual of the topological vector space $(\mathcal{S}_n, \mathcal{S}_{(n)})$, i.e., $\mathcal{S}'_n = (\mathcal{S}_n, \mathcal{S}_{(n)})'$; if $x \in \mathbb{R}^n$, δ_x is the *distribution of Dirac on $\mathcal{S}_n$ centered at x*, i.e., the functional

$$\delta_x : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \phi(x);$$

if $f \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$, where

$$\mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) = \{g \in C^\infty(\mathbb{R}^n, \mathbb{K}) : \forall\phi \in \mathcal{S}_n(\mathbb{K}), \phi g \in \mathcal{S}_n(\mathbb{K})\},$$

then the functional

$$[f] = [f]_n : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \int_{\mathbb{R}^n} f\phi d\mu_n$$

is a tempered distribution, called the *regular distribution generated by* f (see [1] page 110).

2. The $\mathcal{S}$-families

Let I be a non-empty set. We denote by $s\left(I, \mathcal{S}'_n\right)$ the space of all the families in $\mathcal{S}'_n$ indexed by I, i.e., the set of all the surjective maps from I onto a subset of $\mathcal{S}'_n$. Moreover, as usual, if v is one of these families, for each $p \in I$, the distribution $v(p)$ is denoted by v_p, and the family v itself is also denoted by $\left(v_p\right)_{p\in I}$. The set $s\left(I, \mathcal{S}'_n\right)$ is a vector space with respect to the standard operations of addition $+ : s\left(I, \mathcal{S}'_n\right)^2 \to s\left(I, \mathcal{S}'_n\right)$ defined by $v{+}w := \left(v_p + w_p\right)_{p\in I}$, and multiplication by scalars $\cdot : \mathbb{K}\times s\left(I, \mathcal{S}'_n\right) \to s\left(I, \mathcal{S}'_n\right)$ defined by $\lambda v := \left(\lambda v_p\right)_{p\in I}$. In other words, the family $v + w$ is defined by $(v + w)_p = v_p + w_p$, for every p in I, and the family λv is defined by $(\lambda v)_p = \lambda v_p$, for every p in I.

In the theory of superpositions on $\mathcal{S}'_n$ the class of the $\mathcal{S}$-families plays a basic role.

Definition 1.1 (family of tempered distributions of class $\mathcal{S}$). *Let v be a family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. The family v is called **family of class** $\mathcal{S}$ or $\mathcal{S}$-**family** if, for each test function $\phi \in \mathcal{S}_n$, the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$, defined by*

$$v(\phi)(p) := v_p(\phi),$$

for each m-tuple $p \in \mathbb{R}^m$, belongs to the space $\mathcal{S}_m$. We denote the set of all these families by $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. $\square$

Example 1.1 (a family of class $\mathcal{S}$). *The Dirac family in $\mathcal{S}'(\mathbb{R}^n, \mathbb{K})$, i.e., the family $\delta := \left(\delta_y\right)_{y\in\mathbb{R}^n}$, is of class $\mathcal{S}$.* In fact, for each function $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ and for each p in $\mathbb{R}^n$, we have

$$\delta(\phi)(p) = \delta_p(\phi) = \phi(p),$$

and thus $\delta(\phi) = \phi$. So the image of the test function ϕ under the family δ is the function ϕ itself, which lies in $\mathcal{S}_n$. $\triangle$

Example 1.2 (a class of $\mathcal{S}$-families). Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear and continuous operator with respect to the natural topologies of $\mathcal{S}_n$ and $\mathcal{S}_m$ (being these topologies two Fréchet-topologies, this is equivalent to assume A be linear and continuous with respect to the topologies $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$). Let δ be the Dirac family in $\mathcal{S}'_m$ and consider the family

$$A^\vee := (\delta_p \circ A)_{p\in\mathbb{R}^m}.$$

The family $A^\vee$ is a family in $\mathcal{S}'_n$, since each $A^\vee_p$ is the composition of two linear and continuous mappings; moreover, $A^\vee$ is of class $\mathcal{S}$, in fact, for every ϕ in $\mathcal{S}_n$ and p in $\mathbb{R}^m$,

$$\begin{aligned} A^\vee(\phi)(p) &= A^\vee_p(\phi) = \\ &= \left(\delta_p \circ A\right)(\phi) = \\ &= \delta_p(A(\phi)) = \\ &= A(\phi)(p), \end{aligned}$$

so $A^{\vee}(\phi) = A(\phi)$, and it belongs to $\mathcal{S}_m$.

We shall see that every $\mathcal{S}$-family has the form considered in this example.

Example 1.3 (a family that is not of class $\mathcal{S}$). Let u be in $\mathcal{S}'_n$ and let v be the family in $\mathcal{S}'_n$ defined by $v_y = u$, for each $y \in \mathbb{R}^m$. Then, if u is different from zero, v is not of class $\mathcal{S}$. In fact, let $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ be such that $u(\phi) \neq 0$, for every $y \in \mathbb{R}^m$, we have

$$v(\phi)(y) \quad = \quad v_y(\phi) = u(\phi) \cdot 1_{(\mathbb{R}^m,\mathbb{K})}(y),$$

where, $1_{(\mathbb{R}^m,\mathbb{K})}$ is the constant $\mathbb{K}$-functional on $\mathbb{R}^m$ of value 1. Thus, $v(\phi)$ is a constant $\mathbb{K}$ -functional on $\mathbb{R}^m$ different from zero, and so it cannot be in $\mathcal{S}(\mathbb{R}^m, \mathbb{K})$. $\triangle$

The preceding example induces us to consider other classes of families in addition to the $\mathcal{S}$-families, for this reason, we give the following definition.

Definition 1.2 (family of class E). *Let m be a positive integer, E be a subspace of the space $C^0(\mathbb{R}^m, \mathbb{K})$ (of continuous functions) containing $\mathcal{S}(\mathbb{R}^m, \mathbb{K})$. If v is a family in $\mathcal{S}(\mathbb{R}^n, \mathbb{K})$ indexed by $\mathbb{R}^m$, we say that v is an E-family if, for every test function ϕ in $\mathcal{S}(\mathbb{R}^n, \mathbb{K})$, the function $v(\phi)$ is in E.*

With this new definition, the family of the above example is a $C^{\infty}(\mathbb{R}^n, \mathbb{K})$-family.

Remark. In the conditions of the above definition, let w be a Hausdorff locally convex topology on E. If the space $(\mathcal{S}(\mathbb{R}^m, \mathbb{K}))$ is continuously imbedded in (E, w) and such that $\mathcal{S}(\mathbb{R}^m, \mathbb{K})$ is w-dense in E, then, the topological dual of (E, w) is continuously imbedded in $\mathcal{S}'(\mathbb{R}^m, \mathbb{K})$.

Remark. In the conditions of the preceding remark, if (E, w) is continuously imbedded in the space $\left(C^0(\mathbb{R}^m, \mathbb{K})\right)$, then the dual of $\left(C^0(\mathbb{R}^m, \mathbb{K})\right)$ is contained in $(E, w)'$. In other terms, every mensural distribution with compact support is in $(E, w)'$ and, in particular, the Dirac family is contained in (E, w).

3. The operator generated by an $\mathcal{S}$-family

Definition 2.1 (operator generated by an $\mathcal{S}$-family). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of class $\mathcal{S}$. We call* ***operator generated by the family*** *v (or* ***associated with*** *v) the operator*

$$\widehat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi). \,\square$$

Example 2.1 (on the Dirac family). The operator on $\mathcal{S}_n$ generated by the Dirac family, i.e., by the family $\delta = (\delta_y)_{y \in \mathbb{R}^n}$, is the identity operator on $\mathcal{S}_n$. In fact, for each $y \in \mathbb{R}^n$, we have

$$\begin{aligned} \widehat{\delta}(\phi)(y) &= \delta_y(\phi) = \\ &= \phi(y) = \\ &= \mathbb{I}_{\mathcal{S}_n}(\phi)(y). \; \triangle \end{aligned}$$

The set $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is a subspace of the vector space $(s(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$. Moreover, for each $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, the operator $\widehat{v}$ is linear and the map

$$(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathrm{Hom}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$$

is an injective linear operator.

In the following we shall denote by $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ the set of all the linear and continuous operators among the two topological vector spaces $(\mathcal{S}_n)$ and $(\mathcal{S}_m)$. Moreover, consider a linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, we say that A is *transposable* if its algebraic adjoint ${}^*A : \mathcal{S}^*_m \to \mathcal{S}^*_n$ (X^* denote the algebraic adjoint of X), defined by ${}^*A(a) = a \circ A$, maps $\mathcal{S}'_m$ into $\mathcal{S}'_n$.

Theorem 2.1 (basic properties of $\mathcal{S}$-families). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of tempered distributions. Then, the following assertions hold and are equivalent:*
i) for every $a \in \mathcal{S}'_m$ the composition $u = a \circ \widehat{v}$, i.e., the functional

$$u : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(\widehat{v}(\phi)),$$

is a tempered distribution;
ii) $\widehat{v}$ is transposable;
iii) $\widehat{v}$ is $(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m))$-continuous from $\mathcal{S}_n$ to $\mathcal{S}_m$;
iv) $\widehat{v}$ is strongly-continuous from $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$.

Proof. Let us prove (i). Let $a \in \mathcal{S}'_m$ and let δ be the Dirac family in $\mathcal{S}'_m$. Since the subspace $\mathrm{span}(\delta)$ is $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$-sequentially dense in $\mathcal{S}'_m$ (see [2] page 205), there is a sequence of distributions $(\alpha_k)_{k\in\mathbb{N}}$ in $\mathrm{span}(\delta)$ such that

$${}^{\sigma(\mathcal{S}'_m, \mathcal{S}_m)}\lim_{k\to+\infty} \alpha_k = a.$$

Now, since $\alpha_k \in \mathrm{span}(\delta)$ there exist a finite set $\{y_i\}_{i=1}^h$ in $\mathbb{R}^m$ and a finite set $\{\lambda_i\}_{i=1}^h$ in $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^h \lambda_i \delta_{y_i},$$

and consequently

$$\alpha_k \circ \widehat{v} = \sum_{i=1}^h \lambda_i v_{y_i}.$$

Hence, for every $k \in \mathbb{N}$, the composition $\alpha_k \circ \widehat{v}$ belongs to $\mathcal{S}'_n$.

Let τ be the topology of the pointwise convergence in $\mathrm{Hom}(\mathcal{S}_n, \mathbb{K})$, one has

$${}^{\tau}\lim_{k\to+\infty} \alpha_k \circ \widehat{v} = a \circ \widehat{v}.$$

In fact, for every ϕ in $\mathcal{S}_n$, we obtain

$$\lim_{k\to+\infty} (\alpha_k \circ \widehat{v})(\phi) = \lim_{k\to+\infty} \alpha_k (\widehat{v}(\phi)) = a(\widehat{v}(\phi)),$$

so we have that

$$(\alpha_k \circ \widehat{v})_{k\in\mathbb{N}} \xrightarrow{\tau} a \circ \widehat{v}.$$

At this point, being $(\alpha_k \circ \widehat{v})_{k\in \mathbb{N}}$ a sequence in $\mathcal{S}'_n$, then, by the Banach-Steinhauss theorem (that is true since $\mathcal{S}_n$ is barreled), we have $a \circ \widehat{v} \in \mathcal{S}'_n$. So (i) holds.

(i) is equivalent to (ii) by definition of transposable operator.

(ii) is equivalent to (iii) because the space $\mathcal{L}\left(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m)\right)$ is also the space of all the transposable linear operators from $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$ (see [3], chap. 3, § 12, Proposition 1, page 254).

(iii) is equivalent to (iv). In fact, since the space $(\mathcal{S}_n)$ is an $\mathcal{F}$-space (and then its topology coincides with the Mackey topology $\tau(\mathcal{S}_n, \mathcal{S}'_n)$), the space $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ contains the space $\mathcal{L}\left(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m)\right)$, of all the linear and $(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m))$-continuous operators from $\mathcal{S}_n$ to $\mathcal{S}_m$ (see [6] page 91, Corollary). Moreover, the space $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ is contained in the space $\mathcal{L}\left(\sigma(\mathcal{S}_n, \mathcal{S}'_n), \sigma(\mathcal{S}_m, \mathcal{S}'_m)\right)$ by the proposition 3, page 256 of [3], so the two space coincide. ■

It's, at this point, obvious that the two vector spaces $\mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ and $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ are isomorphic, being the map

$$(\cdot)^{\wedge} : \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right) \to \mathcal{L}\left(\mathcal{S}_n, \mathcal{S}_m\right) : v \mapsto \widehat{v}$$

an isomorphism, moreover, its inverse is the map

$$(\cdot)^{\vee} : \mathcal{L}\left(\mathcal{S}_n, \mathcal{S}_m\right) \to \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right) : A \mapsto A^{\vee} := \left(\delta_x \circ A\right)_{x\in\mathbb{R}^m}.$$

As we proved the above theorem, in a perfectly analogous way, it can be proved the following theorem. The $\mathcal{D}$-families in $\mathcal{D}'_n$ can be defined analogously to the $\mathcal{S}$-families and the Corollary of page 91 of [6] holds because $\mathcal{D}'_n$ is an $\mathcal{LF}$-space.

Theorem 2.2 (basic properties on $\mathcal{D}$-families). *Let $v \in \mathcal{D}\left(\mathbb{R}^m, \mathcal{D}'_n\right)$ be a family of distributions. Then, the following assertions hold and are equivalent:*

i) for every $a \in \mathcal{D}'_m$ the composition $u = a \circ \widehat{v}$, i.e., the functional

$$u : \mathcal{D}_n \to \mathbb{K} : \phi \mapsto a\left(\widehat{v}(\phi)\right),$$

is a distribution;

ii) $\widehat{v}$ is transposable;

iii) $\widehat{v}$ is $(\sigma(\mathcal{D}_n, \mathcal{D}'_n), \sigma(\mathcal{D}_m, \mathcal{D}'_m))$-continuous from $\mathcal{D}_n$ to $\mathcal{D}_m$;

iv) $\widehat{v}$ is a strongly-continuous from $(\mathcal{D}_n)$ to $(\mathcal{D}_m)$.

4. The superpositions of an $\mathcal{S}$-family

Now we can give a first generalization to the concept of linear combination.

Definition 3.1 (linear superpositions of an $\mathcal{S}$ -family). *Let $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ and $a \in \mathcal{S}'_m$. The distribution $a \circ \widehat{v} = {}^t(\widehat{v})(a)$ is called **the $\mathcal{S}$-linear superposition of v with respect to (the system of coefficients) a** and we denote it by*

$$\int_{\mathbb{R}^m} av.$$

Moreover, if $u \in \mathcal{S}'_n$ *and there exists an* $a \in \mathcal{S}'_m$ *such that*

$$u = \int_{\mathbb{R}^m} av,$$

u *is said an* $\mathcal{S}$-***linear superposition of*** v. $\square$

As a particular case, we can consider the linear superposition of v with respect to the regular distribution generated by the $\mathbb{K}$ -constant functional on $\mathbb{R}^m$ of value 1, the distribution $\left[1_{(\mathbb{R}^m,\mathbb{K})}\right]$, we denote it simply by $\int_{\mathbb{R}^m} v$, and then

$$\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} \left[1_{(\mathbb{R}^m,\mathbb{K})}\right] v.$$

Example 3.1 (the Dirac family). Let δ be the Dirac family in $\mathcal{S}'_n$. Then, for each tempered distribution $u \in \mathcal{S}'_n$,

$$\begin{aligned} \int_{\mathbb{R}^n} u\delta &= u \circ \widehat{\delta} = \\ &= u \circ \mathbb{I}_{\mathcal{S}_n} = \\ &= u, \end{aligned}$$

thus each tempered distribution is an $\mathcal{S}$-linear superposition of the Dirac family and the coefficients system is the distribution itself, these are properties of the canonical basis of $\mathbb{R}^n$. $\triangle$

An alternative definition of superposition can be obtained defining the superposition of a family of numbers (real or complex) with respect to a distributional system of coefficients.

Definition 3.2. *We say that a family of real or complex number* $x = (x_i)_{i\in\mathbb{R}^m}$ *is a family of class* $\mathcal{S}$ *if the function* $f_x : \mathbb{R}^m \to \mathbb{K}$*, defined by* $f_x(i) = x_i$*, for each* i *in* $\mathbb{R}^m$*, is a function of class* $\mathcal{S}$*. We call* f_x *the test function associated with the family* x*.*

In this conditions, we put

$$\int_{\mathbb{R}^m} a\, x := a(f_x),$$

for every tempered distribution $a \in \mathcal{S}'_m$ *, and we call the number* $\int_{\mathbb{R}^m} ax$ *superposition of the family* x *with respect to* a*.*

Introducing a notation, the relation between the two kind of superpositions is very natural.

Notation. Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$ and let v be an $\mathcal{S}$-family of tempered distributions in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. For every test function $\phi \in \mathcal{S}_n$ by the symbol $\langle v, \phi \rangle$ we denote the family of numbers defined by $\langle v, \phi \rangle_i := \langle v_i, \phi \rangle$, for every i in $\mathbb{R}^m$.

Theorem 3.1. *Let v be an $\mathcal{S}$-family of tempered distributions in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, let a be a tempered distribution in $\mathcal{S}'_m$ and let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$. Then, for every $\phi \in \mathcal{S}_n$, we have*

$$\left\langle \int_{\mathbb{R}^m} av, \phi \right\rangle = \int_{\mathbb{R}^m} a \langle v, \phi \rangle .$$

Proof. It's a straightforward computation:

$$\begin{aligned} \left\langle \int_{\mathbb{R}^m} av, \phi \right\rangle &= \left(\int_{\mathbb{R}^m} av \right)(\phi) = \\ &= a(v(\phi)) = \\ &= \int_{\mathbb{R}^m} a \left(v_i (\phi) \right)_{i \in \mathbb{R}^m} = \\ &= \int_{\mathbb{R}^m} a \langle v, \phi \rangle . \end{aligned}$$

Note that the test function associated with the family $\langle v, \phi \rangle$ is $v(\phi)$. ■

We shall see that the preceding result can be restated saying that the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$ is $\mathcal{S}$-linear in the first argument.

Definition 3.3. *The operators*

$$\int_{\mathbb{R}^m} (\cdot, \cdot) : \mathcal{S}'_m \times \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right) \to \mathcal{S}'_n : (a, v) \mapsto \int_{\mathbb{R}^m} av,$$

and

$$\int_{\mathbb{R}^m} (\cdot, v) : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto \int_{\mathbb{R}^m} av,$$

are called the ***superposition operator in*** $\mathcal{S}'_n$ *with coefficients-systems in $\mathcal{S}'_m$ and the superposition operator associated to v.*

Definition 3.4 (superpositions of an E-family). *Let E be a subspace of the space $C^0(\mathbb{R}^m, \mathbb{K})$ containing $\mathcal{S}(\mathbb{R}^m, \mathbb{K})$. Let w be a Hausdorff locally convex topology on E such that the space $(\mathcal{S}(\mathbb{R}^m, \mathbb{K}))$ is continuously imbedded in (E, w) and $\mathcal{S}(\mathbb{R}^m, \mathbb{K})$ is w-dense in E and let v be an E-family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. We define, for every distribution a in the topological dual of E, the superposition of v with respect to a as the following functional*

$$\left(\int_{\mathbb{R}^m} av \right)(\phi) := a(v(\phi)).$$

Remark. Note that, if the Dirac family of $C^{0\prime}(\mathbb{R}^m, \mathbb{K})$ is $\sigma(E', E)$-sequentially dense in E', then, by the Banach-Steinhaus Theorem, that superposition belongs to $\mathcal{S}'_n$.

Example 3.2. Consider a C^0-family v in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, and consider a distribution a in $\mathcal{S}'_m$ generated by a measure with compact support K. We can consider the superposition $\int_{\mathbb{R}^m} av$. Let m be the Borel-measure generating a, then we have

$$\begin{aligned}\left(\int_{\mathbb{R}^m} av\right)(\phi) &= a(v(\phi)) = \\ &= [m]\,(v(\phi)) = \\ &= \int_{\mathbb{R}^m} v\,(\phi)\;dm = \\ &= \int_K v\,(\phi)\;dm.\end{aligned}$$

5. Some basic properties of superpositions

The aim of this section is to show that the basic properties of the superpositions extend the basic ones of linear combinations. First we prove a property of the superpositions analogous to the following property: $\sum \delta_{(i,\cdot)} v = \sum_{j=1}^{k} \delta_{ij} v_j = v_i$, where $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is the Kronecker's delta and v is a finite family of vectors.

Theorem 4.1 (selection property of Dirac's distributions). *Let $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$. Then, for each $p \in \mathbb{R}^m$,*

$$\int_{\mathbb{R}^m} \delta_p v = v_p.$$

Proof. For every $\phi \in \mathcal{S}_n$ and every index p,

$$\begin{aligned}\delta_p\left(\widehat{v}\,(\phi)\right) &= \delta_p\left(v\,(\phi)\right) = \\ &= v\,(\phi)\,(p) = \\ &= v_p\,(\phi)\,,\end{aligned}$$

and consequently

$$\int_{\mathbb{R}^m} \delta_p v = \delta_p \circ \widehat{v} = v_p. \;\blacksquare$$

In the sense of the above theorem, the Dirac family is a continuous version of the Kronecker delta. Now we shall see that the superposition operators are bilinear operators.

Proposition 4.1 (bi-homogeneity). *Let $a \in \mathcal{S}'_m$, $\lambda \in \mathbb{K}$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then,*

$$\int_{\mathbb{R}^m} (\lambda a) v = \lambda \int_{\mathbb{R}^m} av = \int_{\mathbb{R}^m} a\,(\lambda v)\,.$$

Proof. For any $\phi \in \mathcal{S}_n$,

$$\begin{aligned}\left(\int_{\mathbb{R}^m} (\lambda a) v\right)(\phi) &= (\lambda a)\left(\widehat{v}\,(\phi)\right) = \\ &= \lambda a\left(\widehat{v}\,(\phi)\right) = \\ &= \lambda\left(\int_{\mathbb{R}^m} av\right)(\phi)\,,\end{aligned}$$

and

$$
\begin{aligned}
\left(\int_{\mathbb{R}^m} a(\lambda v)\right)(\phi) &= a\left(\widehat{\lambda v}(\phi)\right) = \\
&= a(\lambda \widehat{v}(\phi)) = \\
&= \lambda a(\widehat{v}(\phi)) = \\
&= \lambda\left(\int_{\mathbb{R}^m} av\right)(\phi). \blacksquare
\end{aligned}
$$

Proposition 4.2 (bi-additivity). *Let $k \in \mathbb{N}$, $(v)_{i=1}^{n}$ be a family in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, $a = (a_i)_{i=1}^{k}$ be a finite sequence in $\mathcal{S}'_m$ and $b \in \mathcal{S}'_m$. Then*

$$\int_{\mathbb{R}^m}\left(\sum_{i=1}^{k} a_i\right) v = \sum_{i=1}^{k}\int_{\mathbb{R}^m} a_i v, \quad \int_{\mathbb{R}^m} b\sum_{i=1}^{k} v_i = \sum_{i=1}^{k}\int_{\mathbb{R}^m} b v_i.$$

Proof. It follows immediately by the basic properties of the transpose of a linear operator, but we see it. For any $\phi \in \mathcal{S}_n$, one has

$$
\begin{aligned}
\int_{\mathbb{R}^m}\sum_{i=1}^{k} a_i v &= \left(\sum_{i=1}^{k} a_i\right)\circ \widehat{v} = \\
&= \sum_{i=1}^{k} a_i \circ \widehat{v} = \\
&= \sum_{i=1}^{k}\int_{\mathbb{R}^m} a_i v(\phi).
\end{aligned}
$$

For the second relation we have

$$
\begin{aligned}
\int_{\mathbb{R}^m} b\sum_{i=1}^{k} v_i &= b \circ \left(\sum_{i=1}^{k} v_i\right)^{\wedge} = \\
&= \sum_{i=1}^{k} b \circ \widehat{v_i} = \\
&= \sum_{i=1}^{k}\int_{\mathbb{R}^m} b v_i. \blacksquare
\end{aligned}
$$

We shall prove a theorem that gives an infinite-continuous version of the additivity expressed by the above proposition. But we first shall introduce the concept of superposition of an $\mathcal{S}$-family with respect to a family of distributions.

Theorem 4.2 (about finite linear combinations). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let δ be the Dirac family in $\mathcal{S}'_m$. Then, a tempered distribution u belongs to* $\mathrm{span}(v)$ *if and only if there* *exists a distribution* Λ *in* span(δ) *such that*

$$u = \int_{\mathbb{R}^m} \Lambda v.$$

Proof. If the condition holds u is a finite combination of v by the selection property of Dirac distributions. Viceversa, let u be a finite linear combination of v, then there exist $k \in \mathbb{N}$, a finite sequence $\lambda \in \mathbb{C}^k$ and $\alpha \in (\mathbb{R}^m)^k$ such that $u = \sum_{i=1}^k \lambda_i v_{\alpha_i}$. Put $\Lambda = \sum_{i=1}^k \lambda_i \delta_{\alpha_i}$, then, one has

$$\begin{aligned}
\int_{\mathbb{R}^m} \Lambda v &= \int_{\mathbb{R}^m} \left(\sum_{i=1}^k \lambda_i \delta_{\alpha_i} \right) v = \\
&= \sum_{i=1}^k \int_{\mathbb{R}^m} \lambda_i \delta_{\alpha_i} v = \\
&= \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v = \\
&= \sum_{i=1}^k \lambda_i v_{\alpha_i} = \\
&= u. \blacksquare
\end{aligned}$$

Now we pass to properties of continuity. Note that

$$\int_{\mathbb{R}^m} (\cdot, v) = {}^t\widehat{v}$$

so the operator $\int_{\mathbb{R}^m}(\cdot,\cdot)$ is continuous in the first argument, with respect to the pairs of topologies $(\beta(\mathcal{S}'_m, \mathcal{S}_m), \beta(\mathcal{S}'_n, \mathcal{S}_n))$ and $(\sigma(\mathcal{S}'_m, \mathcal{S}_m), \sigma(\mathcal{S}'_n, \mathcal{S}_n))$ (see [3] Corollary, page 256). For example we can state the following.

Theorem 4.3 (about denumerable linear combinations). *Let* $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. *Assume a series* $\sum (a_i)_{i=1}^\infty$ *is convergent in* $\mathcal{S}'_m$, *with respect to* $\beta(\mathcal{S}'_m, \mathcal{S}_m)$. *Then, the series* $\sum \left(\int_{\mathbb{R}^m} a_i v \right)_{i=1}^\infty$ *converges in* $\mathcal{S}'_n$, *with respect to* $\beta(\mathcal{S}'_n, \mathcal{S}_n)$ *and*

$$^{\beta(\mathcal{S}'_m,\mathcal{S}_m)} \sum_{i=1}^\infty \int_{\mathbb{R}^m} a_i v = \int_{\mathbb{R}^m} {}^{\beta(\mathcal{S}'_m,\mathcal{S}_m)} \sum_{i=1}^\infty a_i v.$$

In particular, if δ *is the Dirac family of* $\mathcal{S}'_m$ *and the series* $\sum (\delta_{p_i})_{i=1}^\infty$ *is* $\beta(\mathcal{S}'_m, \mathcal{S}_m)$*-convergent in* $\mathcal{S}'_m$. *Then, the series* $\sum (v_{p_i})_{i=1}^\infty$ $\beta(\mathcal{S}'_n, \mathcal{S}_n)$*-converges in* $\mathcal{S}'_n$ *and*

$$^{\beta(\mathcal{S}'_n,\mathcal{S}_n)} \sum_{i=1}^\infty v_{p_i} = \int_{\mathbb{R}^m} {}^{\beta(\mathcal{S}'_m,\mathcal{S}_m)} \sum_{i=1}^\infty \delta_{p_i} v.$$

6. Properties of $\mathcal{S}$-linearity of the superpositions

In this section we generalize the linearity of the operator of superposition. To this end we have to introduce the concept of superposition of a family with respect to a family.

Definition 5.1 (superposition of a family with respect to a family). *Let v be a family in $\mathcal{S}'_m$ indexed by $\mathbb{R}^k$ and let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The family in $\mathcal{S}'_n$ defined by*

$$\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w \right)_{p \in \mathbb{R}^k},$$

is called ***the superposition of*** w ***with respect to*** v. $\square$

Theorem 5.1 ($\mathcal{S}$-linearity of the $\mathcal{S}$-linear combinations). *Let $a \in \mathcal{S}'_k$, $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_m\right)$ and $w \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ be two families of distributions. Then, the family*

$$\int_{\mathbb{R}^m} vw$$

is an $\mathcal{S}$-family, and, concerning the associated operator, we have

$$\left(\int_{\mathbb{R}^m} vw \right)^{\wedge} = \widehat{v} \circ \widehat{w}.$$

Moreover, the following $\mathcal{S}$-linearity property holds

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} av \right) w = \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} vw \right).$$

Proof. For every $\phi \in \mathcal{S}'_n$ and for every index p,

$$\begin{aligned}
\left(\int_{\mathbb{R}^m} vw \right)(\phi)(p) &= \left(\int_{\mathbb{R}^m} vw \right)_p (\phi) = \\
&= \left(\int_{\mathbb{R}^m} v_p w \right)(\phi) = \\
&= v_p \left(\widehat{w}(\phi) \right) = \\
&= \widehat{v} \left(\widehat{w}(\phi) \right)(p) = \\
&= (\widehat{v} \circ \widehat{w})(\phi)(p).
\end{aligned}$$

Then the superposition $\int_{\mathbb{R}^m} vw$ is an $\mathcal{S}$-family, and $\left(\int_{\mathbb{R}^m} vw\right)^\wedge = \widehat{v} \circ \widehat{w}$. Moreover, for every $\phi \in \mathcal{S}'_n$, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^k} av\right) w\right)(\phi) &= \left(\int_{\mathbb{R}^m} av\right)(\widehat{w}(\phi)) = \\
&= a(\widehat{v}(\widehat{w}(\phi))) = \\
&= a((\widehat{v} \circ \widehat{w})(\phi)) = \\
&= a\left(\left(\int_{\mathbb{R}^m} vw\right)^\wedge (\phi)\right) = \\
&= \int_{\mathbb{R}^k} a\left(\int_{\mathbb{R}^m} vw\right)(\phi). \ \blacksquare
\end{aligned}$$

7. The generalized distributive laws

Now, we generalize the two distributive laws of the space $\mathcal{S}'_n$.

Let u be in $\mathcal{S}'_n$ and let v be the family in $\mathcal{S}'_n$ defined by $v_y := u$, for every y in $\mathbb{R}^m$. We have already seen that v is a smooth family (it sends $\mathcal{S}$-test functions to smooth functions, and it is also bounded, in the sense that it sends $\mathcal{S}$-function to smooth and bounded function). Then, we can consider, for every tempered distribution a in $\mathcal{S}'_m$ with compact support (and more generally, being v smooth and bounded, when a is a summable distribution) the superposition $\int_{\mathbb{R}^m} av$.

We shall generalize firstly the following distributive law

$$\sum_{i=1}^m (a_i u) = \left(\sum_{i=1}^m a_i\right) u.$$

In fact, in the above assumptions, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^m} av\right)(\phi) &= a(v(\phi)) = \\
&= a(u(\phi)1_{(\mathbb{R}^m,\mathbb{K})}) = \\
&= a(1_{(\mathbb{R}^m,\mathbb{K})})u(\phi) = \\
&= \left(\left(\int_{\mathbb{R}^m} ad\mu\right) u\right)(\phi),
\end{aligned}$$

for every test function ϕ in $\mathcal{S}_n$, where $1_{(\mathbb{R}^m,\mathbb{K})}$ is the constant functional from $\mathbb{R}^m$ to $\mathbb{K}$ of value 1; so we proved that

$$\int_{\mathbb{R}^m} av = \left(\int_{\mathbb{R}^m} ad\mu\right) u,$$

where the integral of a summable distribution is defined, as usual (see [4]), by

$$\int_{\mathbb{R}^m} ad\mu := a(1_{(\mathbb{R}^m,\mathbb{K})}).$$

Let us see the other distribution law:

$$\sum_{i=1}^{m} av_i = a \sum_{i=1}^{m} v_i.$$

Let k be a real or complex number, with $k_{(\mathbb{R}^m,\mathbb{K})}$ we shall denote the constant functional of value k on $\mathbb{R}^m$, (in this case, the constant distribution is that of coefficients); consequently, the generated distribution is denoted by $\left[k_{(\mathbb{R}^m,\mathbb{K})}\right]$. Let v be a smooth and bounded family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. We have,

$$\begin{aligned}\left(\int_{\mathbb{R}^m} \left[k_{(\mathbb{R}^m,\mathbb{K})}\right] v\right)(\phi) &= \left[k_{(\mathbb{R}^m,\mathbb{K})}\right](v(\phi)) = \\ &= k\left[1_{(\mathbb{R}^m,\mathbb{K})}\right](v(\phi)) = \\ &= k\left(\int_{\mathbb{R}^m} v\right)(\phi),\end{aligned}$$

for every test function ϕ in $\mathcal{S}_n$, i.e.,

$$\int_{\mathbb{R}^m} \left[k_{(\mathbb{R}^m,\mathbb{K})}\right] v = k \int_{\mathbb{R}^m} v.$$

We conclude noting that the euristic origin of the algebra exposed in the paper is [7]; moreover, properties showed in this paper already found several applications in many other papers (see [8], ..., [16]).

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