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ON THE SET OF COADIONS OF AN HOPF ALGEBRA ON K-ALGEBRAS

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ABSTRACT. For a K-Hopf algebra H, for a K-algebra A, K a field, we study when the set of coactions of H on A is closed with respect to the product of two coactions.

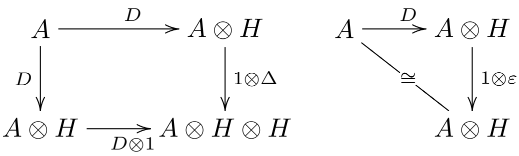
Introduction

Let A be a commutative K-algebra and H be a K-Hopf algebra, K a field. A coaction D of H on A is a morphism of K-algebras such that $(1\otimes\varepsilon)D = 1$, $(1\otimes\Delta)D = (\Delta\otimes 1)D$, where ε and Δ are the counity and the comultiplication maps of H. We can extend D to an endomorfism $\widetilde{D}$ of $A \otimes H$ putting $\widetilde{D} : A \otimes H \rightarrow H \otimes A$ and such that $D(a \otimes 1) = D(a), \forall a \in A$. In this way we can consider the product $\widetilde{D^2}\widetilde{D^1}$ of $\widetilde{D^1}$ and $\widetilde{D^2}$ coming from two coaction D^1 and D^2 of H on A. We ask, if we consider the restriction of $\widetilde{D^2}\widetilde{D^1}$ to A, when we obtain a coaction D of H on A, in other words, if the product of two coactions is again a coaction. The answer is positive if D^1 and D^2 satisfy a permutability condition that we are going to define in the paper. Classically, the result was established by H. Matsumura [1], if H is the I-adic completion of a finitely generated Hopf algebra with augumentation ideal I, described by a formal group. In this case coactions are differentiations of the K-algebra A (see Refs. [2], [3], and [4] for the n-dimensional case). As a consequence, if H is a finite dimensional (co)commutative Hopf algebra on a separably closed field K, by the structure of finite and connected group schemes (see Ref. [5], Sec. 14.4) and thanks to Oort and Munford [6], our result is true since any coaction of H on A is an action of abelian formal groups.

1. Coactions

Let H be a Hopf algebra over a field K, with comultiplication $\Delta : H \rightarrow H \otimes_K H$, antipode $s : H \rightarrow H$ and counity $\varepsilon : H \rightarrow K$. We recall:

Definition 1.1. *Let H be a K-Hopf algebra and A be a K-algebra. A (right) coaction of H on A is a K-algebras morphism $D : A \rightarrow A \otimes H$ such that:*
1) $(1 \otimes \varepsilon)D = 1$
2) $(D \otimes 1)D = (1 \otimes \Delta)D$



(1) (2)

Notations for coactions.
Given a coaction of H on a K-algebra A and $a \in A$, we can write

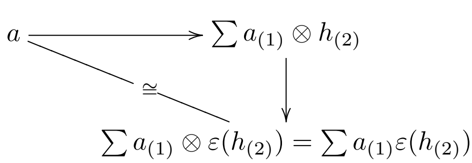
$$D(a) = \sum_{i=1}^{n} a_{1i} \otimes h_{2i} \quad a_{1i} \in A, \quad h_{2i} \in H.$$

Rewriting formally as $D(a) = \sum_{(a)} a_{(1)} \otimes h_{(2)}$, by the diagram (1), we have:
$(\Delta \otimes 1_H)D(a) = (1 \otimes \Delta)D(a)$
$(D \otimes 1_H)(\sum_{(a)} a_{(1)} \otimes h_{(2)}) = (1 \otimes \Delta)(\sum_{(a)} a_{(1)} \otimes h_{(2)})$
$\sum_{(a)} D(a_{(1)}) \otimes h_{(2)} = \sum_{(a)} a_{(1)} \otimes \Delta(h_{(2)})$

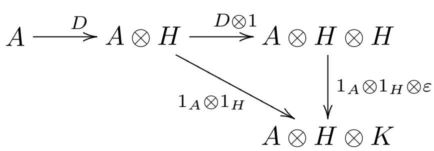
We call this element of $A \otimes H$ by $\sum_{(a)} a_{(1)} \otimes h_{(2)} \otimes h_{(3)}$.
In general, we define:
$D_1 = D$, $D_2 = (D \otimes 1_h)D_1 = (1 \otimes \Delta)D_1$
$D_n = (D \otimes \underbrace{1_H \otimes \cdots \otimes 1_H}_{n-1})D_{n-1} = (\underbrace{1 \otimes \cdots \otimes 1}_{n-1} \otimes \Delta)D_{n-1}$
and we write: $\Delta_n(a) = \sum_{(a)} a_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n+1)}$

Proposition 1.1. *(Equalities for coactions).*
1) *For all* $a \in A$, $a = \sum_{(a)} a_{(1)}\varepsilon(h_{(2)})$.
2) *For all* $a \in A$, $D(a) = \sum_{(a)} D(a_{(1)}) \otimes \varepsilon(h_{(2)}) = \sum_{(a)} D(a_{(1)})\varepsilon(h_{(2)})$ *(by 1).*
3) $D(a) = \sum_{(a)} D(a_{(1)}) \otimes \varepsilon(h_{(2)})h_{(3)} = \sum_{(a)} D(a_{(1)}) \otimes h_{(2)}\varepsilon(h_{(3)})$.
4) *For all* $a \in A$, $a = \sum_{(a)} D(a_{(1)}) \otimes \varepsilon(h_{(2)})\varepsilon(h_{(3)})$.

Proof:
1) By diagram (2)



2) Write now $D(a) = \sum_{(a)} D(a_{(1)}) \otimes \varepsilon(h^{a}_{(2)})$.
3) By the diagram



4) By the diagram

$$\begin{array}{ccc} A & \xrightarrow{D} & A\otimes H \\ & \searrow^{\cong} & \downarrow{\scriptstyle 1\otimes\varepsilon} \\ & & A \end{array}$$

Proposition 1.2. 1) *For any coaction D, for any $a \in A$*

$$D(a) = a \otimes 1_H + \sum_{(a)} a'_{(1)} \otimes h'_{(2)},\ h'_{(2)} \in Ker(\varepsilon)$$

2) *The coaction $D : A \to A \otimes H$ is a an injective map.*
3) *Any coaction $D : A \to A \otimes H$ can be uniquely extended to an endomorphism of $A \otimes H$.*

Proof:
1) From the K-modules isomorphism, $H = K1_H \oplus Ker(\varepsilon)$,
$D(a) = \sum_{(a)} a_{(1)} \otimes h_{(2)} = \sum_{(a)} a_{(1)} \otimes (k_{(2)}1_H + h'_{(2)})$, $h'_{(2)} \in Ker(\varepsilon)$, $k_{(2)} \in K$
$D(a) = \sum_{(a)} a_{(1)} \otimes k_{(2)}1_H + \sum_{(a)} a_{(1)} \otimes h'_{(2)}$ (1).
By the counity diagram, we obtain:

$$\begin{array}{ccc} a & \longrightarrow & D(a) = \sum_{(a)} a_{(1)} \otimes k_{(2)}1_H + \sum_{(a)} a_{(1)} \otimes h'_{(2)} \\ \Big|\cong & \swarrow & \\ \sum_{(a)} a_{(1)} \otimes k_{(2)}1_H = a & & \end{array}$$

From (1), we have:

$$D(a) = a \otimes 1_H + \sum_{(a)} a_{(1)} \otimes h'_{(2)})$$

2) Let $a \neq a'$ and $D(a) = D(a')$. Then:
$\sum_{(a)} a_{(1)} \otimes h_{(2)} = \sum_{(a)} a'_{(1)} \otimes h'_{(2)}$
$(1 \otimes \varepsilon) \sum_{(a)} a_{(1)} \otimes h_{(2)} = (1 \otimes \varepsilon) \sum_{(a)} a'_{(1)} \otimes h'_{(2)}$
$\sum_{(a)} a_{(1)} \otimes \varepsilon(h_{(2)}) = \sum_{(a)} a'_{(1)} \otimes \varepsilon(h'_{(2)})$
Then $a = a'$.
3) Namely, we define $\widetilde{D} : A \otimes H \to A \otimes H$, $\widetilde{D} = (1_H \otimes \mu_H)(D \otimes 1_H)$

$$\begin{array}{ccc} A\otimes H & \xrightarrow{D\otimes 1_H} & A\otimes H\otimes H \\ & \searrow^{\widetilde{D}} & \downarrow{\scriptstyle 1_A\otimes\mu_H} \\ & & A\otimes H \end{array}$$

We consider the restriction $\widetilde{D}_{/A}$ of $\widetilde{D}$.We prove that $\widetilde{D}_{/A} = D$.
$\widetilde{D}(a \otimes 1) = (1_A \otimes \mu_H)(D \otimes 1_H)(\otimes 1) = (1_A \otimes \mu_H)(Da \otimes 1_H)(\otimes 1) =$

$(1_A \otimes \mu_H)(\sum_{(a)} a_{(1)} \otimes h_{(2)} \otimes 1_H) = \sum_{(a)} a_{(1)} \otimes h_{(2)}$

Remark 1.1. *In general $\widetilde{D}$ is not an automorphism of $A \otimes H$.*

2. On the product of two coactions

Let $\widetilde{D^2}\widetilde{D^1} : A \otimes H \to A \otimes H$, where D^1 and D^2 are two coactions of H on A.

Definition 2.1. *(Permutability condition).*
We say that D^1 and D^2commute if

i) $(D^2 \otimes 1_H)D^1 = (D^1 \otimes 1_H)D^2$, *or* $\forall a \in A$

ii) $\sum_{(a)} \sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h_{(2)}^{(2,D^1_{(1)}(a))} h_{(2)}^{(1,a)} =$

$= \sum_{(a)} \sum_{D^2_{(1)}(a)} D^1_{(1)}(D^2_{(1)}(a)) \otimes h_{(2)}^{(1,D^2_{(1)}(a))} h_{(2)}^{(2,a)}$

Theorem 2.1. *Let D^1 and D^2 be two coactions. Suppose that they satisfy i) or ii) of the definition* 2.1. *Then their product is a coaction.*

Proof:
1) *Counity:*
$(\widetilde{D^2}\widetilde{D^1})/A$ is a coaction. We have to verify that $(1 \otimes \varepsilon)(\widetilde{D^2}\widetilde{D^1})(a \otimes 1) = a$, $\forall a \in A$. It results:
$(1 \otimes \varepsilon)(\widetilde{D^2}\widetilde{D^1})(a \otimes 1) = (1 \otimes \varepsilon)\widetilde{D^2}(1_A \otimes \mu_H)(D^1(a) \otimes 1) =$
$= (1 \otimes \varepsilon)\widetilde{D^2}(1_A \otimes \mu_H)(\sum_{(a)} D^1_{(1)}(a) \otimes h_{(2)}^{(1,a)} \otimes 1) =$
$= (1 \otimes \varepsilon)\widetilde{D^2}(\sum_{(a)} D^1_{(1)}(a) \otimes h_{(2)}^{(1,a)}) =$
$= (1 \otimes \varepsilon)(1_A \otimes \mu_H)(\sum_{(a)} \sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h_{(2)}^{(2,D^1_{(1)}(a))} h_{(2)}^{(1,a)}) =$
$= (1 \otimes \varepsilon)(\sum_{(a)} \sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h_{(2)}^{(2,D^1_{(1)}(a))} h_{(2)}^{(1,a)}) =$
$= \sum_{(a)} \sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a))\varepsilon(h_{(2)}^{(1,a)})\varepsilon(h_{(2)}^{(2,D^1_{(1)}(a))})$
But D^2 is a coaction, hence:

$$\sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a))\varepsilon(h_{(2)}^{(2,D^1_{(1)}(a))}) = D^1_1(a)$$

Then we have:

$$\sum_{(a)} D^1_{(1)}(a))\varepsilon(h_{(2)}^{(1,a)}) = a$$

2) *Coassociativity:*
By the diagram (1), we have to prove that $(1 \otimes \Delta)\widetilde{D^2}\widetilde{D^1}(a \otimes 1) = (\widetilde{D^2}\widetilde{D^1} \otimes 1_H)\widetilde{D^2}\widetilde{D^1}(a)$.
1) $(1 \otimes \Delta)\widetilde{D^2}\widetilde{D^1}(a \otimes 1) = (1 \otimes \Delta)\widetilde{D^2}(1_A \otimes \mu_H)(D^1(a) \otimes 1) =$
$= (1 \otimes \Delta)\widetilde{D^2}(1_A \otimes \mu_H)(\sum_{(a)} D^1_{(1)}(a)h_{(2)}^{(1,a)} \otimes 1) =$
$= (1 \otimes \Delta)\widetilde{D^2}(\sum_{(a)} D^1_{(1)}(a)h_{(2)}^{(1,a)}) =$

$= (1 \otimes \Delta)(1_A \otimes \mu_H)(\sum_{D^1_{(1)}(a)} \sum_{(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h^{(2,D^1_{(1)}(a))}_{(2)} \otimes h^{(1,a)}_{(2)}) =$
$= (1 \otimes \Delta) \sum_{D^1_{(1)}(a)} \sum_{(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h^{(2,D^1_{(1)}(a))}_{(2)} h^{(1,a)}_{(2)} =$
$= \sum_{D^1_{(1)}(a)} \sum_{(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes \Delta(h^{(2,D^1_{(1)}(a))}_{(2)})\Delta(h^{(1,a)}_{(2)}).$

2)$(\widetilde{D^2}\widetilde{D^1} \otimes 1_H)\widetilde{D^2}\widetilde{D^1}(a)$=
=$(\widetilde{D^2}\widetilde{D^1} \otimes 1_H)(\sum_{D^1_{(1)}(a)} \sum_{(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h^{(2,D^1_{(1)}(a))}_{(2)} h^{(1,a)}_{(2)}) =$
$= \widetilde{D^2}\widetilde{D^1}(\sum_{D^1_{(1)}(a)} \sum_{(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h^{(2,D^1_{(1)}(a))}_{(2)} h^{(1,a)}_{(2)})$

Since D^1 and D^2 commute:

$\widetilde{D^2}\widetilde{D^1}(\sum_{D^2_{(1)}(a)} \sum_{(a)} D^1_{(1)}(D^2_{(1)}(a)) \otimes h^{(1,D^2_{(1)}(a))}_{(2)} h^{(2,a)}_{(2)}) =$
$= \widetilde{D^2}(1_A \otimes \mu_H)(D^1(\sum_{D^2_{(1)}(a)} \sum_{(a)} D^1_{(1)}(D^2_{(1)}(a)) \otimes h^{(1,D^2_{(1)}(a))}_{(2)} h^{(2,a)}_{(2)})) =$
$= \widetilde{D^2}(1_A \otimes \mu_H)(\sum_{D^1_{(1)}(D^2_{(1)}(a))} \sum_{D^2_{(1)}(a)} \sum_{(a)} D^1_{(1)}(D^1_{(1)} D^2_{(1)}(a)) \otimes h^{(1,D^1_{(1)} D^2_{(1)}(a))}_{(2)} \otimes$
$h^{(1,D^2_{(1)}(a))}_{(2)} h^{(2,a)}_{(2)})$

But D^1 is a coaction, then:

$\widetilde{D^2}(1_A \otimes \mu_H)(\sum_{D^2_{(1)}(a)} \sum_{(a)} D^1_{(1)}(D^2_{(1)}(a)) \otimes \Delta h^{(1,D^2_{(1)}(a))}_{(2)} (h^{(2,a)}_{(2)} \otimes 1) =$
$= (1_A \otimes \mu_H)[D^2(\sum_{D^2_{(1)}(a)} D^1_{(1)}(D^2_{(1)}(a))) \otimes h^{(1,D^2_{(1)}(a))}_{(2)} \otimes h^{(1,D^1_{(1)} D^2_{(1)}(a))}_{(2)} (h^{(2,a)}_{(2)} \otimes 1)] =$
$= (1_A \otimes \mu_H)[D^2(\sum_{D^2_{(1)}(a)} D^1_{(1)}(D^2_{(1)}(a))) \otimes h^{(1,D^2_{(1)}(a))}_{(2)} h^{(2,a)}_{(2)} \otimes h^{(1,D^1_{(1)} D^2_{(1)}(a))}_{(2)}] =$
$= (1_A \otimes \mu_H)$
$[D^2(\sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a))) \otimes h^{(2,D^1_{(1)}(a))}_{(2)} h^{(1,a)}_{(2)} \otimes h^{(2,D^2_{(1)} D^1_{(1)}(a))}_{(2)}) h^{(2,D^2_{(1)} D^2_{(1)}(a))}_{(2)}] =$
$= \sum_{D^2_{(1)} D^1_{(1)}(a)} D^1_{(1)} D^2_{(1)}(D^1_{(1)}(a)) \otimes h^{(2,D^1_{(1)}(a))}_{(2)} h^{(2,D^2_{(1)} D^1_{(1)}(a))}_{(2)} (h^{(1,a)}_{(2)} \otimes 1)$
Since D^2 is a coaction, we have

$\sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes \Delta h^{(2,D^1_{(1)}(a))}_{(2)} (h^{(1,a)}_{(2)} \otimes h^{(2,D^1_{(1)}(a))}_{(2)})$=
$\sum_{D^1_{(1)}(a)} D^2_{(1)}(D^1_{(1)}(a)) \otimes \Delta h^{(2,D^1_{(1)}(a))}_{(2)} \Delta h^{(1,a)}_{(2)}.$

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