BUFFON TYPE PROBLEMS WITH MULTIPLE INTERSECTIONS 
FOR REGULAR LATTICES

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ABSTRACT. In this paper we study Buffon type problems with multiple intersections for 
lattices of equilateral triangles and a circle as test body.

1. Introduction

In papers [1] and [2] A. Duma and M. Stoka studied Buffon type problems with mul-
tiple intersections for lattices of the Euclidian plane \( \mathbb{E}_2 \), with a parallelogram \( \mathcal{P} \) and an 
equilateral triangle \( \tau \), elementary tile respectively, and a segment of constant length.

In this paper, we consider two lattices, \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), with the same fundamental cell, an 
equilateral triangle \( \tau \) of side \( a \). Hence, we determine the probability of multiple intersec-
tions of the test body, a circle of constant radius \( r \) with the sides of the lattices \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), 
respectively.

2. Geometric probability of multiple intersections for the lattice \( \mathcal{R}_1 \)

Considering the lattice \( \mathcal{R}_1 \), we denote by \( p_{1i} \), \( (i = 1, 2, \ldots, 6) \) the probability that the 
body test intersects the sides of the lattice \( i \)-times.

**Theorem 1.** If \( r < a \frac{\sqrt{3}}{6} \), the probabilities that a circle \( C \) of constant radius \( r \), uniformly 
distributed in a bounded region of the plane, intersects, the sides of the lattice \( \mathcal{R}_1 \), \( i \)-times, 
\( (i=1, \ldots, 6) \), are respectively

\[
\begin{align*}
p_{11} &= p_{13} = p_{15} = 0, \\
p_{12} &= \frac{12}{\sqrt{3} a} \left( \frac{r}{a} \right)^2, \\
p_{14} &= p_{16} = 4 \frac{r^2}{a^2}.
\end{align*}
\]
Figure 1.

Proof. We denote by $\mathcal{M}$ the set of circles of radius $r$ with center $C(x, y)$ which belongs to $\tau$. Then, as in figure 2, the test body intersects the sides of the triangles $\tau$ $i$-times, $(i=1, \ldots, 6)$, (i.e. the sides of the lattice $R_1$), if and only if, $C \in \tau_i$, $i = 1, \ldots, 6$. Hence, putting $\mathcal{N}_i = \{(x, y) \in \tau_1\}, (i=1, \ldots, 6)$, we have

$$p_{1i} = \frac{\mu(\mathcal{N}_i)}{\mu(\mathcal{M})}, (i = 1, \ldots, 6),$$

where $\mu$ is the Lebesgue measure in the Euclidean plane. We can compute the previous measures using the elementary kinematic measure of Poincaré ([4])

$$dK = dx \wedge dy \wedge d\varphi;$$

where $x$ and $y$ are the coordinates of the center of $C$ (or the components of a translation) and $\varphi$ is the angle of rotation. We have

$$\mu(\mathcal{M}) = area \tau = \frac{a^2 \sqrt{3}}{4}.$$
We compute $\mu(\mathcal{N}_{12})$ observing that $\tau_{12}$ is union of three congruent trapeziums with bases of lengths $a - \frac{4r}{\sqrt{3}}$ and $a - \frac{6r}{\sqrt{3}}$ and height $r$. Then
\[ \mu(\mathcal{N}_{12}) = \text{area } \tau_{12} = 3ar - \frac{15}{\sqrt{3}}r^2. \] (6)

The sets $\tau_{14}$ and $\tau_{16}$ are the union of three congruent equilateral triangles of side $\frac{2r}{\sqrt{3}}$, therefore
\[ \mu(\mathcal{N}_{14}) = \mu(\mathcal{N}_{16}) = \text{area } \tau_{14} = \text{area } \tau_{16} = r^2\sqrt{3}. \] (7)

We observe that the circle never intersects once, three times or five times the sides of the lattice $\mathcal{R}_1$. From formulas (4), (5), (6) and (7) we have the probabilities (1), (2) and (3).}

Theorem 2. If $r < a\frac{\sqrt{3}}{6}$, the probabilities that a circle $C$ of constant radius $r$, uniformly distributed in a bounded region of the plane, intersects, $i$ sides $(i=1,\ldots,6)$, of the lattice $\mathcal{R}_1$ are respectively

\begin{figure}
\centering
\includegraphics{figure3}
\caption{Figure 3.}
\end{figure}

\begin{align}
\rho_{11} &= \frac{12}{\sqrt{3}} \frac{r}{a} - 20 \left( \frac{r}{a} \right)^2, \\
\rho_{12} &= 4 \frac{r^2}{a^2} \\
\rho_{13} &= \left( 4 - \frac{2\pi}{\sqrt{3}} \right) \left( \frac{r}{a} \right)^2 \\
\rho_{14} &= \rho_{15} = 0 \\
\rho_{16} &= \frac{2\pi}{\sqrt{3}} \frac{r^2}{a^2}.
\end{align}

\textit{Proof.} With the same notations as in the previous theorem, as in Figure 3, the test body intersects \(i\) sides (\(i=1, \ldots, 6\)) of the lattice \(R_1\) if, and only if, \(C \in \tau_{l_1}i, i = 1, \ldots, 6\). Hence, putting \(N_{1i} = \{(x, y) \in \tau_{l_1}i\}, (i=1, \ldots, 6)\), we have

\[\rho_{1i} = \frac{\mu(N_{1i})}{\mu(M)}, \ (i = 1, \ldots, 6).\]

We compute \(\mu(N_{13})\) observing that \(\tau_{l_1}3\) is the union of three congruent surfaces, given as the difference between the area of an equilateral triangle of side \(\frac{2r}{\sqrt{3}}\) and the area of the
circular sector of radius \( r \) and angle \( \frac{\pi}{3} \), therefore
\[
\mu(\mathcal{N}_{13}) = \text{area } \tau_{l13} = r^2 \sqrt{3} - \frac{\pi}{2} r^2. \tag{14}
\]
The sets \( \tau_{l16} \) are the union of three congruent circular sectors of radius \( r \) and angle \( \frac{\pi}{3} \). Then
\[
\mu(\mathcal{N}_{16}) = \text{area } \tau_{l16} = \frac{\pi}{2} r^2. \tag{15}
\]
We observe that the circle never intersects four or five sides of the lattice \( \mathcal{R}_1 \), hence (11) follows. Since \( p_{l11} = p_{l12} = p_{l14} \) and from formulas (13), (14) and (15), we have the probabilities (8), (9), (10), (11) and (12).

**Corollary 3.** The probability that a circle \( \mathcal{C} \) of constant radius \( r < \frac{a \sqrt{3}}{6} \) intersects one of the sides of the lattice \( \mathcal{R}_1 \) is
\[
p = \frac{12}{\sqrt{3}} \alpha - 12 \left( \frac{r}{a} \right)^2. \tag{16}
\]

**Proof.** Taking into account that \( p = p_{l11} + p_{l12} + p_{l13} + p_{l14} + p_{l15} + p_{l16} \), formulas (8), (9), (10), (11) and (12) give the probability (16). \( \square \)

**Remark** Applying formula
\[
p_{3; a, \alpha} = 4 \left( \frac{1 + \cos \alpha}{\sin \alpha} \right) \left( \frac{r}{a} \right)^2 - 4 \left( \frac{1 + \cos \alpha}{\sin \alpha} \right)^2 \left( \frac{r}{a} \right)^2 \tag{17}
\]
of the probability that a circle of constant radius \( r \), uniformly distributed in a bounded region of the plane, intersects a straight line of the lattice \( \mathcal{R}_{3; a, \alpha} \) of lines, having an isosceles triangle as elementary tile with basis of length \( a \) and angles \( \alpha, \alpha \) and \( \pi - 2\alpha \), [3] with \( \alpha = \frac{\pi}{3} \), we obtain (16).

### 3. Geometric probability of multiple intersections for the lattice \( \mathcal{R}_2 \)

Now we consider the lattice \( \mathcal{R}_2 \) and we denote by \( p_{2i} \), \( i=1,2,3,4 \) the probability that the test body intersects the sides of the lattice \( i \)-times.

**Theorem 4.** If \( r < \frac{a \sqrt{3}}{6} \), the probabilities that a circle \( \mathcal{C} \) of constant radius \( r \), uniformly distributed in a bounded region of the plane, intersects the sides of the lattice \( \mathcal{R}_2 \), \( i \)-times, \( (i=1,2,3,4) \), are respectively
\[
p_{21} = p_{23} = 0, \tag{18}
\]
\[
p_{22} = 4 \sqrt{3} \frac{r}{a} - \left( 20 + \frac{2 \sqrt{3} \pi}{3} \right) \left( \frac{r}{a} \right)^2 \tag{19}
\]
\[
p_{24} = \left( 8 + \frac{2 \pi \sqrt{3}}{3} \right) \left( \frac{r}{a} \right)^2 \tag{20}
\]

**Proof.** With the same notations as theorem 1, as in figure 4, the test body intersects the sides of the lattice \( \mathcal{R}_2 \) \( i \)-times \( (i=1,\ldots,4) \) if, and only if, \( C \in \tau_{2i} \), \( i = 1, \ldots, 4 \). We compute \( \mu(\mathcal{N}_{22}) \) observing that \( \tau_{22} \) is the union of two congruent trapeziums with bases of lengths \( a - \frac{4r}{\sqrt{3}} \) and \( a - \frac{6r}{\sqrt{3}} \) and height \( r \) and a surface given as the difference between the area...
of a trapezium with bases of lengths $a - \frac{4r}{\sqrt{3}}$ and $a - \frac{6r}{\sqrt{3}}$ and height $r$, and the area of a semicircle of radius $r$ and angle $\frac{\pi}{3}$. Then

$$\mu(\mathcal{N}_{22}) = \text{area } \tau_{22} = 3ar - \frac{15}{\sqrt{3}}r^2 - \frac{\pi}{2}r^2. \quad (21)$$

The sets $\tau_{24}$ is the union of three congruent rhombs of side $\frac{2r}{\sqrt{3}}$ and a semicircle of radius $r$. Then

$$\mu(\mathcal{N}_{24}) = \text{area } \tau_{24} = \left(2\sqrt{3} + \frac{\pi}{2}\right)r^2. \quad (22)$$

We observe that the circle never intersects the sides of the lattice $\mathcal{R}_2$ once or three times. From formulas (4), (5), (21) and (22) we have the probabilities (18), (19) and (20).

Considering the lattice $\mathcal{R}_2$, we denote by $p_{l_{2i}}$, ($i=1,2,3,4$) the probability that the test body intersects $i$ sides of the lattice.

**Theorem 5.** If $r < a\frac{\sqrt{3}}{6}$, the probabilities that a circle $C$ of constant radius $r$, uniformly distributed in a bounded region of the plane, intersects, $i$ sides ($i=1,2,3,4$) of the lattice.
Buffon type problems . . .

$R_2$, are respectively

$$pl_{21} = 4\sqrt{3} \frac{r}{a} - \left(20 + \frac{2\sqrt{3}\pi}{3}\right) \left(\frac{r}{a}\right)^2,$$

(23)

$$pl_{22} = \left(8 - \frac{2\pi\sqrt{3}}{3}\right) \left(\frac{r}{a}\right)^2,$$

(24)

$$pl_{23} = 0,$$

(25)

$$pl_{24} = \frac{4\sqrt{3}\pi}{3} \left(\frac{r}{a}\right)^2.$$

(26)

**Proof.** With the same notations as theorem 1, as in figure 5, the test body intersects $i$ sides ($i=1,\ldots,4$) of the lattice $R_2$ if, and only if, $C \in \tau l_{2i}$, $i = 1, \ldots, 4$. We compute $\mu(N_{2i})$ observing that $\tau l_{21}$ is equal to $\tau l_{22}$. The set $\tau l_{22}$ is the union of six congruent equilateral triangles of sides $\frac{2r}{\sqrt{3}}$ minus three circular sector of radius $r$ and angle $\frac{\pi}{3}$. Then

$$\mu(N_{22}) = \text{area } \tau l_{22} = \left(2\sqrt{3} - \frac{\pi}{2}\right) r^2.$$

(27)
Finally, the set \( \tau_{l_{24}} \) is the union of three congruent circular sectors of radius \( r \) and angle \( \frac{\pi}{3} \), and the area of a semicircle of radius \( r \). Therefore

\[
\mu(N_{24}) = \text{area } \tau_{l_{24}} = \pi r^2.
\]  

(28)

We observe that the circle never intersects the sides of the lattice \( \mathcal{R}_2 \) once or three times. From formulas (4), (5), (27) and (28) we have the probabilities (23), (24), (25) and (26).

\[\square\]

**Corollary 6.** The probability that a circle \( C \) of constant radius \( r < a \sqrt[3]{ \frac{3}{6} } \) intersects one of the sides of the lattice \( \mathcal{R}_2 \) is

\[
p = \frac{12}{\sqrt{3} a} r - 12 \left( \frac{r}{a} \right)^2.
\]  

(29)

**Proof.** Taking into account that \( p = pl_{11} + pl_{12} + pl_{13} + pl_{16} \), formulas (23), (24) and (26) give the probability (29).

\[\square\]

**References**


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