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QUALITATIVE ANALYSIS BEHAVIOUR OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL SYSTEMS

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ABSTRACT. By using two-parametric scale of increasing functions the qualitative analysis of the behaviour of nonlinear impulsive differential systems solutions are investigated.

1. Introduction

In the papers [1-5] were investigated qualitative characteristics of solutions of impulsive systems of differential equations

$$\frac{dx}{dt} = A(t)x + r(t,x),\ t \neq t_i(x), \Delta x|_{t=t_i(x)} = B_i x + J_i(x), \tag{1}$$

with linear approximation

$$\frac{dx}{dt} = A(t)x,\ t \neq t_i(x), \Delta x|_{t=t_i(x)} = B_i x \tag{2}$$

and their generalizations in [6-10]. In [7-19] systems (1) was studied under such assumptions :

$$||r(t,x)|| \leqslant \overline{r}(t)||x||^{\alpha^*}, \tag{3}$$

$$||J_i(x)|| \leqslant\ k_i||x||^{\beta}. \tag{4}$$

The conditions of stability, practical stability, attraction of solutions of (1) with some nonlinearity index α^*, β, have been obtained. The case of system (1) in which nonlinearities $r(t,x)$ and $J_i(x)$ satisfy the following conditions:

$$0 < \alpha^* < 1, 0 < \beta < 1,\ or\ \alpha^* > 1, \beta > 1, if\ \alpha^* \neq \beta \tag{5}$$

is already open. This paper is devoted to solve this question. At first we study system (1) with general conditions (3), (4), (5), (g). In section 2 we obtain new analogy Bihari's result for discontinuous functions (Lemma). By using the result of Lemma in section 3 we obtain new conditions of boundedness stability, attracting, practical stability of solutions of the system (1).

2. Setting of the problem and preliminary results

By using the results and definitions in [5],[9] let us consider system (1) with such assumptions:

$a)$ functions $f(t,x) = A(t)x + r(t,x)$ and $I_i(x) = B_i x + J_i(x)$ are defined in the domain $\Omega = \{(t,x) : t \in J = [t_0, T_0],\ T_0 \leqslant \infty,\ t_0 \geqslant 1,\ ||x|| \leqslant h\}$ and $\exists a = const > 0$ $: ||A(t)|| \leqslant a, \forall t \in J$;
$b)$ nonlinearities $r(t,x)$ and $J_i(x)$ satisfy conditions (3), (4) and $\overline{r}(t) \geqslant 0, \overline{r} \in C(R^+), \exists r = const : \overline{r}(t) \leqslant r < \infty, k_i = const \geqslant 0, i \in N, (0 < \alpha^* < 1, 0 < \beta \leqslant 1)$ or $(\alpha^* > 1, \beta \geqslant 1)$;
$c)$ $\exists \theta = const > 0 : \inf\limits_{||x|| \leqslant h} t_i(x) - \sup\limits_{||x|| \leqslant h} t_{i-1}(x) = \theta > 0,\ \forall i \in N$;
$d)$ if $x(t) = x(t, t_0, x_0)$ is the solution of Cauchy problem for system (1) and t_i^* : $t_i^* = x(t) \bigcap (t = t_i(x)),\ \inf\limits_{||x|| \leqslant h} t_i^*(x) \geqslant t_i^* \geqslant \sup\limits_{||x|| \leqslant h} t_{i-1}(x),\ \forall i \in N$;
$e)$ $\exists \theta_i = const > 0 : \theta_1 \tau < i(t, t+\tau) < \theta_2 \tau;\ t_0 < t_1^* < t_2^* < ..., \lim\limits_{i \to \infty} t_i^* = \infty$; where $i(a,b)$ is the number of points $t_i^* \in [a,b] \subset [t_0, T_0]$, ($\theta_j$ depends only on $\tau, j = 1,2$);
$f)$ $\exists L = const > 0 : ||\frac{\partial t_i(x)}{\partial x}|| \leqslant L\ \forall i \in N,\ x \in \Omega,\ \sup\limits_{0 \leqslant \sigma \leqslant 1} \left(\frac{\partial t_i(x + \sigma I_i(x))}{\partial x}, I_i(x) \right) \leqslant 0, L < \frac{1}{(ah^{1-\alpha^*} + r)h^{\alpha^*}}$;
$g)$Cauchy matrix $C(t,t_0)$ of shorted system

$$\frac{dx}{dt} = A(t)x,\ t \neq t_i^*,\ \Delta x = B_i x,\ t = t_i^* \tag{6}$$

satisfies such estimate $||C(t,t_0)|| \leqslant c\, exp[(\tilde{\alpha} + \theta_i \ln \alpha)(t - t_0)] \left[\frac{t}{t_0}\right]^{\tilde{\beta}}$, where $\tilde{\alpha}$ is parameter of characteristic index of Lyapunov nontrivial solution of the system $\frac{dx}{dt} = A(t)x$, $\tilde{\beta}$ is a parameter connected with characteristic degree of Lyapunov of this system. Moreover,

$\alpha^2 = \max\limits_j \lambda_j[(B_j + E)^T (B_j + E)], \theta_i \ln \alpha = \begin{cases} \theta_1 \ln \alpha, & 0 < \alpha < 1; \\ \theta_2 \ln \alpha, & \alpha > 1, \end{cases}$

$det(B_i + E) \neq 0, \forall i \in N$.

Remark 1. As $\alpha^* = 1$ *and* $\beta > 0$ system (1) is investigated in [7-19]; for $\alpha^* > 0$ *and* $\beta = 1$ in [1-5].
Condition f) guarantees that solutions of system (1) with conditions (3), (4) are not "beating" about hypersurfaces $t_i(x)$.

In order to estimate solutions of system (1) we use next result:

Lemma. Let a nonnegative, piecewise-continuous on $J = [t_0, \infty[$ function $W(t)$, with 1-st kind discontinuities in the points $\{t_i\} : t_1 < t_2 < ..., \lim\limits_{i \to \infty} t_i = \infty$, satisfy the integro-sum inequality:

$$W(t) \leq \phi(t) + \int_{t_0}^{t} p(\tau) W^m(\tau) d\tau + \sum_{t_0 < t_i < t} \beta_i W^n(t_i - 0), \tag{7}$$

where $\phi(t) > 0, p(t) \geq 0, p \in C(J), \beta_i \geqslant 0, m, n > 0, m \neq 1$, $\phi(t)$ nondecreasing on J.
Then

$$W(t) \leq \phi(t) \prod_{t_0<t_i<t} \left(1+\beta_i \phi^{n-1}(t_i)\right) \left[1+(1-m)\int_{t_0}^{t} \phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}},$$

$$0<m<1, 0<n\leqslant 1 \ \ \forall t\in J; \tag{8}$$

$$W(t) \leq \phi(t) \prod_{t_0<t_i<t} \left(1+\beta_i m \phi^{n-1}(t_i)\right) \left[1-(m-1)\left[\prod_{t_0<t_i<t} \left(1+\beta_i m\phi^{n-1}(t_i)\right)\right]^{m-1} \times \right.$$

$$\left. \times \int_{t_0}^{t} p(\tau)\phi^{m-1}(\tau)d\tau\right]^{-\frac{1}{m-1}}, m>1, n\geqslant 1 \tag{9}$$

$$\forall t \in J:$$

$$\int_{t_0}^{t} p(\tau)\phi^{m-1}(\tau)d\tau \leqslant \frac{1}{m},$$

$$\prod_{t_0<t_i<t} \left(1+\beta_i m\phi^{n-1}(t_i)\right) < \left(\frac{m}{m-1}\right)^{\frac{1}{m-1}}. \tag{10}$$

Proof. . Let us consider the interval $[t_0, t_1[$; inequality (7) reduces itself to:

$$\forall t \in [t_0, t_1[\quad W(t) \leq \phi(t) + \int_{t_0}^{t} p(\tau)W^m(\tau)d\tau.$$

from which, thanks to (8), it follows:

$$\forall t \in [t_0, t_1[\quad \frac{W(t)}{\phi(t)} \leq 1 + \int_{t_0}^{t} \frac{p(\tau)W^m(\tau)}{\phi(t)}d\tau \leq 1 + \int_{t_0}^{t} \frac{p(\tau)W^m(\tau)}{\phi(\tau)}d\tau =$$
$$= 1 + \int_{t_0}^{t} p(\tau)\phi^{m-1}(\tau)\left[\frac{W(\tau)}{\phi(\tau)}\right]^m d\tau.$$

Let it be $u(t) = \frac{W(t)}{\phi(t)}$. Hence by using Bihari lemma for inequality (9), we have:

$$u(t) \leq \left[1+(1-m)\int_{t_0}^{t} \phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}, 0<m<1,$$

$$u(t) \leq \left[1-(m-1)\int_{t_0}^{t} \phi^{m-1}(\tau)p(\tau)d\tau\right]^{-\frac{1}{m-1}}, m>1,$$

$\forall t \in [t_0, t_1[$ such that

$$\int_{t_0}^{t} p(\tau)\phi^{m-1}(\tau)d\tau \leq \frac{1}{m} < \frac{1}{m-1}.$$

Then inequalities (5)-(7) are justified in the interval $[t_0,t_1[$. Passing to $[t_1,t_2[$ we study only the case $0<m<1, 0<n\leqslant 1$ (the proof is analogous when $m>1, n\geqslant 1$). From the inequality

$$u(t)\le 1+\int_{t_0}^{t}p(\tau)\phi^{m-1}(\tau)u^m(\tau)d\tau+\beta_1\phi^{n-1}(t_1-0)u^n(t_1-0),$$

since $\phi(t_k-0)<\phi(t_k), \forall k=1,2,...$, we get:

$$u(t)\le 1+\int_{t_0}^{t_1}p(\tau)\phi^{m-1}(\tau)\left[1+(1-m)\int_{t_0}^{\tau}\phi^{m-1}(\sigma)p(\sigma)d\sigma\right]^{\frac{m}{1-m}}d\tau+$$

$$+\beta_1\phi^{n-1}(t_1)\left[1+(1-m)\int_{t_0}^{t_1}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{n}{1-m}}+\int_{t_1}^{t}p(\tau)\phi^{m-1}(\tau)u^m(\tau)d\tau=$$

$$=\left[1+(1-m)\int_{t_0}^{t_1}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}+\beta_1\phi^{n-1}(t_1)\times$$

$$\times\left[1+(1-m)\int_{t_0}^{t_1}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{n}{1-m}}+$$

$$+\int_{t_1}^{t}p(\tau)\phi^{m-1}(\tau)u^m(\tau)d\tau\le\left(1+\beta_1\phi^{n-1}(t_1)\right)\times$$

$$\times\left[1+(1-m)\int_{t_0}^{t_1}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}+$$

$$+\int_{t_1}^{t}p(\tau)\phi^{m-1}(\tau)u^m(\tau)d\tau\Rightarrow$$

$$u(t)\le\left\{\left(1+\beta_1\phi^{n-1}(t_1)\right)^{1-m}\left[1+(1-m)\int_{t_0}^{t_1}\phi^{m-1}(\tau)p(\tau)d\tau\right]+\right.$$

$$\left.+(1-m)\int_{t_1}^{t}\phi^{m-1}(\tau)p(\tau)d\tau\right\}^{\frac{1}{1-m}}\le$$

$$\le\left(1+\beta_1\phi^{n-1}(t_1)\right)\left[1+(1-m)\int_{t_0}^{t}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}.$$

By used the scheme described above for the arbitrary interval $[t_k,t_{k+1}[$, we have

$$u(t)\le 1+\sum_{i=1}^{k}\beta_i\phi^{n-1}(t_i)\left[\prod_{j=1}^{i-1}\left(1+\beta_j\phi^{n-1}(t_j)\right)\right]^n\times$$

$$\times\left[1+(1-m)\int_{t_0}^{t}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{n}{1-m}}+$$

$$+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} p(\tau)\phi^{m-1}(\tau)\left[\prod_{j=1}^{i-1}\left(1+\beta_j\phi^{n-1}(t_j)\right)\right]^m \times$$

$$\times\left[1+(1-m)\int_{t_0}^{\tau}\phi^{m-1}(\sigma)p(\sigma)d\sigma\right]^{\frac{n}{1-m}} d\tau+$$

$$+\int_{t_k}^{t} p(\tau)\phi^{m-1}(\tau)u^m(\tau)d\tau \le \prod_{i=1}^{k}\left(1+\beta_i\phi^{n-1}(t_i)\right)\times$$

$$\times\left[1+(1-m)\int_{t_0}^{t_k}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}+\int_{t_k}^{t} p(\tau)\phi^{m-1}(\tau)u^m(\tau)d\tau$$

from which it follows thet:

$$u(t)\le\left\{\left[\prod_{i=1}^{k}\left(1+\beta_i\phi^{n-1}(t_i)\right)\right]^{1-m}\left[1+(1-m)\int_{t_0}^{t_k}\phi^{m-1}(\tau)p(\tau)d\tau\right]+\right.$$

$$\left.+(1-m)\int_{t_k}^{t}\phi^{m-1}(\tau)p(\tau)d\tau\right\}^{\frac{1}{1-m}}\le\prod_{i=1}^{k}\left(1+\beta_i\phi^{n-1}(t_i)\right)\times$$

$$\times\left[1+(1-m)\int_{t_0}^{t}\phi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}.$$

Taking into account that $u(t)=\frac{W(t)}{\phi(t)}$, we obtained the required result. □

Remark 2. Lemma is a new analogy of Bellman-Bihari result for integral inequalities with discontinuous functions. From the result of lemma, in particular case, we obtain such classical results:

a) If $\varphi(t)=c=const>0,\ n=1$ lemma coincides with lemma by S. D. Borysenko, Kiev (Preprint) N 82-35 Acad of Sc. Ukr. SSR, Inst Matematiki, 1982; (see also S. D. Borysenko Ukr. Math. Journ. 35(2) 1983)(Later we write: lemma ⇒ lemma [·](or theorem N));

b) If $n=1$ lemma ⇒ Theorem 3.1.2 (Monograph [19]);

c) If $m=n$ lemma ⇒ lemma 2 [1], Proposition 2.11 in [16]; Theorem 2.2.2 ($\varphi(\tau(s))=\varphi(s)$) [19], [13]; Lemma 1 ($\varphi(\tau(s))=\varphi(s)$) Proposition 2.13 [16].

3. Main results

In the present section we obtain following new conditions: A) of boundedness of solutions of system (1) by using the property of boundedness of shorted system (2); B) stability by Lyapunov, attracting and practical stability (uniformly, attractive) by Chetaev of trivial solution of system (1).

Under conditions (3), (4) and Hölder type $r(t,x), J_i(x)$ $(0<\alpha^*<1, 0<\beta<1)$, we can not guarantee the uniqueness of the solution of Cauchy problem $x(t,t_0,x_0)$ connected to (1).Therefore, in qualitative analysis of the behavior of solutions in Theorems 1-3 we point out on the issues of boundedness and practical stability $(T_0<\infty)$.

About the case $\alpha^*>1, \beta>1$, qualitative analysis of the behaviour of solutions are obtained in Theorems 4-6.

Theorem 1. *Let assumptions a)-g) be valid and following conditions be fulfilled:*

$a^1)$ $\bar{\alpha}+\theta_i\ln\alpha=\tilde{\beta}=0;\ 0<\alpha^*<1,\ 0<\beta<1$;

$b^1)$ $\exists m_1(t_0)=const>0:\prod\limits_{t_0<t_i^*<t}\big(1+c^\beta k_i||x_0||^{\beta-1}\big)\leqslant\big(1+m_1(t_0)||x_0||^{\beta-1}\big), \forall t\in J=[t_0,T_0]$;

$c^1)$ $\exists m_2(t_0)=const>0:\int\limits_{t_0}^{t}\bar{r}(\tau)d\tau\leqslant m_2(t_0)<\infty\ , \forall t\in J$;

$d^1)$ $c\big(1+m_1(t_0)\lambda^{\beta-1}\big)[1+(1-\alpha^*)c^{\alpha^*}\lambda^{\alpha^*-1}m_2(t_0)]^{\frac{1}{1-\alpha^*}}<\frac{\Lambda}{\lambda}$;

$e^1)$ $\exists m_3(t_0)=const>0:||x_0||\big(1+m_1(t_0)||x_0||^{\beta-1}\big)\leqslant m_3(t_0)||x_0||^\beta$;

$f^1)$ $\lambda<\Big[\Lambda(cm_3(t_0)[1+(1-\alpha^*)c^{\alpha^*}\lambda^{\alpha^*-1}m_2(t_0)]^{\frac{1}{1-\alpha^*}})^{-1}\Big]^{\frac{1}{\beta}}$.

Then:

I) $a^2)$ *all solutions of (1) are bounded in* Ω, *if conditions* $a^1)-c^1)$ *hold ;*

II) Trivial solution (t.s.) of (1) is

$b^2)$ *practical stable (p.s.) relative to* (λ,Λ,J), *if conditions* $a^1)-d^1)$, *or* $a^1)-c^1), e^1), f^1)$ *hold;*

$c^2)$ *is uniformly practical stable (u.p.s.) relative to* t_0, *if condition* $b^2)$ *holds and* $m_i(t_0)(i=1,2,3)$ *are independent of* t_0.

Proof. . The proving that phenomena "beating" of the solutions of (1) on $t_i(x)$ is absent guarantees the conditions $c), f)$. By using $d), e)$ let us consider t_i^* be instants when solution $x(t,t_0,x_0)$ contacts the surface $t_i(x)$. It is obvious to see that $x(t,t_0,x_0)$ satisfies the system of differential equations

$\frac{dx}{dt}=A(t)x(t)+r(t,x),\ t\neq t_i^*,$

$\triangle x|_{t=t_i^*}=B_ix+J_i(x).$

Then for

$$t \in J : x(t,t_0,x_0) = C(t,t_0)x_0 + \int\limits_{t_0}^{t} C(t,\tau)r(\tau,x(\tau,t_o,x_o))d\tau +$$
$$+ \sum_{t_0<t_i^*<t} C(t,t_i^*)J_i(x(t_i^*-0,t_0,x_0)) \Rightarrow ||x(t,t_0,x_0)|| \leqslant$$
$$\leqslant ||C(t,t_0)||||x_0|| + \int\limits_{t_0}^{t} ||C(t,\tau)||||r(\tau,x(\tau,t_o,x_o))||d\tau +$$
$$+ \sum_{t_0<t_i^*<t} ||C(t,t_i^*)||||J_i(x(t_i^*-0,t_0,x_0))|| \leqslant$$
$$\leqslant c\left\{ ||x_0|| + \int\limits_{t_0}^{t} \bar{r}(\tau)||x(\tau,t_o,x_o)||^{\alpha^*}d\tau + \sum_{t_0<t_i^*<t} k_i||x(t_i^*-0,t_0,x_0)||^{\beta} \right\}.$$

Then Lemma, allows us to obtain:

$$||x(t)|| \leqslant c||x_0|| \prod_{t_0<t_i^*<t}\left(1+c^{\beta}k_i||x_0||^{\beta-1}\right)\left[1+(1-\alpha^*)\int\limits_{t_0}^{t}\bar{r}(\tau)c^{\alpha^*}||x_0||^{\alpha^*-1}d\tau\right]^{\frac{1}{1-\alpha^*}}. \tag{11}$$

Statement $a^2)$ follows from assumptions $a^1) - c^1)$ and (11); statement $b^2)$ is a consequence of $a^1) - d^1)$ and (11) (or $a^1) - c^1), e^1), f^1)$). Statement $c^2)$ follows from $b^2)$ and (11). □

Theorem 2. *Let following conditions be valid:*

$a^3)$ *assumptions* $a) - g)$ *hold;*

$b^3)$ $\bar{\alpha} + \theta_i \ln\alpha = 0,\ \tilde{\beta} < 0,\ 0 < \alpha^* < 1,\ 0 < \beta < 1$;

$c^3)$ $\exists m_4(t_0) = const > 0 :$
$$\prod_{t_0<t_i^*<t}\left\{1 + c^{\beta}\left[\frac{t_i^*}{t_0}\right]^{\tilde{\beta}(\beta-1)} k_i||x_0||^{\beta-1}\right\} \leqslant \left(1 + m_4(t_0)||x_0||^{\beta-1}\right), \forall t \in J;$$

$d^3)$ $\exists m_5(t_0) = const > 0 : \int\limits_{t_0}^{t}\left[\frac{\tau}{t_0}\right]^{\tilde{\beta}(\beta-1)}\bar{r}(\tau)d\tau \leqslant m_5(t_0) < \infty\ , \forall t \in J;$

$e^3)$ $c\big(1 + m_4(t_0)\lambda^{\beta-1}\big)[1 + (1-\alpha^*)c^{\alpha^*}\lambda^{\alpha^*-1}m_5(t_0)]^{\frac{1}{1-\alpha^*}} < \frac{\Lambda}{\lambda}$;

$f^3)$ $\exists m_6(t_0) = const > 0 : ||x_0||\big(1 + m_4(t_0)||x_0||^{\beta-1}\big) \leqslant m_6(t_0)||x_0||^{\beta}$;

$g^3)$ $\lambda < \big(\Lambda(cm_6(t_0)[1 + (1-\alpha^*)c^{\alpha^*}\lambda^{\alpha^*-1}m_5(t_0)]^{\frac{1}{1-\alpha^*}})^{-1}\big)^{\frac{1}{\beta}}$.

Then ((t.s.) of (1) is:

$a^4)$ (λ,Λ,J)*-stable; moreover it is attractive practical stable (a.p.s.) relative to* $(\lambda,\Lambda,\Lambda^*,J)$*, where* $\lambda < \Lambda^* < \Lambda$ *if only* $a^3) - e^3)$ *or* $a^3) - c^3), f^3), g^3)$ *hold ;*

$a^5)$ *(u.p.s.) relative to* (λ,Λ,J)*, if only* m_4, m_5, m_6 *are independent of* t_0*; moreover, it is attractive (u.p.s.) relative to* $(\lambda,\Lambda,\Lambda^*,J)$.

Proof. Analogously to previous theorem, it is easy to see that

$$||x(t,t_0,x_0)|| \leqslant c\left\{\left[\frac{t}{t_0}\right]^{\tilde\beta}||x_0|| + \int\limits_{t_0}^{t}\left[\frac{t}{\tau}\right]^{\tilde\beta}\bar r(\tau)||x(\tau,t_o,x_o)||^{\alpha^*}d\tau + \sum_{t_0<t_i^*<t}\left[\frac{t}{t_i^*}\right]^{\tilde\beta}||x(t_i^*-\right.$$

$$\left.0,t_0,x_0)||^{\beta}\right\} \Rightarrow w(t) \leqslant c\left\{||x_0||t_0^{-\tilde\beta} + \int\limits_{t_0}^{t}\tau^{\tilde\beta(\alpha^*-1)}(w(\tau))^{\alpha^*}\bar r(\tau)d\tau + \right.$$

$$\left. + \sum_{t_0<t_i^*<t}k_i(t_i^*)^{\tilde\beta(\beta-1)}\left[w(t_i^*)\right]^{\beta}\right\} \Longrightarrow$$

$$w(t) \leqslant c||x_0||t_0^{-\tilde\beta}\prod_{t_0<t_i^*<t}\left\{1 + c^{\beta}\left[\frac{t_i^*}{t_0}\right]^{\tilde\beta(\beta-1)}k_i||x_0||^{\beta-1}\right\}\times$$

$$\times\left[1 + (1-\alpha^*)\int\limits_{t_0}^{t}\bar r(\tau)c^{\alpha^*}||x_0||^{\alpha^*-1}\left[\frac{\tau}{t_0}\right]^{\tilde\beta(\alpha^*-1)}d\tau\right]^{\frac{1}{1-\alpha^*}},$$

where $w(t) \stackrel{def}{=} ||x(t,t_0,x_0)||t^{-\tilde\beta}$.

The statements $a^4), a^5)$ follow from such estimates:

$$||x(t,t_0,x_0)|| \leqslant c\left[\frac{t}{t_0}\right]^{\tilde\beta}\left(1+m_4(t_0)||x_0||^{\beta-1}\right)||x_0||\left[1+(1-\alpha^*)c^{\alpha^*}||x_0||^{\alpha^*-1}m_5(t_0)\right]^{\frac{1}{1-\alpha^*}},$$

$$||x(t,t_0,x_0)|| \leqslant c\left[\frac{t}{t_0}\right]^{\tilde\beta}m_6(t_0)||x_0||^{\beta}\left[1+(1-\alpha^*)c^{\alpha^*}||x_0||^{\alpha^*-1}m_5(t_0)\right]^{\frac{1}{1-\alpha^*}}. \quad \square$$

Theorem 3. *Let the condition $a^3)$ hold and following conditions be fullfilled:*
$b^4)$ $\tilde\alpha + \theta_i\ln\alpha < 0,\ \tilde\beta \leqslant 0,\ 0 < \alpha^* < 1,\ 0 < \beta < 1;$

$^4)$ $\exists m_7(t_0) = const > 0:$
$$D(t_0,t) \stackrel{def}{=} \prod_{t_0<t_I^*<t}\left(1 + c^{\beta}\left[\frac{t_i^*}{t_o}\right]^{\tilde\beta(\beta-1)}exp\big[(\tilde\alpha+\theta_i\ln\alpha)(\beta-1)(t_i^*-t_0)\big]||x_0||^{\beta-1}k_i\right) \leqslant$$
$\leqslant \left(1 + m_7(t_0)||x_0||^{\beta-1}\right),\ \forall t \in J;$

$d^4)$ $\exists m_8(t_0) = const > 0: \int\limits_{t_0}^{t}exp\big[(\tilde\alpha+\theta_i\ln\alpha)(\beta-1)(\tau-t_0)\big]\left[\frac{\tau}{t_0}\right]^{\tilde\beta(\beta-1)}\bar r(\tau)d\tau \leqslant$
$m_8(t_0) < \infty\ , \forall t \in J;$

$e^4)$ $c\big(1 + m_7(t_0)\lambda^{\beta-1}\big)[1 + (1-\alpha^*)c^{\alpha^*}\lambda^{\alpha^*-1}m_8(t_0)]^{\frac{1}{1-\alpha^*}} < \frac{\Lambda}{\lambda};$

$f^4)$ $\exists m_9(t_0) = const > 0: ||x_0||\big(1 + m_7(t_0)||x_0||^{\beta-1}\big) \leqslant m_9(t_0)||x_0||^{\beta};$

$g^4)$ $\lambda < \big(\Lambda(cm_9(t_0)[1 + (1-\alpha^*)c^{\alpha^*}\lambda^{\alpha^*-1}m_8(t_0)]^{\frac{1}{1-\alpha^*}})^{-1}\big)^{\frac{1}{\beta}}$

Then (t.s.) of system (1) is:
$a^6)$ (λ,Λ,J)*-stable;moreover attractive practical stable (a.p.s.) relative to* $(\lambda,\Lambda,\Lambda^*,J)$, *if only* $a) - g)$, $b^4) - e^4)$, *or* $b^4) - d^4), f^4), g^4)$ *take place;*
$a^7)$ *(u.p.s) relative to* (λ,Λ,J), *if* $a^6)$ *is valid and* m_7, m_8, m_9 *are independent of* t_0, *moreover, it is (a.u.p.s.) relative to* $(\lambda,\Lambda,\Lambda^*,J)$.

Proof. The statements of the theorem are obtained from such estimate of solutions of system (1):

$$||x(t,t_0,x_0)|| \leqslant c\left[\frac{t}{t_0}\right]^{\tilde{\beta}} \exp[(\tilde{\alpha}+\theta_i \ln\alpha)(t-t_0)]||x_0||D(t_0,t)\times$$

$$\times\left[1+(1-\alpha^*)c^{\alpha^*}||x_0||^{\alpha^*-1}\int\limits_{t_0}^{t}\exp\big[(\tilde{\alpha}+\theta_i\ln\alpha)(\beta-1)(\tau-t_0)\big]\left[\frac{\tau}{t_0}\right]^{\tilde{\beta}(\beta-1)}\bar{r}(\tau)d\tau\right]^{\frac{1}{1-\alpha^*}}. \tag{12}$$

□

Let us denote by $S(N,M)=\alpha^*c^{\alpha^*}N[||x_0||M)]^{\alpha^*-1}$. By reasoning analogously to theorems 1-3, we obtain next statements for the case $\alpha^*>1, \beta\geqslant 1$:

Theorem 4. *Let us assume that*

$a^{1*})$ $\tilde{\alpha}+\theta_i\ln\alpha=\tilde{\beta}=0,\ \alpha^*>1,\ \beta\geqslant 1;$

$b^{1*})$ *assumptions* $a)-g), c^1$ *hold;*

$c^{1*})$ $\exists m_1^*(t_0)=const>0: \prod\limits_{t_0<t_i^*<t}\left(1+c^\beta k_i||x_0||^{\beta-1}\right)\leqslant m_1^*(t_0)<f(\alpha^*), \forall t\in J;$

$d^{1*})$ $S(m_2(t_0), m_1^*(t_0))\leqslant 1;$

$e^{1*})$ $cm_1^*(t_0)\left[1-(\alpha^*-1)(m_1^*(t_0))^{\alpha^*-1}c^{\alpha^*}\lambda^{\alpha^*-1}m_2(t_0)\right]^{-\frac{1}{\alpha^*-1}}<\frac{\Lambda}{\lambda}.$

Then we have: I) all solutions of (1) are bounded in Ω*, if conditions* $a^{1*})-d^{1*})$ *hold;*

II) (t.s.) of the system (1) is:

$b^{2*})$ *stable by Lyapunov, if conditions* $a^{1*})-d^{1*})$ *hold (uniformly, if* m_1^*, m_2 *are independent of* t_0*);*

$c^{2*})$ *(u.p.s.), if* $d^{1*}), e^{1*}), b^{2*})$ *hold and* m_1^*, m_2 *are independent of* t_0.

Theorem 5. *Let us suppose that conditions* $a^3)$ *is valid, and moreover*

$a^{3*})$ $\tilde{\alpha}+\theta_i\ln\alpha=0,\ \tilde{\beta}<0,\ \alpha^*>1,\ \beta\geqslant 1;$

$b^{3*})$ $\exists m_2^*(t_0)=const>0: \prod\limits_{t_0<t_i^*<t}\left(1+c^\beta\left[\frac{t_i^*}{t_0}\right]^{\tilde{\beta}(\beta-1)}||x_0||^{\beta-1}k_i\right)\leqslant m_2^*(t_0)<f(\alpha^*), \forall t\in J;$

$c^{3*})$ $\exists m_3^*(t_0)=const>0: \int\limits_{t_0}^{t}\left[\frac{\tau}{t_0}\right]^{\tilde{\beta}(\beta-1)}\bar{r}(\tau)d\tau\leqslant m_3^*(t_0)<\infty, \forall t\in J;$

$d^{3*})$ $S(m_3^*(t_0), m_2^*(t_0))\leqslant 1;$

$e^{3*})$ $cm_2^*(t_0)\left[1-(\alpha^*-1)(m_2^*(t_0))^{\alpha^*-1}c^{\alpha^*}\lambda^{\alpha^*-1}m_3^*(t_0)\right]^{-\frac{1}{\alpha^*-1}}<\frac{\Lambda}{\lambda}.$

Then (t.s.) of system (1) is:
$i)$ *asymptotically stable by Lyapunov, only if* $a^{3*})-d^{3*})$ *take place (uniformly, if* $m_2^*(t_0), m_3^*(t_0)$ *are independent of* t_0*);*

$ii)$ *(p.s.) if only* $a^{3*})-e^{3*})$ *take place (uniformly, if* $m_2^*(t_0), m_3^*(t_0)$ *are independent of* t_0*),*

Theorem 6. *Let us assume that for system (1):*

$a^{4*})$ *assumptions* $a)-g)$*, conditions* $b^1), c^1), e^1)$ *hold;*

$b^{4*})$ $\exists m_4^*(t_0) = const > 0 : D(t_0,t) \leqslant m_4^*(t_0) < f(\alpha^*), \forall t \in J$*;*

$c^{4*})$ $\tilde{\alpha} + \theta_i \ln\alpha < 0,\ \tilde{\beta} < 0,\ \alpha^* > 1,\ \beta \geqslant 1;$

$d^{4*})$ $S(m_5^*(t_0), m_4^*(t_0)) \leqslant 1;$
$e^{4*})$ $\exists m_5^*(t_0) = const > 0 : \int\limits_{t_0}^{t} exp\big[(\tilde{\alpha}+\theta_i\ln\alpha)(\beta-1)(\tau-t_0)\big]\left[\frac{\tau}{t_0}\right]^{\tilde{\beta}(\beta-1)} \bar{r}(\tau)d\tau \leqslant m_5^*(t_0) < \infty\ \forall t \in J;$
$f^{4*})$ $cm_4^*(t_0)\left[1-(\alpha^*-1)(m_4^*(t_0))^{\alpha^*-1}c^{\alpha^*}\lambda^{\alpha^*-1}m_5^*(t_0)\right]^{-\frac{1}{\alpha^*-1}} < \frac{\Lambda}{\lambda}.$
Then (t.s.) of system (1) is:
$iii)$ *asymptotically stable by Lyapunov, if only* $a^{4*})-e^{4*})$ *take place (uniformly, if* $m_4^*(t_0),\ m_5^*(t_0)$ *are independent of* t_0*);*

$iv)$ *(u.p.s.) if* $m_4^*(t_0),\ m_5^*(t_0)$ *are independent of* t_0*, and* $iii),\ f^{4*})$ *hold.*

Remark 3. The complete qualitative analysis of system (1) with assumptions (3), (4), g), where we use two parametric scales of increasing functions is firstly considered in this article. From Theorems 1-6, in particular case (by using lemma), we obtain such classical results in the theory of Differential Systems with Impulse Influence:
d) if $\beta = 1,\ A(t) = A = const\ (n \times n)$ matrix $0 < \alpha^* < 1$ Theorems 1-3 coincide with Theorems 4.3.11-4.3.13 [19]

e) if $\beta^* = 1,\ A(t) = A,\ \alpha^* = 1$, from the results Theorems 1-3 $\Rightarrow$ Theorems 4.3.16-4.3.18 [19].

f) $\beta = 1,\ \alpha^* = 1,\ \eta(t) = e^{(\widetilde{\alpha}+\theta_i ln\alpha)t}t^{\widetilde{\beta}},\ l(t_0) = ce^{(\widetilde{\alpha}+\theta_i ln\alpha)t}t^{\widetilde{\beta}},\ t_i(x) = t_i = const :$ $t_0 < t_1 < t_2 < ...,\ \lim\limits_{i\to\infty},$ then from Theorems 1-3 $\Rightarrow$ Theorems 4.4.1, 4.4.2 [19]

g) $\beta = 1,\ \alpha^* = 1,\ A(t) = A$, from Theorems 1-3 $\Rightarrow$ Theorems 4.3.19, 4.3.20 [19]

h) If $m = 1$, then $W(t) = \varphi(t)\prod\limits_{t_0<t_i<t}(1+\beta_i\varphi^{n-1}(t_i))exp[\int\limits_{t0}^{t} p(\tau d\tau)],$ where $0 < n \leqslant 1,\ \forall t \geqslant t_0,$

$W(t) = \varphi(t) \prod\limits_{t_0<t_i<t} (1 + \beta_i \varphi^{n-1}(t_i)) exp[m \int\limits_{t0}^{t} p(\tau d\tau)]$, where $n \geqslant 1$, $\forall t \geqslant t_0$, (Yu. A. Mitropolskiy, S.D. Borysenko, S. Toscano, Reports of the National Academy of Sciences of Ukraine, N7, 2008, also Nonlinear Analysis N12, 2009). Theorems 1-6 generalize Theorems 1-6 from Nonlinear Analysis N12, 2009, where system (1) is investigated when nonlinearity $r(t, x)$ is not Lipschitz type ($\alpha^* > 0, r^* \neq 1$) (Hölder type of nonlinearity $r(t, x)$) . See also [8] (Theorems 16.3.1-16.3.6).

References

[1] D. S. Borysenko, "About one integral inequality for piece-wise continuous functions", *Proc. X Int. Kravchuk Conf.*, Kyiv, 2004, p. 323.

[2] S. Borysenko, "On some generalizations Bellman-Bihari result for integro-functional inequalities for discontinuous functions and their applications", *The Nonlinear Analysis and Applications 2009: Materials of the International scientific conference* (April 02-04th 2009, Kyiv), Kyiv: NTUU "KPI", 2009, p. 83.

[3] D. S. Borysenko, A. Gallo, and R. Toscano, "Integral inequalities Gronwall- Bellman-Bihari Type for discontinuous functions and rstimates of solutions impulsive systems", *Bulletin of the University of Kiev, Series: Physics & Mathematics* (4) (2005), 60-66.

[4] D. S. Borysenko, A. Gallo, and R. Toscano, "Integral inequalities Gronwall-Bellman-Bihari Type for discontinuous functions and estimates of solutions impulsive systems", in *Proc. DE&CAS*, Brest, 2005, 5-9.

[5] S. D. Borysenko, M. Ciarletta, and G. Iovane, "Integro-sum inequalities and motion stability of systems with impulse perturbations", *Nonlinear Analysis* **62**, 417-428 (2005).

[6] S. Borysenko and G. Iovane, "Integro-sum inequalities and qualitative analysis dynamical systems with perturbations", *Tipogr. "Elda"-Mercato*, S. Severino, 2006.

[7] S. D.Borysenko and G. Iovane, "About some integral inequalities of Wendroff type for discontinuos functions", *Nonlinear Analysis* **66**, 2190-2203 (2007).

[8] S. Borysenko and G. Iovane, *Differential Models: Dynamical Systems with Perturbations* (Aracne, Rome, 2009).

[9] S. D. Borysenko, G. Iovane, and P. Giordano, "Investigations of the properties motion for essential nonlinear systems perturbed by impulses on some hypersurfaces", *Nonlinear Analysis* **62**, 345 - 363 (2005).

[10] A. Galo and A. M. Piccirillo, *Nonlinear Analysis* **67**, 1550-1559 (2007).

[11] A. Galo and A. M. Piccirillo, *Nonlinear Analysis* **71**, 2276-2288 (2009).

[12] G. Iovane, "Some new integral inequalities of Bellman-Bihari type with delay for discontinuous functions", *Nonlinear Analysis* **66**, 498-508 (2007).

[13] G. Iovane and S. D. Borysenko, "Boundedness, stability, practical stability of motion impulsive systems", *Proc. DE&CAS*, Brest, 2005, 15-21.

[14] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of impulsive differential equations* (World Scientific, Singapore, 1989).

[15] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities. Theory and Applications* (Academy Press, New York, 1969).

[16] Yu. A. Mitropolsky, G. Iovane, and S. D.Borysenko, "About a generalization of Bellman-Bihari type inequalities for discontinuous functions and their applications", *Nonlinear Analysis* **66**, 2140-2165 (2007).

[17] J. J. Nieto, "Imresonance periodic problems of first order", *Applied Mathematics Letters* **15 (4)**, 489-493 (2002).

[18] J. J. Nieto, "Periodic boundary value problems for first-order impulsive ordinary differential equations", *Nonlinear Analysis* **51(7)**, 1223-1232 (2002).

[19] A. M.Samoilenko, S. D. Borysenko, C. Catani, G. Matarazzo, and V. Yasinsky, *Differential Models: Stability* (Naukova Dumka, Kyiv, 2001).

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