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BASIC STATEMENTS OF RELATIVITY THEORY

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ABSTRACT. Some basic statements of relativity theory, starting out with geometry and observers up to Einstein's field equations, are collected in a systematical order without any proof, to serve as a short survey of tools and results.

1. Introduction: Observer and standard coordinates

Def.: An *observer* is locally represented by meters generating position coordinates $\boldsymbol{\xi}$ and a clock generating a time coordinate ϑ. The 4-tupel $(\boldsymbol{\xi}, \vartheta)$ is called an *event*.

Def.: An observer is called *locally inertial* (IS), if there exists an *observer-invariant transformation* of its event coordinates

$$x^\alpha \;=\; f^\alpha(\boldsymbol{\xi}),\ \alpha = 1,2,3, \quad t \;=\; f^4(\boldsymbol{\xi}, \vartheta) \tag{1}$$

into new ones (x^α, t) in which a light flash generated at $(x^\alpha = 0, \wedge\alpha, t = 0)$ can be later represented by a sphere

$$(ct)^2 - \boldsymbol{x}^2 = 0. \tag{2}$$

These coordinates (x^α, t) are called *standard coordinates* generated by *standard meters* and *standard clocks*.

Proposition: The observer-invariant transformations form a group.

Conclusion: Because observer-invariant transformations form a group, all possible coordinate systems of an IS are connected by observer-invariant transformations. One special coordinate system of an IS are the standard coordinates which are only defined for locally inertial observers.

2. Special Relativity

2.1. Michelson-Morley experiment. Empirem: A first interpretation of the Michelson-Morley experiment is the statement *The velocity of light "c" in vacuum is independent of the arbitrary motion of the light source with respect to a locally inertial observer.*

Empirem: A second interpretation of the Michelson-Morley experiment is the statement: *If a light flash is generated in an IS Σ at* $\boldsymbol{x} = \mathbf{0},\ t = 0$ *which correspond to* $\boldsymbol{x}^\star = \mathbf{0},\ t^\star = 0$ *in an other IS* $\Sigma^\star$*, both the observers see a light sphere at* $t > 0, t^\star > 0$ (in standard coordinates !)

$$(ct)^2 - \boldsymbol{x}^2 = 0 \iff (ct^\star)^2 - \boldsymbol{x}^{\star 2} = 0. \tag{3}$$

Proposition: According to (2) an infinitesimal light sphere in an IS is represented by

$$(ds)^2 := \eta_{ik} dx^i dx^k \ = \ 0,\ i,k \ = \ 1,..,4, \tag{4}$$
$$x^4 := ct,\ \eta_{11} = \eta_{22} = \eta_{33} = -1,\ \eta_{44} = 1, \tag{5}$$
$$\eta_{ik} = 0,\ i \neq k, \quad \text{sign}(\eta_{ik}) = -2. \tag{6}$$

According to the second interpretation (3) of the Michelson-Morley experiment we have

$$(ds)^2 \ = \ 0 \iff (ds^\star)^2 \ = \ 0. \tag{7}$$

2.2. Lorentz transformation. Proposition: The transformation of the standard coordinates of two IS observers Σ and $\Sigma^\star$ moving to each other

$$\Sigma :\ (\boldsymbol{x}, t) \quad \iff \quad \Sigma^\star :\ (\boldsymbol{x}^\star, t^\star), \tag{8}$$
$$x^{\star a} \quad = \quad f^a(x^b), \tag{9}$$

1: does not affect the *distance element*

$$(ds)^2 \ = \ \text{const} \iff (ds^\star)^2 \ = \ \text{const}, \tag{10}$$

2: is linear, one-to-one, and called *inhomogeneous Lorentz transformation* (Poincaré group)

$$x^{\star a} \ = \ A^{a\cdot}_{\cdot b} x^b \ + \ C^a,\ \ x^a \ = \ A^{\star a \cdot}_{\cdot b} x^{\star b} \ + \ C^{\star a}, \tag{11}$$

3: is unimodular

$$\det(A^{a\cdot}_{\cdot b}) \ = \ \pm 1. \tag{12}$$

Proposition: If the relative velocity of $\Sigma^\star$ in Σ is $\boldsymbol{v} = \boldsymbol{const}$ or $\boldsymbol{v}^\star \equiv -\boldsymbol{v}$, respectively, and if we choose without any restriction of generality

$$\boldsymbol{v} \ = \ v\boldsymbol{e}_x \tag{13}$$

the x-axis into the direction of the relative velocity, and if

$$(\boldsymbol{x}, t) \ = \ (\mathbf{0}, 0) \iff (\boldsymbol{x}^\star, t^\star) \ = \ (\mathbf{0}, 0) \tag{14}$$

correspond to each other, the transformation of the standard coordinates of Σ into those of $\Sigma^\star$ is

$$x^\star \ = \ \alpha[x - vt],\ y^\star \ = \ y,\ z^\star \ = \ y, \tag{15}$$
$$ct^\star \ = \ \alpha[ct - \beta x], \tag{16}$$
$$\alpha \ = \ (1 - \beta^2)^{-1/2},\ \beta \ = \ v/c, \tag{17}$$

or in matrix formulation

$$\begin{pmatrix} x^\star \\ y^\star \\ z^\star \\ ct^\star \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & -\alpha\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha\beta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}, \tag{18}$$

and is called a *special Lorentz transformation*:

$$x^{\star k} = A^{k\cdot}_{\cdot l} x^l, \qquad k = 1,2,3,4. \tag{19}$$

2.3. Minkowski space. Def.: A 4-dimensional space spanned by the Cartesian standard coordinates $(\boldsymbol{x}, ct) \in \mathcal{M}$ and everywhere equipped with the constant metric (5) and (6) of signature -2 is called a *Minkowski space*.
According to (4) we have

$$(ds)^2 = dx_i dx^i, \quad dx_i := \eta_{ik} dx^k. \tag{20}$$

Def.: We call

$$(d\tau)^2 := \frac{1}{c^2} dx_k dx^k \tag{21}$$

the square of the differential of the *proper time* τ.

Conclusion: The differential of proper time is invariant under Lorentz transformation.

Proposition: For a resting clock is the differential of proper time equal to the differential of the coordinate time,

$$(d\boldsymbol{x})^2 = 0 \quad \to (d\tau)^2 = (dt)^2, \tag{22}$$

for a moving clock with constant velocity v in IS Σ and resting in IS $\Sigma^\star$

$$d\tau = dt^\star = dt\sqrt{1 - (v^2/c^2)} \tag{23}$$

is valid.

Def.: The 4-vector

$$u^k := \frac{dx^k}{d\tau} \tag{24}$$

is called the *4-velocity*.

Proposition: The 4-velocity is a normalized tensor of 1st order (it is covariant!)

$$u^{\star k} = A^{k\cdot}_{\cdot l} u^l, \qquad u_k u^k = c^2. \tag{25}$$

Proposition:

$$u^\alpha = \frac{dx^\alpha/dt}{\sqrt{1-\beta^2}}, \qquad \alpha = 1,2,3, \tag{26}$$

$$u^4 = \frac{c}{\sqrt{1-\beta^2}}, \qquad \beta := v/c. \tag{27}$$

2.4. Geodesic. Proposition: A motion in an locally inertial observer Σ is described by (relativistic Newton's law)

$$m_0\frac{du^k}{d\tau} \;=\; m_0\frac{d^2x^k}{d\tau^2} \;=\; K^k. \tag{28}$$

K^k is called the *imposed 4- force*. For an arbitrary observer $\Sigma^\star$ characterized by the one-to-one observer transformation $x^k = x^k(x^{\star l})$, (28) transfers to

$$m_0\frac{d^2x^{\star k}}{d\tau^2} + m_0\frac{\partial x^{\star k}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^{\star q}\partial x^{\star r}}\frac{dx^{\star q}}{d\tau}\frac{dx^{\star r}}{d\tau} \;=\; \frac{\partial x^{\star s}}{\partial x^k}K^k. \tag{29}$$

Conclusion: According to (29) we see, that to the transformed imposed forces on the right-hand side, additional forces by changing the observer arise in $\Sigma^\star$ which are not present in IS Σ.

Def.: According to (28) the force-free motion of a mass point in IS Σ is uniform and on a straight line. Transformed to (29) the motion is not uniform anymore and is not on a straight line in $\Sigma^\star$. This curve describing a *imposed force-free motion* in IS ($K^k = 0$) and being transformed to $\Sigma^\star$ is called a *geodesic*.

Proposition: Inserting (29) into the covariant differential of the 4-velocity, we obtain the *acceleration*

$$\frac{Du^{\star s}}{d\tau} = \frac{\partial x^{\star s}}{\partial x^k}\frac{K^k}{m_0} - \frac{\partial x^{\star s}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{\star t}\partial x^{\star l}}u^{\star t}u^{\star l} + \Gamma^{\star s}_{(pq)}u^{\star q}u^{\star p}. \tag{30}$$

Def.: Here, the *covariant differential* is defined by

$$DA^k \;:=\; dA^k + \Gamma^k_{rs}A^s dx^r, \tag{31}$$

with the *connexion* Γ^k_{rs}.

Def.: The symmetric part of the connexion in (30) is now chosen in such a way, that along the geodesic the 4-velocity is parallel displaced

$$\Gamma^{\star s}_{(pq)} \quad := \quad \frac{\partial x^{\star s}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{\star p}\partial x^{\star q}}, \tag{32}$$

$$\frac{Du^{\star s}}{d\tau} \quad = \quad 0, \qquad \text{(geodesic)}. \tag{33}$$

The antisymmetric part of the connexion is not determined by the geodesic equation of motion (33). Therefore the setting

$$\Gamma^{\star s}_{[pq]} \doteq 0 \tag{34}$$

is not physically induced by the geodesical motion of a mass point. It is here an additional assumption beyond physics of mass points. (32) and (34) define a *Riemann space*

3. General Relativity

3.1. Equivalence principle. Empirem: A locally inertial observer is not able to detect gravitation locally (free falling). This is due to the experimental fact, that gravitational mass and inertial mass are equal.
If an observer $\Sigma^\star$ is not locally inertial, the geodesic motion of a mass point is influenced by forces originated by the change of the observer from IS to $\Sigma^\star$. In contrast to IS the observer $\Sigma^\star$ can detect gravitational forces along the geodesic motion.

Equivalence Principle: An observer which moves accelerated with respect to an IS cannot decide, if a gravitational field or an accelerated motion of himself is present.
By the equivalence principle the *gravitational forces are "geometrized"* (other forces are not !).

3.2. Curvature. The Riemann space is not flat, as the Minkowski space is, there is a non-vanishing covariant *curvature tensor* of 4-th order R^s_{kpj}.

Proposition: In a 4-dimensional Riemann space, the covariant curvature tensor has 20 independent components.

Def.: The *Ricci tensor* and the *curvature scalar* are defined by contractions

$$R_{ip} \;:=\; R^m_{imp} \;=\; g^{mk} R_{kimp}, \tag{35}$$

$$R \;:=\; R^i_i = \; g^{ik} R_{ki}. \tag{36}$$

Proposition: In a Riemann space, the Ricci tensor is symmetric and depends on the metric and its derivatives

$$R^{pq} \;=\; F^{pq}(g_{ik}, \partial_j g_{ik}, \partial_j \partial_l g_{ik}). \tag{37}$$

3.3. Energy-momentum tensor. Definition: In special relativity theory, the symmetric *energy-momentum tensor* T^{ik} is generating the balances of energy and momentum

$$\partial_k T^{ik} \;=\; f^i. \tag{38}$$

3.4. Einstein field equations. Transfer: special $\longrightarrow$ general relativity
1: Minkowski space $\longrightarrow$ Riemann space

$$\partial_k \longrightarrow_{\|k} \tag{39}$$

2: Gravitational forces are "geometrized"

$$f^i \;=\; 0 \longrightarrow T^{kj}_{\|k} \;=\; 0 \tag{40}$$

and replaced by the metric of the curved Riemann space.
3: Except of the free choice of an observer, the energy-momentum tensor determines the metric

$$R^{ik} - \frac{1}{2} g^{ik} R - \lambda g^{ik} \;=\; \kappa T^{ik} \;\rightarrow\; T^{kj}_{\|k} \;=\; 0. \tag{41}$$

Given: T^{ik}, wanted: g^{ik}. Additionally we need initial conditions and constraints. λ is called the *cosmological constant* which is usually set to zero.

3.5. Structure of the field equations.

a) 10 partial non-linear differential equations of second order
b) Only 6 of them are independent of each other because of $T^{kj}_{\|k} = 0$
c) $\longrightarrow$ 4 of the g^{ik} can be chosen arbitrarily $\longrightarrow$ choice of a special observer
d) $\longrightarrow$ the g^{ik} cannot be determined uniquely from the field equations
e) The cosmological constant makes possible, that the empty space ($T^{ik} \equiv 0$) may be curved.

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