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# ACTION OF A HEAT SOURCE AND INFLUENCE OF INITIAL CONDITION ON THE PLANE STATE OF TEMPERATURE

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ABSTRACT. The work takes an advantage of the temperature component method for some heat conduction problems. Simplifications of the method for 2D problems are considered. Some examples of calculations are quoted.

## 1. Introduction

The temperature component method enables solving linear problems of heat conduction for homogeneous and isotropic bodies. The method required the space value problem to be formulated. Fictitious extension of the body region to the whole space leads to the space problem with known solution. The field obtained this way turn out a particular solution of the primary initial-boundary value problem. The temperature component is determined in an approximate form, enabling to satisfy the boundary conditions with imposed accuracy, and, at the same time, exactly observing the governing equations and the initial condition. Intensity values of the fictitious components located beyond the body region are determined in fictitious instants. For this purpose a bounded time interval was introduced in which the boundary condition is verified. The testing instants divide the bounded time into particular sections. The verifying algorithm of the boundary condition, which is of recursive character with regard to the testing time, is based on the properties of the fundamental solution for the parabolic operator.

The theory of heat conduction relates to a homogeneous and isotropic model of a body pro-vided that the heat sources are subject to time-changes. The thermal process is described by the temperature field: $T(\vec{(x)}, t),\ x \in D,\ t > 0$, where x determines the place with regard to the reference system, and t is the current instant. The interval $(0, \infty)$ is referred to as the time, $(0, t]$ is the current time. A homogeneous and isotropic material is described by Fourier relation

$$q_i\left(\vec{x}, t\right) = -\lambda T_{,i}\left(\vec{x}, t\right), \qquad \vec{x} \in D, \quad t > 0, \quad i = \overline{1,3} \tag{1}$$

where $q_i$ is the heat flux vector and $\lambda$ is thermal conductivity. Thermal flow through a face determined by a normal versor $\nu_i$ is defined by

$$^{\nu}q\left(\vec{x}, t\right) = q_j\left(\vec{x}, t\right)\nu_j\left(\vec{x}, t\right). \tag{2}$$

This thermal field illustrates the rate of heat conduction in an arbitrary chosen direction.

## 2. Heat conduction problem

Let $\nu_i$ denotes the outward normal to the boundary $\partial D$. For the process undergoing in the time $(0, \infty)$ in the $D$ domain the limiting condition is considered as an initial condition for $t = 0$ and boundary condition at $\partial D$. The heat conduction problem includes governing equation

$$T_{,jj}\left(\vec{x}, t\right) = \frac{1}{\kappa}\frac{\partial}{\partial t}T\left(\vec{x}, t\right) - \frac{1}{\lambda}W\left(\vec{x}, t\right), \qquad \vec{x} \in D, \quad t > 0, \tag{3}$$

initial condition

$$T\left(\vec{x}, 0\right) = g\left(x\right), \qquad \vec{x} \in D, \tag{4}$$

and boundary condition

$$\lambda T_{,j}\left(\vec{r}, t\right)\nu_j\left(\vec{r}, t\right) + \alpha T\left(\vec{r}, t\right) = V\left(\vec{r}, t\right) + \alpha_b T\left(\vec{r}, t\right), \quad \vec{r} \in \partial D, \quad t > 0. \tag{5}$$

The symbol $W$ is the heat generated per unit volume and time, $g$ stands here for the initial temperature state, thermal diffusivity $\kappa$ determines the rate of temperature changes, $\alpha$ surface conductance ([1], p. 19), $V$ is the heat generated per unit surface and time ${}_bT$ surrounding medium temperature. Formulae (3)-(5) are referred to as the heat conduction problem.

The approximate solution of the heat conduction problem obtained by temperature component method is as follows [2]

$$
\begin{aligned}
{}^aT\left(\vec{x}, t\right) &= \frac{\kappa}{\lambda}W(D \times (0, t)) * F\left(\vec{x}, t\right) \\
&+ g(D) * F\left(\vec{x}, t\right) + \sum_{l=1}^{p}\sum_{m=1}^{n}\tilde{w}_l^m F(\vec{x} - {}_m\vec{y}, t - {}^l s).
\end{aligned}
\tag{6}
$$

Here $F$ means a fundamental solution to the parabolic operator for the three-dimensional space $Z$ [3]

$$F\left(\vec{z}, t\right) = \begin{cases} 0, & t < 0,\ \vec{z} \in Z \\ 0, & t = 0,\ \vec{z} \neq \vec{0} \\ \left(2\sqrt{\pi\kappa t}\right)^{-3} exp\left(-\frac{|\vec{z}|^2}{4\kappa t}\right), & t > 0,\ \vec{z} \in Z \end{cases} \tag{7}$$

The denotation $(D)*$ will be called a space convolution multiplication on the set $D \times D$ , and its result is a space convolution

$$f\left(D\right) * F\left(\vec{z}, t\right) \equiv \int\limits_D f\left(\vec{y}\right) F\left(\vec{z} - \vec{y},\ t\right) d\vec{y}. \tag{8}$$

Symbol is named a space-time convolution multiplication on the set $D \times (0, t)$, and its result a space-time convolution $(D \times (0, t))*$

$$f\left(D \times (0, t)\right) * F\left(\vec{z}, t\right) \equiv \int\limits_0^t \int\limits_D f\left(\vec{y}, s\right) F\left(\vec{z} - \vec{y},\ t - s\right) d\vec{y} ds. \tag{9}$$

In the formula (6) the first two components depend on heat source and initial thermal state. Component is responsible for fulfill the boundary condition (5). Let us to sign the third component by ${}_fW$. It should to meet the necessary condition

$$ {}_fW_{,jj}\left(\vec{x},t\right)=\frac{1}{\kappa}\frac{\partial}{\partial t}{}_fW\left(\vec{x},t\right),\qquad {}_fW(\vec{x},0)=0,\quad \vec{x}\in D,\quad t>0, \tag{10} $$

which is called the component admissibility condition.
The boundary condition enables determining of the temperature component

$$ \lambda {}_fW_{,j}\left(\vec{r},t\right)\nu_j\left(\vec{r},t\right)+\alpha {}_fW\left(\vec{r},t\right)=f\left(\vec{r},t\right),\quad \vec{r}\in\partial D,\quad t>0. \tag{11} $$

It is here sufficient condition for temperature component. The following denotation is used here

$$ f\left(\vec{r},t\right)\equiv V\left(\vec{r},t\right)+\alpha_b T\left(\vec{r},t\right)-\lambda[g(D)*F(\vec{r},t)]\nu_j\left(\vec{r}\right)-\alpha g(D)*F(\vec{r},t). \tag{12} $$

Nevertheless, the temperature component determined this way usually does not satisfy the component admissibility condition (10).

Let $\bar{D}=D\cup\partial D$ is a body region, inclusive of its boundary. The time $(0,\infty)$ will be contracted to the interval $(0,\infty)$ called the current time. An approximate component is introduced instead of the temperature one

$$ {}^a_fW\left(\vec{z},\tilde{t}\right)\equiv\sum_{l=1}^{p}\sum_{m=1}^{n}w_l^m F\left(\vec{z}-{}_m\vec{y},\tilde{t}-{}^ls\right),\quad \vec{z}\in\bar{Z},\ \tilde{t}\in(0,t]\,,{}_m\vec{y}\in Z\setminus\bar{D}. \tag{13} $$

The elements in the sum (13) are called fictitious components, ${}_my$ a fictitious place of the m-th fictitious component, ${}^ls$ the l-th fictitious instant, $w_l^m$ the component intensity in the ${}_my$ place at the ${}^ls$ instant, the set $\{{}_m\vec{y}:m=\bar{1,n}\}$ fictitious location, and $\{{}^l\vec{s}:l=\bar{1,p}\}$ fictitious time. This shape of the temperature component we can find in the solution (6).

## 3. Two - dimensional problems

The physical processes are three dimensional. But in many cases one could assume an independence of thermal flow from one or two directions. In this way the solutions can be obtained that have some technical applications. Of course, some calculations are usually much simpler than in a general three dimensional case. We will consider the heat conduction problem with a thermal load depend on time and 2-dimensional position vector ${}_2\vec{x}=(x_1,x_2,0)=\vec{x}(x_3=0)$ only. We also will suppose that each cross-section of the body model perpendicular to the third axis can not be distinguished. One of these cross-sections will be sign by $\tilde{D}$ and called reference section. The region should be long enough in the third direction (Fig.1).

For such assumptions we can speak about two-dimensional heat conduction problem

$$ T_{,jj}\left(\vec{x},t\right)=\frac{1}{\kappa}\frac{\partial}{\partial t}T\left(\vec{x},t\right)-\frac{1}{\lambda}{}_2W\left({}_2\vec{x},t\right),\qquad \vec{x}\in D,\quad t>0, \tag{14} $$

$$ T\left(\vec{x},0\right)={}_2g\left({}_2x\right),\qquad \vec{x}\in D, \tag{15} $$

$$ \lambda T_{,j}\left(\vec{r},t\right)\nu_j\left(\vec{r},t\right)+\alpha T\left(\vec{r},t\right)={}_2V\left(\vec{r},t\right)+\alpha_{2b}T\left(\vec{r},t\right),\quad \vec{r}\in\partial D,\quad t>0. \tag{16} $$

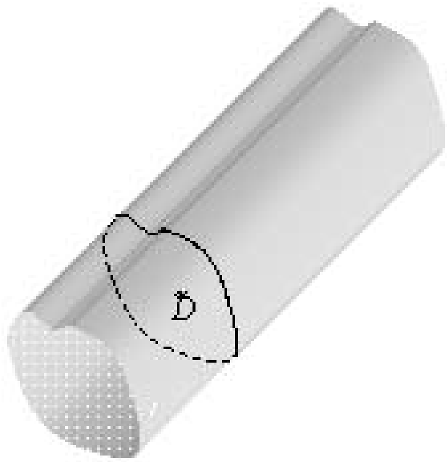


FIGURE 1. Reference cross-section.

Here ${}_2W, {}_2g, {}_2V, {}_{2b}T$ denotes the two-dimensional thermal load. We will apply the temperature component method to derive the two-dimensional heat conduction problem. To do that we can take the three-dimensional approximate solution

$$ {}^a_2T\,(\vec{x},t) = \frac{\kappa}{\lambda}{}_2W(D\times(0,t)) * F(\vec{x},t) + {}_2g(D) * F(\vec{x},t) $$

$$ + \sum_{l=1}^{p}\sum_{m=1}^{n} \tilde{w}_l^m F\left(\vec{x} - {}_m\vec{y}, t - {}^l s\right). \tag{17} $$

The thermal load appeared in (17) has two-dimensional character. It leads to the situation that the integrals along $x_3$ in convolutions contain the fundamental solution only. The results of the improper integrations along $x_3$ are constant. The above assumption about integrate limits is justified by the properties of the fundamental solution. In consequence we have

$$ {}^a_2T\,(\vec{x},t) = \frac{\kappa}{\lambda}{}_2W(\tilde{D}\times(0,t)) * {}_2F({}_2\vec{x},t) + {}_2g(\tilde{D}) * {}_2F({}_2\vec{x},t) $$

$$ + \sum_{l=1}^{p}\sum_{m=1}^{n} \tilde{w}_l^m F\left(\vec{x} - {}_m\vec{y}, t - {}^l s\right). \tag{18} $$

Here ${}_2F$ means a fundamental solution to the parabolic operator for the two-dimensional space

$$ {}_2F\,(\vec{z},t) = \begin{cases} 0, & t<0,\ \vec{z}\in\mathbf{R}^2 \\ 0, & t=0,\ \vec{z}\neq\vec{0} \\ \frac{1}{4\pi\tilde{\kappa}t} exp\left(-\frac{|\vec{z}|^2}{4\kappa t}\right), & t>0,\ \vec{z}\in\mathbf{R}^2. \end{cases} \tag{19} $$

The temperature component method requires of wrapping the region by some regular faces parallel to the boundary. So, we have to take a bounded region in two-dimensional problems. But, following the popularly two-dimensional problems, the boundary condition will be defined on reference section only. In consequence, the temperature field does not depend on the third variable

$$
{}^a_2T\,(\vec{x},t) = \frac{\kappa}{\lambda}{}_2W(\tilde{D}\times(0,t)) * {}_2F({}_2\vec{x},t) + {}_2g(\tilde{D}) * {}_2F({}_2\vec{x},t)
$$

$$
+\sum_{l=1}^{p}\sum_{m=1}^{n}\tilde{w}_l^m F\left({}_2\vec{x} - {}_m\vec{y}, t - {}^l s\right). \tag{20}
$$

Thanks to three-dimensional function $F$ in (20) the admissibility condition is fulfilled. In (20) the intensity values $\tilde{w}_l^m$ were calculated in the same way as in three-dimensional case [2]. One of the ways of approximate satisfaction of boundary conditions is provided by the method of boundary collocation ([5], p. 8). We shall apply a modification of this method named the method of boundary collocation in the least-squares approach ([5], p. 13). For two-dimensional problems the body model region in boundary condition is represented by the reference section. Now the boundary of the reference section is a closed curve $\partial\tilde{D}$ , which should be split into regular arches

$$
\vec{r} = \vec{\bar{r}}(\omega). \tag{21}
$$

A close interval in the set of the real parameters $\omega$ is a domain of the function, whose derivative is denoted by $\vec{\bar{r}}_{,\omega}$. Unit normal to the regular arch for the fixed ${}_0\omega$ is defined by the formula

$$
\tilde{\nu}_i = \pm\,(h)^{-1}\,h_i, \quad h_i \equiv \epsilon_{ijk}a_j b_k, \quad \vec{a} \equiv \vec{r_{,\omega}}\,({}_0\omega)\,, \quad \vec{b} \equiv (0,0,1). \tag{22}
$$

Let us choose the sign of the normal $\tilde{\nu}$ placed outside the regular arch cut out from the closed curve ([4], p. 108). Relation $h \neq 0$ is always valid for the regular arch ([4], p. 96). A parallel arch located in the distance $d$ from the regular arch (21) is defined by the formula

$$
\vec{y} = \vec{\bar{r}}(\omega) + d\tilde{\nu}(\vec{\bar{r}}(\omega)). \tag{23}
$$

Let us defined the collocation nodes ${}_h\vec{r},\ h = \overline{1, {}_1N}$ on the first regular arch (21) cut out from $\partial\tilde{D}$. The fictitious places ${}_m\vec{y},\ m = \overline{1, {}_1n},\ {}_1n < {}_1N$ shall be determined on the part of the first parallel arch (23) in the distance ${}_1d$, that does not intersect $\partial\tilde{D}$. Let us defined the collocation nodes ${}_h\vec{r},\ h = \overline{{}_1N+1, {}_2N}$ on the second arch (21) cut out from the boundary $\partial\tilde{D}$. The next fictitious places ${}_m\vec{y},\ m = \overline{{}_1n+1, {}_2n},\ {}_2n < {}_2N$ corresponding to it will be selected at the part of the second parallel arch (23) in the distance ${}_2d$, that does not intersect the curve $\partial\tilde{D}$ nor the first parallel arch. All the collocation nodes ${}_h\vec{r},\ h = \overline{1, {}_1N}$ and the fictitious location $\{{}_m\vec{y} : m = \overline{1, n}\}$ may be obtained by a recurrent method with regard to the other regular arches on $\partial\tilde{D}$, assuming fictitious locations for the parts of parallel arches that do not intersect $\partial\tilde{D}$ nor previous parallel arches. Similar to the three-dimensional approach verifying the boundary condition leads to a set of linear equations defined at nodes on $\partial\tilde{D}$ at the chosen test instant.

The Fourier law (1) allows to write the heat flux vector from (20)

$$
{}^a_2q_i\,(\vec{x},t) = -\kappa{}_2W(\tilde{D}\times(0,t)) * {}_2F_{,i}({}_2\vec{x},t) - \lambda{}_2g(\tilde{D}) * {}_2F_{,i}({}_2\vec{x},t)
$$

$$\lambda \sum_{l=1}^{p} \sum_{m=1}^{n} \tilde{w}_l^m F_{,i} \left( {}_2\vec{x} - {}_m\vec{y}, t - {}^l s \right), \quad i = \overline{1,2}, \quad {}^a_2 q_3 \left( \vec{x}, t \right) = 0. \tag{24}$$

The thermal flux changes in two dimensions only.
The thermal flow (2) through a face determined by a normal versor $\nu_i$ is

$$^{\nu} q_i \left( \vec{x}, t \right) = -\kappa_2 W(\tilde{D} \times (0,t)) * {}_2F_{,\nu}({}_2\vec{x}, t) - \lambda_2 g(\tilde{D}) * {}_2F_{,\nu}({}_2\vec{x}, t)$$

$$-\lambda \sum_{l=1}^{p} \sum_{m=1}^{n} \tilde{w}_l^m F_{,\nu} \left( {}_2\vec{x} - {}_m\vec{y}, t - {}^l s \right). \tag{25}$$

As we can see the third component of a unit normal has no influence on the thermal flow.

## 4. Influence of the initial condition on heat conduction solutions

Let us admit of ${}_2W = 0$ in (14)

$${}^a_2 T \left( \vec{x}, t \right) = {}_2 g(\tilde{D}) * {}_2F({}_2\vec{x}, t) + \sum_{l=1}^{p} \sum_{m=1}^{n} \tilde{w}_l^m F \left( {}_2\vec{x} - {}_m\vec{y}, t - {}^l s \right). \tag{26}$$

The expression (26) represents the approximate solution to the problem (14 - 16). It is characterized by local dependence of the heat generated per unit surface and time and surrounding medium temperature, the dependence being intermediated by the boundary condition. The solution meets the boundary condition only in collocation nodes and in test instants. Thus, we can introduce the measurement data for the thermal load on this points only. Thermal flow through a face determined by a normal versor $\nu_i$ is

$$^{\nu} q_i \left( \vec{x}, t \right) = -\lambda_2 g(\tilde{D}) * {}_2F_{,\nu}({}_2\vec{x}, t) - \lambda \sum_{l=1}^{p} \sum_{m=1}^{n} \tilde{w}_l^m F_{,\nu} \left( {}_2\vec{x} - {}_m\vec{y}, t - {}^l s \right). \tag{27}$$

In (26) formula the global dependence of the initial state is of direct character. On the other hand, the indirect local dependence on the boundary condition adjusted by the intensities of fictitious components enables the surroundings temperature and the boundary heat source to affect the distribution heat flow in the region $D$ in any instant $t > 0$.

## 5. Example

Let us consider a rectangular parallelepiped body, with heat generated per unit volume and time equal zero. The initial condition is given by the function (Fig. 2).

$$g(x, y) = 4 - 0.1x^2 - 0.125y^2.$$

The boundary condition is assumed different on the various parts of the boundary. On the cross-section various kinds of boundary conditions are presented (Fig.3). Solution obtained by the Temperature Component Method is presented below (Fig.4).

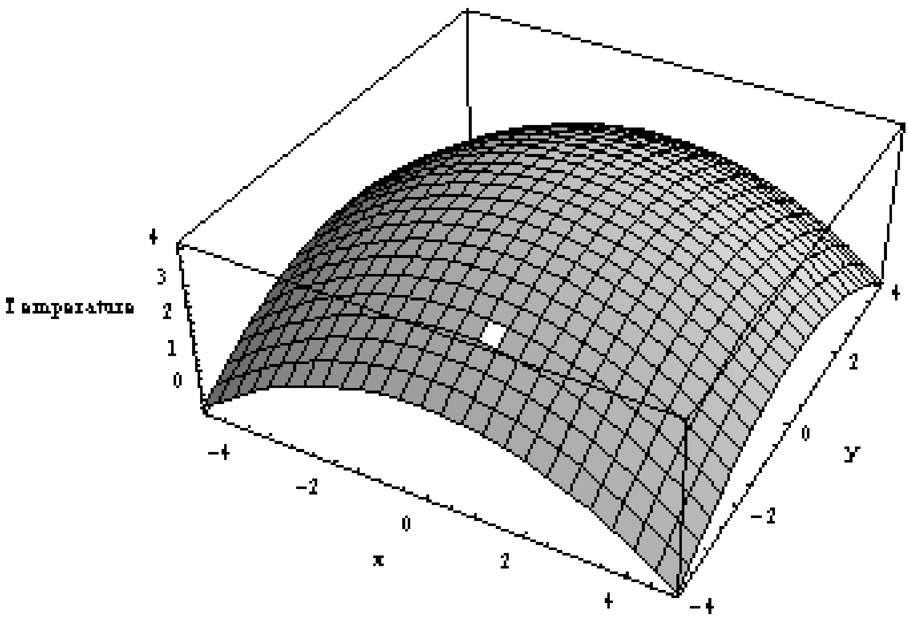


FIGURE 2. Graph of the initial condition.

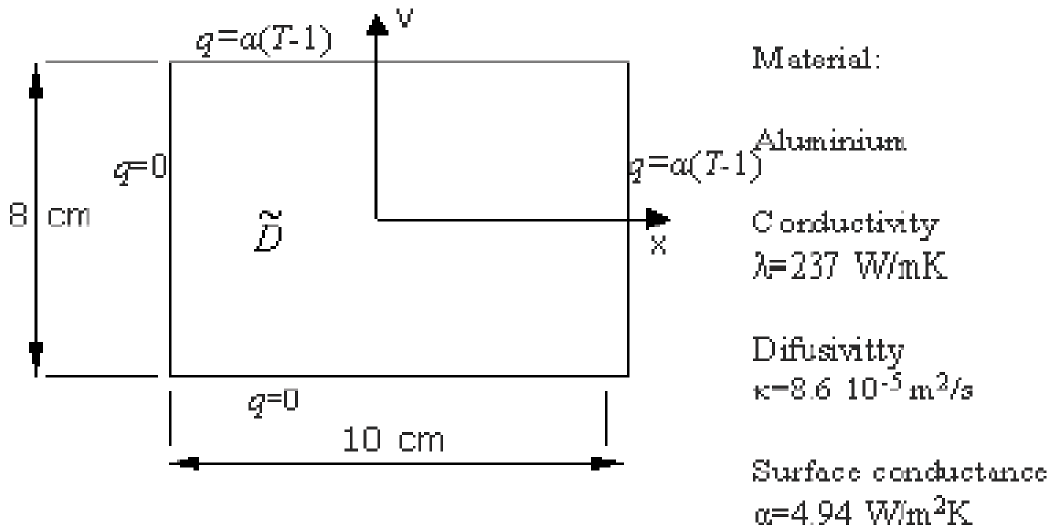


FIGURE 3. Mixed boundary conditions on the reference section.

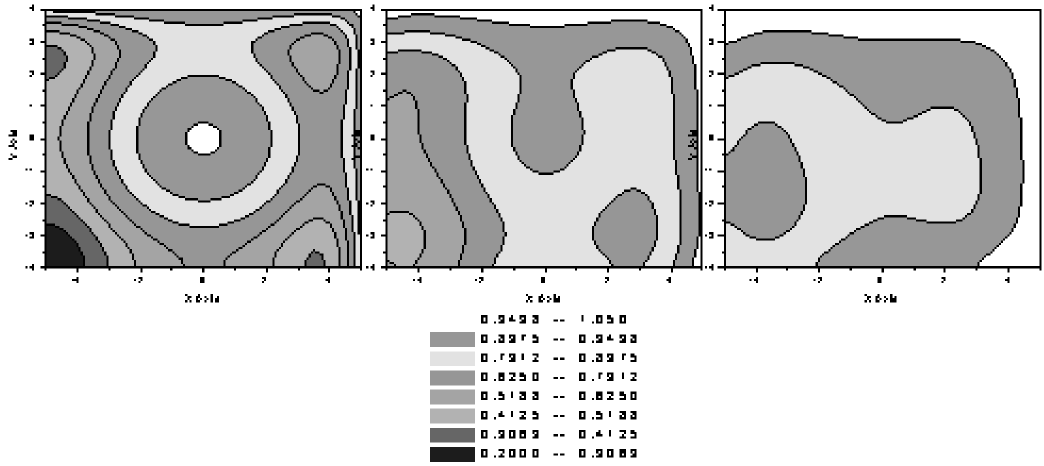


FIGURE 4. Temperature distributions for 0.5, 1.5 and 2.5 seconds.

## 6. Action of a Heat Source

The method could be useful to observe the evolution of the temperature field caused by some heat source. Let us consider a rectangular parallelepiped with a height, load on the boundary, and the heat generated per unit volume and time that allows us to treat the heat conduction as a two-dimensional problem. At the moment the body has a uniform temperature distribution compatible with surround temperature. On the boundary we use the condition of the third kind. The heat generated per unit volume is time independent. For the case we intend to ob-serve changes of temperature field and thermal flows in chosen directions. The reference section is a rectangle of dimensions .

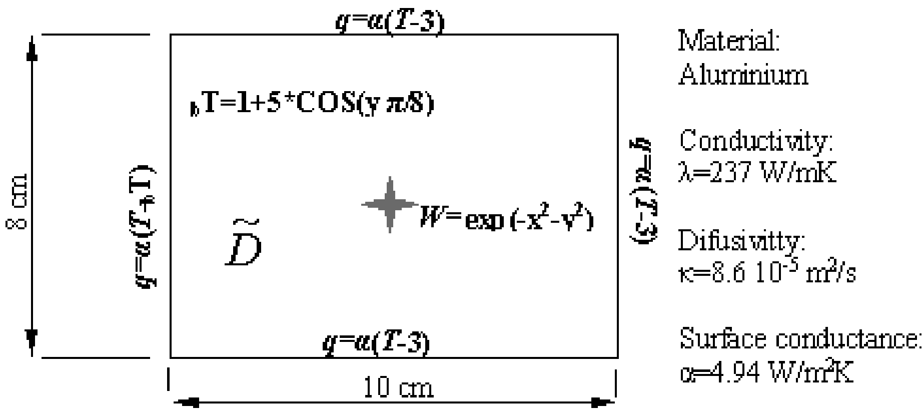


FIGURE 5. The reference section with source W.

**Solution.**

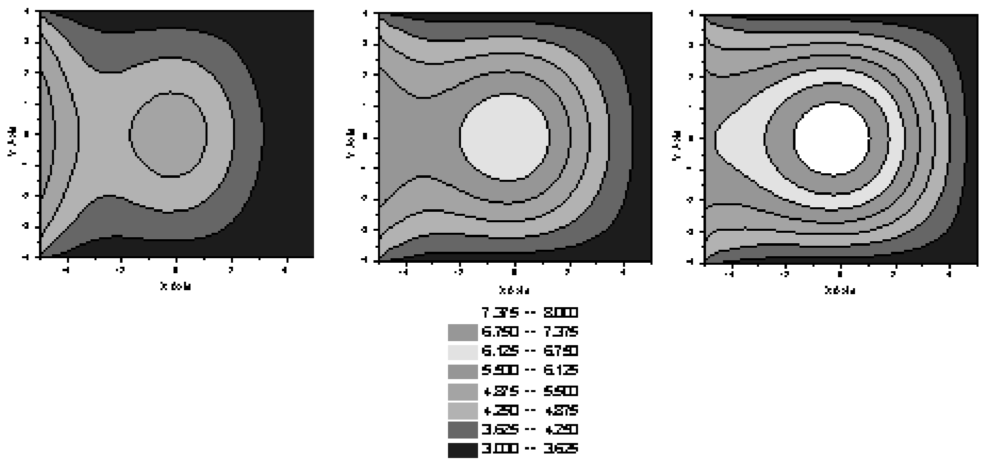


FIGURE 6. Temperature distribution for 3, 4 and 5 seconds.

## 7. Conclusions

The temperature component method elaborated for three-dimensional problems was applied to solve some problems with a plane state of temperature. The reference cross-section was introduced in order to take into consideration two-dimensional heat conduction and fictitious components were located at the arches parallel to the boundary curve. Temperature distributions through the body region caused by the initial condition and action of the heat source were illustrated. The method is effective and numerically stable. Computational time for presented examples is comparable with other well-known methods such as FEM, BEM [5]. The method is quite easy to implement.

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