

Necessary and sufficient conditions for constrained nonconvex and noncoercive autonomous variational problems

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Let us consider the classical autonomous one-dimensional Lagrange problem

$$(P) \quad \text{minimize } F(v) := \int_a^b f(v(t), v'(t)) dt, \quad v \in \Omega$$

where $\Omega := \{v \in W^{1,1}(a, b) : v(a) = \alpha, v(b) = \beta, v'(t) \geq 0 \text{ a.e. in } (a, b)\}$ and $f : [\alpha, \beta] \times [0, +\infty) \rightarrow \mathbb{R}$ is a lower semicontinuous, non-negative integrand. Hence, f can be nonsmooth, nonconvex, noncoercive.

As it is well-known, the lack of convexity and coercivity does not allow the use of the classical direct methods of the Calculus of Variations and this problem can have no solution. On the other hand, in some classical examples of noncoercive problems, such as the problem of the surface of revolution of minimal area, it is well known that the existence of the minimum is related to the position of the prescribed endpoints. This suggests that also in general a sufficient condition for the existence of the minimum could be obtained in terms of the boundary data $(a, \alpha), (b, \beta)$.

Indeed, in [1] Botteron and Dacorogna achieved existence and non-existence results for constrained functionals with non-autonomous integrand having the sum-type structure $f(t, s, z) = \phi(t, z) + \psi(t, s)$ just relatively to the prescribed mean slope $\xi_0 = \frac{\beta - \alpha}{b - a}$. They showed that under some technical assumptions on functions ϕ and ψ , if ξ_0 is sufficiently small the minimum exists, while if it is too large the minimum does not exist. A similar investigation was carried out in [2] for integrands not depending on the state variable, obtaining a necessary and sufficient condition for the existence of the minimum expressed in term of a limitation on the assigned slope ξ_0 . The relevance of the boundary conditions and other specific parameters of the problem was also discussed by B. Mordukhovich in [4], where *individual existence theorems* were presented in the framework of optimal control problems.

In [3] we investigated autonomous constrained problems, without a specific structure, under very mild assumptions on the integrand (lower semicontinuity and boundedness by below). The first aim of that paper is to analyze necessary conditions for the optimality of a given admissible trajectory $v_0 \in \Omega$ for problem (P). In particular, we proved the following nonsmooth version of the DuBois-Reymond necessary condition, expressed in terms of an inclusion involving the subdifferential of the Convex Analysis, which turns to be both necessary and sufficient for the optimality of v_0 .

Theorem 1. *Let $v_0 \in \Omega$ be a given trajectory and set $A_{v_0} := \{t \in [a, b] : \text{there exists } v_0'(t) > 0\}$. Then, v_0 is a minimizer for problem (P) if and only if $\partial f(v_0(t), v_0'(t)) \neq \emptyset$ for a.e. $t \in A_{v_0}$ and the following DuBois-Reymond condition (DBR) (expressed according to the measure of the set A_{v_0}) holds:*

(DBR)₁ (when $|A_{v_0}| = b - a$): there exists $c \leq \mu$ such that

$$f(v_0(t), v'_0(t)) - c \in v'_0(t) \partial f(v_0(t), v'_0(t)) \quad \text{a.e. in } (a, b)$$

(DBR)₂ (when $|A_{v_0}| < b - a$): $f(v_0(t), 0) = \mu$ for a.e. $t \notin A_{v_0}$ and

$$f(v_0(t), v'_0(t)) - \mu \in v'_0(t) \partial f(v_0(t), v'_0(t)) \quad \text{a.e. in } A_{v_0}$$

where $\mu := \min_{s \in [\alpha, \beta]} f(s, 0)$ and $\partial f(v_0(t), v'_0(t))$ denotes the subdifferential of $f(v_0(t), \cdot)$, restricted to $(0, +\infty)$, at $v'_0(t)$.

This result is obtained by introducing a suitable non-autonomous Bolza problem whose integrand does not depend on the state variable, which is equivalent to problem (P). An arrangement of the theorems in [2] to this Bolza problem leads to the result.

The second step of our investigation consists in a relaxation result.

Theorem 2. Set $C_s := \{z > 0 : f(s, z) = f^{**}(s, z)\}$ and let $Bd(C_s)$ denote the boundary of C_s . Assume that the origin is not a cluster point for $Bd(C_s)$ and $f(s, \cdot)$ is continuous at the origin, for every $s \in [\alpha, \beta]$. Then, problem (P) admits minimum if and only if problem (P^{**}) admits a minimizer $v_0 \in \Omega$ such that

$$(0.1) \quad v'_0(t) \in co(C_{v_0(t)}) \quad \text{for a.e. } t \in A_{v_0}$$

where $co(C_{v_0(t)})$ denotes the convex envelope of the set $C_{v_0(t)}$.

So, the study of necessary and sufficient conditions for problem (P) is reduced to the study of the solvability of the relaxed problem (P^{**}), by a minimizer satisfying (0.1). To this aim, in [3] a necessary and sufficient condition for the existence of the minimum of problem (P^{**}) expressed in terms of an upper limitation for the assigned mean slope $\xi_0 = (\beta - \alpha)/(b - a)$ is proved (see [3: Theorems 12, 13, 15]).

Here we just quote two simple applications of such results to the particular case when f^{**} has one of the following structures:

$$f^{**}(s, z) = a(s)h(z) \quad \text{or} \quad f^{**}(s, z) = a(s) + h(z)$$

with $a \in C[\alpha, \beta]$ and $h \in C[0, +\infty)$ convex (but not necessarily coercive).

Indeed, set

$$\ell := \inf_{z > 0} (h(z) - zh^+(z)) = \inf_{z > 0} (h(z) - zh^-(z)),$$

$$M := \max_{s \in [\alpha, \beta]} a(s), \quad m := \min_{s \in [\alpha, \beta]} a(s), \quad \gamma^-(y) = \max\{z > 0 : h(z) - zh^-(z) \geq y\}$$

where $h^-(z), h^+(z)$ respectively denote the left and right derivative of h at z . In other words, ℓ is the infimum of the values at the origin of the affine support functions of h .

Corollary 1. Let $f^{**}(s, z) = a(s)h(z)$, with $a \in C[\alpha, \beta]$ positive almost everywhere, $h \in C[0, +\infty)$ non-negative, convex, but not affine. Then,

- If $\ell = -\infty$ the minimum (of (P^{**})) exists for every slope $\xi_0 > 0$; moreover, if $m > 0$ the minimizers are Lipschitz continuous.

- If $\ell = 0$, $m > 0$ and h is not affine in any half-line, then the minimum exists for every slope $\xi_0 > 0$ and the minimizers are Lipschitz continuous. Instead, if $\ell = 0$, $m = 0$ and h is strictly convex, then the minimum does not exist for any slope $\xi_0 > 0$.
- If $\ell > 0$ and $a(s) < M$ a.e. in (α, β) , then the minimum exists if and only if $M\ell \leq mh(0)$ and

$$\int_{\alpha}^{\beta} \frac{1}{\gamma^{-}(M\ell/a(s))} ds \leq b - a.$$

Moreover, if $M\ell < mh(0)$ and

$$\lim_{c \rightarrow (M\ell)^+} \int_{\alpha}^{\beta} \frac{1}{\gamma^{-}(c/a(s))} ds < b - a$$

then the minimizers are Lipschitz continuous.

- If $-\infty < \ell < 0$ and $a(s) > m$ a.e. in (α, β) , then the minimum exists if and only if

$$\int_{\alpha}^{\beta} \frac{1}{\gamma^{-}(m\ell/a(s))} ds \leq b - a.$$

Moreover, if $m > 0$ and

$$\lim_{c \rightarrow (m\ell)^+} \int_{\alpha}^{\beta} \frac{1}{\gamma^{-}(c/a(s))} ds < b - a$$

then the minimizers are Lipschitz continuous.

Corollary 2. Let $f^{**}(s, z) = a(s) + h(z)$, with $a \in C[\alpha, \beta]$ non-negative, $h \in C[0, +\infty)$ convex, non-negative. Then,

- if $\ell = -\infty$, then the minimum exists for every slope $\xi_0 > 0$ with Lipschitz continuous minimizers;
- if $\ell > -\infty$ and $a(s) < M$ a.e. in (α, β) , then the minimum exists if and only if $M + \ell \leq m + h(0)$ and

$$\int_{\alpha}^{\beta} \frac{1}{\gamma^{-}(M + \ell - a(s))} ds \leq b - a.$$

Moreover, if $M + \ell < m + h(0)$ and

$$\lim_{c \rightarrow (M + \ell)^+} \int_{\alpha}^{\beta} \frac{1}{\gamma^{-}(c - a(s))} ds < b - a$$

then the minimizers are Lipschitz continuous.

REFERENCES

1. B. Botteron and B. Dacorogna, *Existence and non-existence results for non-coercive variational problems and applications in ecology*, J. Diff. Eqns. **85** (1990), 214-235.

2. C. Marcelli, *Variational problems with non-convex, non-coercive, highly discontinuous integrands: characterization and existence of minimizers*, SIAM J. Control Optim., **40** (5) (2002), 1473-1490.
3. C. Marcelli, *Necessary and sufficient conditions for optimality of nonconvex, non-coercive autonomous variational problems with constraints*, submitted
4. B. Mordukhovich, *Existence theorems in non-convex optimal control*, in *Calculus of Variations and Optimal Control*, Chapman and Hall/CRC res. Notes Math. 411, Boca Raton, FL, 1999, 173-197.