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On Nonautonomous Bifurcation

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In recent years there has been consistent interest in the process of stability breakdown and bifurcation of compact invariant sets in skew-product flows determined by nonautonomous differential equations. Roughly speaking, two classes of equations have been studied.

The first consists of equations of the form

$$x' = \varepsilon f(t, x, \varepsilon) \tag{0.1}$$

where $x \in \mathbb{R}^2$ and f is a nonautonomous vector field depending on a small parameter ε. It is assumed that $f(t, 0, \varepsilon) = 0$ for all $t \in \mathbb{R}$ averaging method, one can discuss certain bifurcation patterns when $\psi(t) \equiv 0$ loses stability as ε passes through 0.

The second class contains in particular certain control systems studied by Colonius and Kliemann [2]. These systems will be considered in our talk. Consider a control system of the form

$$x' = f(x, \varepsilon) + u(t)g(x, \varepsilon) \tag{0.2}$$

where $x \in \mathbb{R}^2$. Here $u : \mathbb{R} \to [-1, 1]$ is a measurable control function, ε takes values in an open interval $I = (a, b) \subset \mathbb{R}$, and ρ is a real parameter which assumes small values. We will always assume that $f(0, \varepsilon) = g(0, \varepsilon) = 0$ for all $\varepsilon \in I$, so that $x = 0$ is an equilibrium point of (0.1) for all choices of ε, ρ, and $u(\cdot)$.

When $\rho = 0$ one obtains a one-parameter family of ordinary differential equations

$$x' = f(x, \varepsilon) \qquad (x \in \mathbb{R}^2).$$

each of which admits $x = 0$ as an equilibrium point. The classical Poincaré-Bendixon theory imposes strong constraints on the asymptotic behavior of the solutions of such equations.

We pose a bifurcation problem of "Arnold type" [1]. Let us explain what we mean by this term. For each $\varepsilon \in D$ let $B_\varepsilon = D_x f(0, \varepsilon)$ be the Jacobian matrix of $f(\cdot, \varepsilon)$ at $x = 0$. Suppose that there exist numbers $\varepsilon_1 < \varepsilon_2 \in I$ with the following properties. First, the eigenvalues of B_ε lie in the left half-plane for $\varepsilon < \varepsilon_1$; i.e., B_ε is a Hurwitz matrix. Second, the eigenvalues of B_ε lie in the right half-plane if $\varepsilon > \varepsilon_2$. Finally, in the intermediate regime $\varepsilon_1 < \varepsilon < \varepsilon_2$, we will assume that B_ε admits one negative (real) eigenvalue and one positive eigenvalue.

The problem we pose is the following. Suppose that the parameter ρ in (0.1) is small but non-zero. First, describe the qualitative change in the behavior of the solutions of (1) as ε passes through $[\varepsilon_1, \varepsilon_2]$. Then, determine the number and structure of the control sets of (1) when $\varepsilon < \varepsilon_1$, $\varepsilon_1 < \varepsilon < \varepsilon_2$, and $\varepsilon > \varepsilon_2$.

Classical bifurcation theory offers only limited insight into this problem because of the nonautonomous nature of equation (0.2) However L. Arnold has offered hypotheses concerning the expected behavior of solutions of 1-parameter families like (0.2) [2]. His insights have been developed by later authors [4]. We will apply and amplify previous methods and results regarding the Arnold bifurcation pattern in the context of the family (0.2).

We will in particular make systematic use of the fact that the nonautonomous part of (0.2) is small, i.e., proportional to ρ. Specifically, we will use a method of Mel'nikov type to study the behavior of solutions of (0.2) when $\varepsilon_1 < \varepsilon < \varepsilon_2$, and we will use the continuation properties of the Conley index to study invariant sets and control sets for (0.2) when $\varepsilon > \varepsilon_2$. These methods seem particularly suited to the study of nonautonomous equations which are perturbations of autonomous equations.

After we have studied the solutions of (0.2) for small $\rho \neq 0$, we will analyze the corresponding control sets [2]. These sets provide basic information about the local controllability properties of the nonlinear control system (0.2). We will use results of Colonius-Kliemann [2].

The talk is based on joint work with F. Colonius, R. Fabbri, and M. Spadini [3].

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