

## Particle trajectories in linear water waves

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### 1 Introduction

The aim of this paper is to show that for a linear water wave no particle trajectory is closed, unless the free surface is flat. Each trajectory involves over a period a backward/forward movement of the particle, and the path is an elliptical arc (which degenerates on the flat bed) but with a forward drift. Our result can be viewed as a more detailed version of the classical theory for the trajectories of particles below a water wave, as it was summarized by Longuet-Higgins :

“In progressive gravity waves of very small amplitude it is well known that the orbits of the particles are either elliptical or circular. In steep waves, however, the orbits become quite distorted, as shown by the existence of a mean horizontal drift or mass-transport in irrotational waves.”

We show that within linear water wave theory, the particle paths are almost closed and the more we approach the free surface, the more pronounced the deviation from a closed orbit becomes. This is in agreement with Stokes’ observation :

“It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of the propagation of the waves ...”

The conclusions reached in this paper must have occurred to many people interested in water waves, but, to our knowledge, the point has never been clearly put on record. We do so in this paper, with the simplest means. We expect that similar conclusions hold for the governing equations (without linearization), that is, for Stokes waves (irrotational flow) and for periodic traveling waves with vorticity. While explicit formulas are not available for these wave motions, qualitative features of the underlying flow are known and could possibly lead to a confirmation of the features we prove here within the confines of linear water waves.

## 2 Preliminaries

### 2.1 The governing equations

Most of the waves propagating on the surface of the sea or on a lake are, as a matter of common experience, approximately two-dimensional. That is, the motion is identical in any direction parallel to the crest line. To describe these waves consider a cross section of the flow that is perpendicular to the crest line with Cartesian coordinates  $(x, y)$ , the  $y$ -axis pointing vertically upwards and the  $x$ -axis being the direction of wave propagation, while the origin lies on the flat bed. Let  $(u(t, x, y), v(t, x, y))$  be the velocity field of the flow over a flat bed  $y = 0$  and let  $y = h_0 + \eta(t, x)$  be the water's free surface. Here  $h_0 > 0$  is the mean water level.

Homogeneity (constant density) is a physically reasonable assumption for gravity waves, and it implies the equation of mass conservation

$$(2.1) \quad u_x + v_y = 0.$$

Under the assumption of inviscid flow (experimental evidence confirms that the length scales associated with an adjustment of the velocity distribution due to laminar viscosity or turbulent mixing are long compared to typical wavelengths, in support of the inviscid setting), the equation of motion is Euler's equation

$$(2.2) \quad \begin{cases} u_t + uu_x + vv_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g, \end{cases}$$

where  $P(t, x, y)$  denotes the pressure and  $g$  is the gravitational constant of acceleration. The free surface decouples the motion of the water from that of the air, a fact that is expressed by the dynamic boundary condition

$$(2.3) \quad P = P_0 \quad \text{on} \quad y = h_0 + \eta(t, x),$$

if we neglect surface tension, where  $P_0$  is the (constant) atmospheric pressure. Since the same particles always form the free surface, we also have the kinematic boundary condition

$$(2.4) \quad v = \eta_t + u\eta_x \quad \text{on} \quad y = h_0 + \eta(t, x).$$

On the flat bed we have the kinematic boundary condition

$$(2.5) \quad v = 0 \quad \text{on} \quad y = 0,$$

expressing the fact that the flow is tangent to the horizontal bed (or, equivalently, that water cannot penetrate the rigid bed). The governing equations for water waves are encompassed by the nonlinear free-boundary problem. In our discussion we suppose that at some distant point in the past a disturbance of the flat surface of still water was created and we analyze the subsequent motion of the water. The balance between the restoring gravity force and the inertia of the system governs the evolution of the mass of water and our primary objective is to understand the behaviour of the water particles below the free surface.

An important category of flows are those of zero vorticity (irrotational flows), characterized by the additional assumption

$$(2.6) \quad u_y = v_x.$$

The vorticity of a flow,  $\omega = u_y - v_x$ , measures the local spin or rotation of a fluid element so that in irrotational flows the local whirl is completely absent. This relation ensures the existence of a velocity potential  $\phi(t, x, y)$  defined up to a constant via

$$\phi_x = u, \quad \phi_y = v.$$

In view of (2.1),  $\phi$  is a harmonic function, i.e.  $(\partial_x^2 + \partial_y^2)\phi = 0$  so that the powerful methods of complex analysis become available for the study of irrotational flows. Concerning the physical relevance of irrotational water flows, field evidence indicates that for waves entering a region of still water the assumption of irrotational flow is realistic. Moreover, as a consequence of Kelvin's circulation theorem, a water flow that is irrotational initially has to be irrotational at all later times. It is thus reasonable to consider that water motions starting from rest will remain irrotational at later times.

### 3 Particle trajectories

If  $(x(t), y(t))$  is the path of the particle below the linear wave then

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v,$$

so that finally, after having taken in account the linearized problem, the motion of the particle is described by the system

$$(3.1) \quad \begin{cases} \frac{dx}{dt} = M \cosh(ky) \cos k(x - ct), \\ \frac{dy}{dt} = M \sinh(ky) \sin k(x - ct), \end{cases}$$

with initial data  $(x_0, y_0)$ . We denoted

$$(3.2) \quad M = \frac{\varepsilon \omega h_0}{\sinh(kh_0)}.$$

#### 3.1 Classical approximation theory

We seek approximations of the solution in terms of the small parameter  $M$ . In this subsection we mainly describe the classical approach.

We restrict our attention to a the fixed time interval  $[0, T]$ , where  $T = \lambda/c > 0$  is the wave period. Since  $y$  belongs to a set bounded a priori as  $0 \leq y \leq h_0(1 + \varepsilon)$ , from (11) we readily obtain that

$$x(t) - x_0 = O(M), \quad y(t) - y_0 = O(M), \quad t \in [0, T],$$

where  $O(M)$  denotes an expression of order  $M$ . Using the mean-value theorem, we may write (11) on  $[0, T]$  as

$$\begin{cases} \frac{dx}{dt} = M \cosh(ky_0) \cos(kx_0 - \omega t) + O(M^2), \\ \frac{dy}{dt} = M \sinh(ky_0) \sin(kx_0 - \omega t) + O(M^2), \end{cases}$$

since  $\omega = kc$ . Neglecting terms of second order in  $M$ , we find that

$$\begin{cases} \frac{dx}{dt} \approx M \cosh(ky_0) \cos(kx_0 - \omega t), \\ \frac{dy}{dt} \approx M \sinh(ky_0) \sin(kx_0 - \omega t), \end{cases}$$

so that by integration we obtain

$$(3.3) \quad \begin{cases} x(t) \approx x_0 - \frac{M}{\omega} \cosh(ky_0) \sin(kx_0 - \omega t), \\ y(t) \approx y_0 + \frac{M}{\omega} \sinh(ky_0) \cos(kx_0 - \omega t). \end{cases}$$

Thus

$$(3.4) \quad \frac{[x(t) - x_0]^2}{\cosh^2(ky_0)} + \frac{[y(t) - y_0]^2}{\sinh^2(ky_0)} \approx \frac{M^2}{\omega^2},$$

which is the equation of an ellipse: to a first-order approximation the water particles move in closed elliptic orbits, the centre of the ellipse being  $(x_0, y_0)$ .

### 3.2 Exact theory

To study the exact solution to (3.1) it is convenient to re-write it in a moving frame with scaled independent variables: the transformation

$$(3.5) \quad X = k(x - ct), \quad Y = ky,$$

maps (11) into

$$(3.6) \quad \begin{cases} \frac{dX}{dt} = k M \cosh(Y) \cos(X) - kc, \\ \frac{dY}{dt} = k M \sinh(Y) \sin(X). \end{cases}$$

Working in the phase plane we are able to show that the particle traces a loop that fails to close-up: there is a small forward drift which is minimal on the flat bed.