Simple iterative ways to define Riemann surfaces with prescribed number of sheets and given branching via explicit equations are shown. The analysis is then focused on a class of Riemann surfaces analogue to IFS pre-fractals. Their topology (genus) and monodromy are closed-form computed; the associated self-similar symbolic dynamics is formalized and some convergence issues are presented. They can be used as interesting paradigm of complexity and Chaos for some dynamical systems in Physics and Computer Science which will be presented – with more mathematical details – in the full paper.

1. Riemann surfaces, Iterated Monodromy Groups and Julia sets

Riemann surfaces are usually defined as subsets $\mathbb{P} \subseteq \mathbb{C}^2$ such that, let $R : \mathbb{C}^2 \to \mathbb{C}$,

$$ (w,z) \in \mathbb{P} \iff R(w,z) = 0; \quad (1) $$

explicit definitions involve multi-valued function $w = f(z)$ such that $f^{-1}$ induces a branched covering of $\mathbb{C}^2$. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a $p$-branched covering and $W^p_f(z) = (n_1, n_2, ..., n_r) f^n(z)$ be the forward orbit of any point $z \in \mathbb{C}^2$; the hierarchy tree naturally associated to $W^p_f(z)$ has root $(0, z)$, is connected and $p$-regular. If $B \subseteq \mathbb{C}^2$ is the branch points set then the postcritical points-set, or Fatou set, is

$$ \bigcup_{n_0 \in \mathbb{N}_0} f^{-n}(B); \quad (2) $$

its complement $F_J(f)$ is the filled Julia set and $J(f) = \partial F_J(f)$ is the Julia set. Loops of $F_J(f)$ lift to automorphisms of tree $W^p_f(z)$. $\text{Aut}_f W^p_f(z)$ is isomorphic to the monodromy group of $F_J(f)$, i.e. the iterated monodromy group of $f$ [3]:

$$ \text{IMG} f = \text{MG} F_J(f) = \bigcup_{n_1 \in \mathbb{N}_0} f^{-n}(B) \cup \mathbb{C}^2; \quad (3) $$

2. Prefractal Riemann surfaces An initiator

Let $p \in \mathbb{N} \setminus \{1\}$, let $P = \mathbb{C}[z]/\mathbb{C}$ a monic polynomial with $\deg P = d$, $z_1, z_2, ..., z \in \mathbb{C}$ be its $r \leq d$ distinct roots with multiplicities $m_1, m_2, ..., m_r$ respectively and let $A_{z_i}$ be the Riemann surface defined by the explicit equation $w^p - P(z) = 0$, which is globally equivalent to the explicit, multivalued equation $w = f(z)$:

$$ z \mapsto f(z) = \left( \prod_{j=1}^{m_j} (z - z_j)^{m_j} \right)^{1/p}; \quad (4) $$

By posing $m_j = m_j \text{GCD}(m_j, p)$ and $p = \text{GCD}(m_j, p) = d \text{GCD}(d, p)$:

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* Such a tree has $W_p^i(x)$ as node-set and its incident nodes are consecutive points of the (backward) orbit, i.e. belong to $(n-1, f^{n+1}(x))$ and $(n f^{n}(x))$, respectively, $\forall n \in \mathbb{N}$.

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\[
 f(z) = \bigodot_{j=1}^{r} p_j \sqrt{(z - z_j) /[z_1, z_j]} := (z, 1 - z) \mathbb{P},
 \]

\( z = (z_1, z_2, \ldots, z_{\ell}) \in \mathbb{C}^\ell \), and the following multi-indices in \( \mathbb{N}^\ell \) were used:

\[ I_1 = (1, 1, K, 1), \quad m = (m_1, m_2, K, m_\ell), \quad m = (m_1, m_2, K, m_\ell), \]

\[
p = (p_1, p_2, K, p_\ell), \quad m = \frac{m^p}{p^p} \bigodot (m_1, m_2, K, m_\ell), \quad Q'.
\]

\( A_{\ell} \) has \( \text{LCM} p \) sheets, all sharing the same branch points, which are the roots \( z \) of \( P \) (and possibly \( \infty \)) with ramification indexes \( p - 1, (d - 1) \) [5].

A relevant case is when \( p = 2 \) and \( r = d - 1 \) (\( P \) has simple roots only): \( A_{\ell} \) is a \( 2 \)-sheeted \emph{hyperelliptic} \( \lfloor d/2 \rfloor \)-fold torus; \( \infty \) is a branch point iff \( d \) is odd [4].

Let \( f(z) = (z - z_0, (z - z_1)(z - z_2)^2(z - z_3)^2(z - z_4)^2(z - z_5)^2(z - z_6)^2 : \) the table below reports the action of the monodromy group for \( p = 6, r = 7 \) and \( m = (1, 1, 2, 5, 1, 7) \) thus \( m = (1, 1, 2, 5, 1, 7) \), \( p = (6, 3, 2, 3, 6, 1) \). Rows stand for one of the \( \text{LCM} p = 6 \) sheets, columns for a loop’s monodromy winding counter-clockwise around 6 branch points (\( z_6 \) is not a branch point since \( p | 6 \Rightarrow \infty = 1 \)).

Monodromy group action \( \mathbb{Z}_6 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_5 \otimes \mathbb{Z}_6 \otimes \mathbb{Z}_6 \) is shown taking a sample loop \( \cdot k \cdot \) with \( k = (-1, 2, 1, 0, 1, k_0, 0) \) \( \forall k_0 \in \mathbb{Z} \), starting from \( 3^{\text{rd}} \) sheet and always ending on the \( 2^{\text{nd}} \) one. Whether the winding order of the branch points is \( z_1, \nabla z_2, \nabla z_3, \nabla z_4, \nabla z_5, \nabla z_6 \) or \( \nabla z_1, \nabla z_2, \nabla z_3, \nabla z_4, \nabla z_5, \nabla z_6 \), the corresponding visited sheets’ sequence is either \( 3 = 3 \pm 3 \pm 3 = 3 \) or \( 3 = 3 \pm 3 \pm 3 = 3 \) or \( 3 = 3 \pm 3 \pm 3 = 3 \) in \( \in \mathbb{Z} \), respectively, thus the monodromy group \( \mathbb{M} \mathbb{G} A_{\ell} \) must be abelian.

Any path \( \gamma_{[0,1]} \to A_{\ell} \) winding once [counter-]clockwise around \( z \) \( (1 \leq \gamma \leq r) \) ends to the same point times exp(\( [t \in 2\pi i] \)). As \( \mathbb{Z}_{\text{LCM} p} \) is the index-set of the sheets a monodromy action \( m : p_1(A_{\ell}) \otimes S_{\text{LCM} p} \) exists such that

\[
\text{LS}_s' \cdot a \mathbb{K} \cdot \mathbb{I} \mathbb{M} A_{\ell} = \bigodot_{j=1}^{r} m_j \mathbb{Z}_{p_j}, \quad g(1) = e^{2\pi ik_{\ell}/p} g(0) \quad (6)
\]

(where \( k = (k_1, k_2, \ldots, k_\ell) \in \mathbb{Z}^\ell \) such that \( k = \) the winding number of \( \gamma \) around \( z \)).

Direct sum (6) holds because whenever \( \gamma \) winds \( k \) times clockwise around \( j^{\text{th}} \) branch point the loop cyclically “tunnels” down by \( \mu \) sheets (modulo \( \mathbb{Z}_{\text{LCM} p} \) \( k \) times. This just depends on \( k \), not on the branch points’ looping order. If \( \gamma \) starts from the \( j^{\text{th}} \) sheet (\( \forall s \in \mathbb{Z}_{\text{LCM} p} \) it ends on the \( \text{md} \mathbb{M} \mathbb{G} A_{\ell} \) sheet given by

\[
\text{md} \text{LS}_s' \cdot a \mathbb{K} \cdot \mathbb{I} \mathbb{M} \mathbb{A}_{\ell} \text{LCM} p \text{mod} \text{LCM} p \quad (7)
\]

\[ \text{3. Higher iteration orders: self-similarity} \]

\( \forall \omega \in \mathbb{N}_0 \) let \( A_{\omega} := \{(\omega, z) \in \mathbb{C}^2 \mid \omega = f(z)\} \) be the Riemann surface defined by the \( \omega^{\text{th}} \) self-composition of \( f \); it can be recursively defined over \( A_{\omega-1} \) too, as

\[
A_{\omega} = \{ (\omega, f(\omega, \omega, -1) ) \mid C \to C \} 
\]

Definitions in (8) coincide with \( \omega_{\omega, \infty} \), \( \omega_{\omega, \omega} \), the implicit equation being \( w_0 = P(w, w-1) = 0 \). Each and every \( (\omega, z) \)-sheet of \( A_{\omega} \) ramifies to the \( (\omega, w_{\omega}) \)-sheets of \( \omega \) like the \( (\omega, z) \)-sheets of \( A_{\omega} \) do from the \( C \) plane.

Each of the \( (\omega, z) \)-sheets (1\(^{\text{st}} \)-generation sheets) has the same branch points \( \omega \in \{z, \infty\} \) of \( f \) since \( \omega_1 = f(\omega) \). Other branch points are present on \( \omega^{\text{th}} \)-generation sheets because (8) is multivalued, solutions to \( f(\omega) = \omega \).

\[ ^a \text{The monodromy group for a } p \text{-sheeted Riemann surface is the subgroup of the symmetric group } S_p \text{ (of order } p!) \text{ acting on its fundamental group’s linear presentations.} \]

\[ ^b \text{Hereinafter, the } (\omega, \omega_{\omega-1}) \text{-sheets will be referred to as } \omega^{\text{th}} \text{-generation sheets.} \]
Analyzing the document, the text appears to be discussing mathematical concepts related to fractal geometries, symbolic dynamics, and the properties of certain functions. The text mentions the use of LCM (Least Common Multiple) and other algebraic operations, and references to specific mathematical expressions and theorems. The document also touches on the behavior of functions under different conditions and the implications for their growth and convergence. Overall, it seems to be a detailed exploration of complex mathematical ideas.
References

Figure 4. Left to right, top to bottom: $|f_{13}(z)|$, Arg$f_{13}(z)$, $|f_{23}(z)|$, Arg$f_{23}(z)$.

$$f(z) = \sqrt[3]{\frac{(z + \pi)(z + 2)}{z^3 - 1}}.$$  

Figure 5. Arg$f_{23}(z)$ for various $f(z) = \sqrt[n]{N(z)/D(z)}$ (top: $f$ is R-definite).