

## TOPOLOGICAL CALCULUS: BETWEEN ALGEBRAIC TOPOLOGY AND ELECTROMAGNETIC FIELDS

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The Topological Calculus, based on Algebraic Topology, is introduced as a discrete Field Theory. Diagonalization of simplicial complex adjacency matrices allows to extract information about domain topology and Helmholtz equation eigenfunctions. Electromagnetic analysis of IFS fractals for Sierpinski gasket/carpet is then carried out: self-similar topology deeply influences the type of e.m. fields, as well as its finite TEM modes (as many as the domain's Euler characteristic; represented by harmonic fields) and self-similar distribution of resonating frequencies. This proves that even in such discrete model many features of guided waves depend on the topology rather than metrics.

### 1. Simplicial Cohomology

Let  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$  be  $p+1$  geometrically independent vectors, then the  $p$ -dimensional simplex<sup>b</sup> (or, briefly,  $p$ -simplex) characterized by such vertices is indicated as  $\sigma := (\mathbf{v}_0 | \mathbf{v}_1 | \dots | \mathbf{v}_p)$ . Whenever an ordering of these vertices is provided, the corresponding *oriented*  $p$ -simplex (whose orientation is equivalent to the parity of the permutation of its vertices) is indicated as  $\overset{\uparrow}{s} = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{K}, \mathbf{v}_p]$ , its opposite orientation being  $\overset{\downarrow}{s} = -\overset{\uparrow}{s}$ . An  $n$ -dimensional [oriented] *simplicial complex*  $\Sigma$  [ $S$ ] is a finite or countable union set of [oriented]  $n$ -simplices such that if  $\sigma, \tau \in \Sigma$  and  $\sigma \cap \tau \neq \emptyset$ , their intersection is still a lower-dimensional simplex of  $\Sigma$  (i.e. a boundarying simplex of  $\partial\sigma$ ).

Let  $R(+, \cdot)$  be a ring, let  $C_p(\Sigma; R)$  be the  $R$ -module of all the  $p$ -chains, i.e.  $R$ -valued linear operators  $c$  defined on the oriented  $p$ -simplices of  $S$  such that  $c(\overset{\downarrow}{s}) = -c(\overset{\uparrow}{s})$ , and let  $C^p(\Sigma; R)$  be its dual group (whose elements are called  $p$ -cochains). If '@ <sub>$\Sigma$</sub> ' is the adjacency relation between  $\Sigma$ 's simplices, the  $p$ -boundary  $\partial_p: C_p(\Sigma; R) \rightarrow C_{p-1}(\Sigma; R)$  and  $p$ -coboundary  $\delta_p: C^p(\Sigma; R) \rightarrow C^{p+1}(\Sigma; R)$  linear operators are defined as acting this way respect so single  $p$ -(co-)chains<sup>c</sup>:

$$\begin{aligned} \mathbb{1}_p [\mathbf{v}_0, \mathbf{v}_1, \mathbf{K}, \mathbf{v}_p] &:= \sum_{j=0}^p (-1)^j \langle \mathbf{v}_0, \mathbf{K}, \mathbf{v}_j, \mathbf{K}, \mathbf{v}_p \rangle \mathbf{u}_j \\ d_p [\mathbf{v}_0, \mathbf{v}_1, \mathbf{K}, \mathbf{v}_p] &:= \sum_{\mathbf{u} @_{\Sigma} [\mathbf{v}_0, \mathbf{v}_1, \mathbf{K}, \mathbf{v}_p]} \mathbf{u} \end{aligned} \quad (1)$$

Associated to the chain groups  $\{C_p(\Sigma; R), \partial_p\}_{p \in \mathbb{Z}}$  and  $\{C^p(\Sigma; R), \delta_p\}_{p \in \mathbb{Z}}$  there are the  $p^{\text{th}}$  *simplicial homology* and *cohomology* groups, respectively [6]:

$$H_p(S; R) := \frac{\text{Ker } \mathbb{1}_p}{\text{Im } \mathbb{1}_{p+1}}, \quad H^p(S; R) := \frac{\text{Ker } d_p}{\text{Im } d_{p-1}}. \quad (2)$$

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<sup>b</sup> A simplex is the higher-dimensional analogue to a triangle (2-simplex): so a point is a 0-simplex, a segment is a 1-simplex and a tetrahedron is a 3-simplex.

<sup>c</sup> Here ' $1$ ' is the unity of ring  $R$ , whereas  $p$ -(co-)chains can be described as a linear combination (over  $R$ ) of isomorphisms, one for each  $p$ -simplex of  $\Sigma$ .

If  $K$  is a field (e.g.  $\mathbb{R}$  or  $\mathbb{C}$ ), all these  $K$ -modules become vector spaces, whereas simplicial homology and cohomology are isomorphic, i.e.  $C_p(\Sigma; K) \cong C^p(\Sigma; K)$  [4]. In this case  $\beta_p(\Sigma) = \dim H^p(\Sigma; K)$  is the  $p^{\text{th}}$  Betti number of  $\Sigma$ , i.e. the number of  $p$ -dimensional ‘holes’ inside<sup>d</sup>  $\Sigma$ .

## 2. Topological Calculus

Using a field-valued simplicial cohomology destroys many algebraic properties distinguishing the homology and cohomology groups with each other but allows discrete physical Field Theories biased towards applications to be easily formulated. In fact  $\partial_p$  and  $\delta_p$  become adjoint with each other, the  $\Pi$ -order operators  $\partial_{p+1}\delta_p$  and  $\delta_{p-1}\partial_p$  are self-adjoint, non-negative definite and self-commuting, and the Laplace-Beltrami operators (or  $p$ -beltramians) are defined:

$$D_p := - (\mathbb{I}_{p+1} d_p + d_{p-1} \mathbb{I}_p). \quad (3)$$

All these are the discrete analogue to I- and  $\Pi$ -order differential operators gradient, divergence, curl, Laplace’s and their main integro-differential theorems (e.g. Green’s and Stoke’s formulæ) naturally hold without further assumptions (unlike in most other discrete field theories’ formulations). Furthermore, the Helmholtz decomposition theorem for differential  $p$ -forms is a natural consequence of the Kronecker’s group structure theorem for  $C^p(\Sigma; K)$ :

$$H^p(S; K) = \text{Im } \mathbb{I}_{p+1} \dot{\wedge} \text{Im } d_{p-1} \dot{\wedge} H^p(S; K), \quad (4)$$

i.e. any  $p$ -cochain (discrete analogue to a  $p$ -form) is orthogonally decomposed into a gradient-free, a divergence-free and a *harmonic* part (which accounts for the presence of holes inside the domain), respectively [1], [4], [7].

## 3. Simplicial Electromagnetics formulation

Let  $\Sigma$  be the 2- or 3-dimensional simplicial complex triangulation of a continuum domain for an electromagnetic problem. Maxwell’s equations are<sup>e</sup>:

$$\begin{cases} \dot{\mathbb{I}}_2 d_2 D = r \\ \dot{\mathbb{I}}_2 d_2 B = 0 \end{cases}, \quad \begin{cases} \dot{\mathbb{I}}_1 d_1 E + \dot{\mathbb{I}}_1 B \dot{\wedge} = 0 \\ \dot{\mathbb{I}}_1 d_1 H - \dot{\mathbb{I}}_1 D \dot{\wedge} = J \end{cases}, \quad d_2 J + r \dot{\wedge} = 0. \quad (5)$$

Electric field  $E$  and magnetic field  $H$  are represented by 1-cochains because those fields are, basically, force fields (measured *along* paths), i.e. they are valued on the simplices’ sides; electric displacement  $D$ , magnetic flux density  $B$  and electric current density  $J$  are 2-cochains because they are flux/surface densities (measured *across* surfaces), i.e. they are valued on the simplices’ faces. Electric charge density  $\rho$  is a 3-cochain because it is measured *inside* simplices’ volumes, whereas the electrodynamic potentials  $(V, \mathcal{A})$  are respectively a 0- and a 1-cochain ( $V$  is measured *in* the simplices’ vertices).

By Laplace-transforming the previous equations (and using  $s \in \mathbb{C}$  instead of real time  $t \in \mathbb{R}$ ), using simplicial analogue to the well-known vector identities and setting a Lorenz gauge, the Helmholtz equations are found again. For simplicity’s sake, let us suppose that the domain is filled with an homogeneous medium (permittivity  $\epsilon$  and permeability  $\mu$  are trivial 0-cochains) and no impressed currents are given, then:

<sup>d</sup> The 0<sup>th</sup> Betti number is the number of connected components of  $\Sigma$  instead.

<sup>e</sup> Rightmost is continuity equation for the electric charge. Over-dot ‘ $\dot{\cdot}$ ’ denotes the time derivative for the associated cochain (time is basically continuous in this formulation).

$$\begin{cases} D_1 A(s) - s^2 e m A(s) = 0 \\ D_0 V(s) - s^2 e n V(s) = 0 \end{cases} \quad (6)$$

whose resolvent (formally the same for both  $A$  and  $V$ ) is easily computed starting from  $\Sigma$ 's adjacency matrices, so all the modes of the electromagnetic field with its resonating frequencies can be computed [1], [4].

Of course this model is purely topological (no metric information is carried together with the cochains), although in the case of a regular tetrahedral mesh,  $s$  becomes a 'normalized' parameter and the metric information is regained. Conversely, the domain's cohomology information are gained for free, just computing the vector subspace of harmonic  $p$ -cochains  $H^p(\Sigma; \mathbb{C})$ .

#### 4. Applications to Fractal Electrodynamics

Suppose that  $\Sigma$  is a triangulation of a *prefractal*, i.e. the finite-step iteration of 'just-touching' *Iterated Function System* (IFS) [5]. As shown in [4], the construction of such triangulation can be produced for the  $N^{\text{th}}$  step starting from the adjacency matrix of the initiator set and the resolvent matrix of the beltramian operator strictly depends on that.

For this reason the eigenfrequencies of a prefractal resonator, as well as the cut-off frequencies of a waveguide whose cross-section is such a prefractal (together with all its degenerancies) can be easily estimated, as well as is their asymptotic behaviour whenever  $N \rightarrow \infty$  (i.e. the self-similar fractal limit) [2].

Figures 1 and 2 show this algorithm for a resonating *Sierpinski gasket*: the normalized eigenvalues of (6) have a self-similar distribution, with broader plateaux (Fig. 2) associated to higher multiplicities due to the bigger holes of the gaskets (Fig. 1). The electric potential's eigenfunctions also show some degree of self-similarity, since many of them are present on smaller copies of the gasket (i.e. the similar waveforms at higher frequencies), which is a characteristic of continuum *diaperiodic* modes on self-similar sets (and from which their fractal dimension can be estimated) [3].

Figures 3 and 4 show the same analysis done with a Sierpinski carpet domain: again the fractal topology induces self-similarity of the resolvent matrix and self-similar eigenfunctions.

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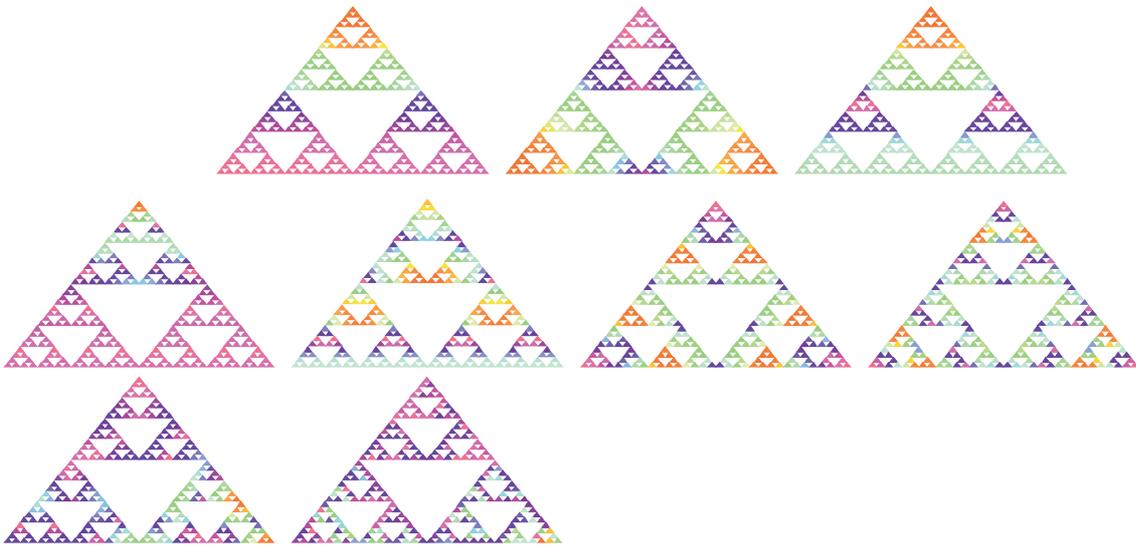


Figure 1. Eigenvectors of the 0<sup>th</sup> Laplace-Beltrami operator on the simplicial Sierpinski gasket's 6<sup>th</sup>-step prefractal.

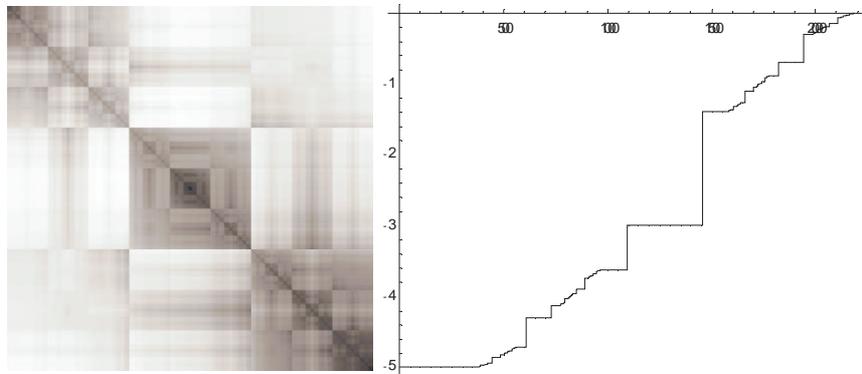


Figure 2. Resolvent matrix<sup>f</sup> of the scalar Laplace-Beltrami operator for the simplicial Sierpinski carpet's 6<sup>th</sup>-step prefractal and its eigenvalues.

<sup>f</sup> Absolute values of the matrix elements (as in Fig. 3) are represented as dots' brightness.

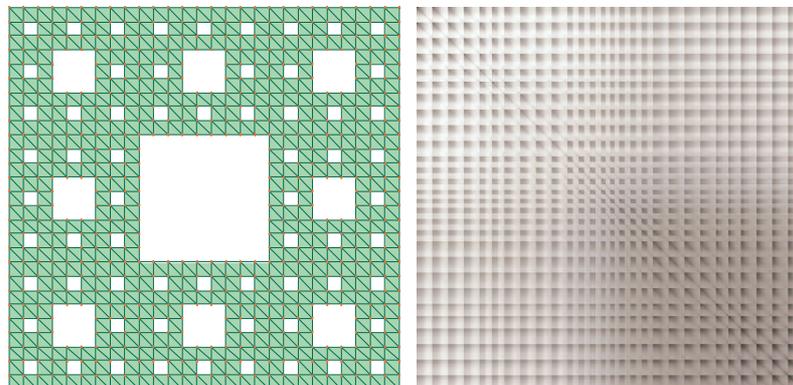


Figure 3. Simplicial Sierpinski carpet's 3<sup>rd</sup>-step prefractal and resolvent matrix of its scalar Laplace-Beltrami operator.

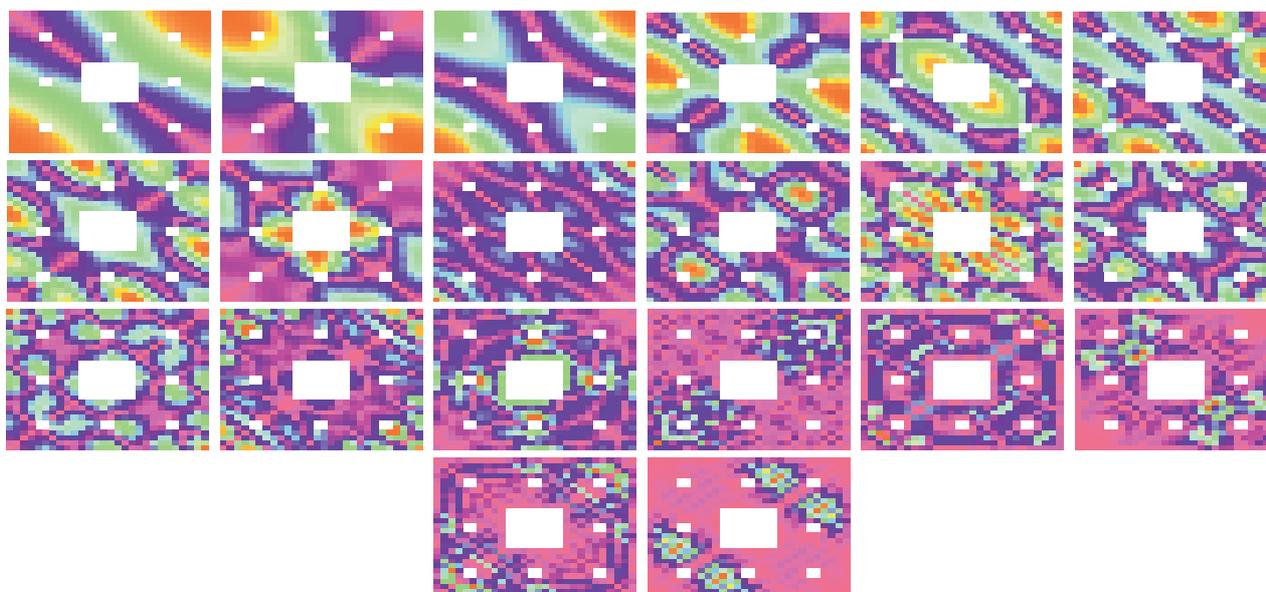


Figure 4. Eigenvectors of the scalar Laplace-Beltrami operator on the simplicial Sierpinski carpet's 3<sup>rd</sup>-step prefractal (increasing eigenvalues).