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Implicit Pseudo-Spectral Methods for Dispersive and Wave Propagation Problems

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For the sake of simplicity, we consider first the derivation of the implicit pseudo-spectral methods in the case of the linear advection equation

$$\frac{\partial u}{\partial t} + \mathrm{D}(au) = 0 \ , \qquad \mathrm{D} = \frac{\partial}{\partial x} \ ,$$

where a is a constant different from zero. We rewrite the equation in the form

$$\frac{\partial u}{\partial t} = -\mathrm{D}(au) \ ,$$

and integrate from t to $t + \Delta t$ to get

$$u(x, t + \Delta t) - u(x, t) = -\int_t^{t+\Delta t} \mathrm{D}(au) dt \ .$$

At this point, we can obtain different methods for the advection equation by using different quadrature rules for the integral appearing in the above equation. The resulting methods have the same discretization error of the applied quadrature rule, because no errors were introduced before. In order to avoid any stability restriction to the time step, Wineberg et al. [4] propose to apply the trapezoid rule, but other possibilities are still available. For instance we can use, to a first order of accuracy, the end-point rectangle rule (implicit Euler) to find out

$$u(x, t + \Delta t) - u(x, t) = -\Delta t \mathrm{D}(au(x, t + \Delta t)) \ .$$

The last equation can be rewritten in the form

$$(1 + a\Delta t \mathrm{D})u(x, t + \Delta t) = u(x, t) \ ,$$

or symbolically, in the step form

$$u(x, t + \Delta t) = \mathrm{R}(\Delta t)u(x, t) \ , \qquad \mathrm{R}(\Delta t) = \frac{1}{\mathrm{I} + a\Delta t \mathrm{D}} \ ,$$

where we have defined the step operator $\mathrm{R}(\Delta t)$. Note that, the inversion of $\mathrm{I} + a\Delta t\mathrm{D}$ is straightforward since D is skew adjoint and its eigenvalues are imaginary.

In the case of the trapezoid rule (Crank-Nicolson) we end up with the same equation but a different step operator

$$\mathrm{R}(\Delta t) = \frac{\mathrm{I} - \frac{1}{2} a \Delta t \mathrm{D}}{\mathrm{I} + \frac{1}{2} a \Delta t \mathrm{D}} \ .$$

Pseudo-spectral methods using the trapezoid rule have been applied successfully to several problems of interest governed by nonlinear PDEs: Korteweg-de Vries (KdV), Klein Gordon, Whitham (the equation for weak dispersion proposed in [3]), etc. As an example let us consider the KdV equation

$$\frac{\partial u}{\partial t} + \mathrm{D}^3 u + \mathrm{D}\left(\frac{u^2}{2}\right) = 0 \ .$$

It is a simple matter to verify that, in the trapezoid rule case, we get symbolically

$$u(x, t+\Delta t) = \mathrm{R}(\Delta t) u(x,t) - \mathrm{S}(\Delta t)\left(u^2(x, t+\Delta t) + u^2(x,t)\right) \ ,$$

where $\mathrm{R}(\Delta t)$ and $\mathrm{S}(\Delta t)$ are symbolic operators defined by

$$\mathrm{R}(\Delta t) = \frac{\mathrm{I} - .5 \Delta t \mathrm{D}^3}{\mathrm{I} + .5 \Delta t \mathrm{D}^3} \ ,$$

and

$$\mathrm{S}(\Delta t) = \frac{.25 \Delta t \mathrm{D}}{\mathrm{I} + .5 \Delta t \mathrm{D}^3} \ .$$

In the time domain, we have

$$v(t+\Delta t) = \mathrm{R}(\Delta t) v(t) - \mathrm{S}(\Delta t)\left(\texttt{fft}(u^2(x, t+\Delta t)) + \texttt{fft}(u^2(x,t))\right) \ ,$$

where $v(t) = \texttt{fft}(u(x,t))$, and $\texttt{fft}(\cdot)$ indicates the fast Fourier transform (FFT). Moreover, the introduced symbolic operators can be computed by the FFT. As usual, the nonlinear terms are best computed in the spatial representation, hence we transform back to the original space, make the multiplication, which is point-wise in x, and transform again. We have here an implicit method. For the solution of the nonlinear system, it is possible to apply the Newton method, but it requires the inversion of full matrices. As a consequence, Newton iterations result to be not suitable for spectral methods. On the other hand, nonlinear spectral methods are usually implemented by using, first order but simpler, successive approximation. That is, we can apply the iterations

$$v_{n+1}(t+\Delta t) = w(t) - \mathrm{S}(\Delta t)\left(\texttt{fft}(u_n^2(x, t+\Delta t))\right) \ ,$$

where

$$w(t) = \mathrm{R}(\Delta t) v(t) - \mathrm{S}(\Delta t)\left(\texttt{fft}(u^2(x,t))\right) \ ,$$
$$u_n(x, t+\Delta t) = \texttt{ifft}(v_n(t+\Delta t)) \ , \qquad u_1^2(x, t+\Delta t) = u^2(x,t) \ ,$$

here $\texttt{ifft}(\cdot)$ is the inverse FFT.

In the following the pseudo-spectral methods are applied to two test problems. Let us suppose that we would like to solve a model problem written here in the general form:

$$\begin{aligned} & q_\tau + f\left(q, q_\xi, q_{\xi\xi}, q_{\xi\xi\xi}\right) = 0 \ , \qquad \xi \in [0, L] \\ & q(\xi, 0) = q_0(\xi) \ , \qquad q(0,\tau) = q(L,\tau) \ , \end{aligned} \tag{0.1}$$

where $q(\cdot,\cdot) : [0, L] \times \mathbb{R}^+ \to \mathbb{R}$, $f(\cdot,\cdot,\cdot,\cdot)$ is a given function of the field variable q and its first, second, and third order derivatives with respect to the space variable ξ, and τ represents the time. In order to be able to apply the FFT, we have to transform this problem into one defined within the space domain $[0, 2\pi]$. This can be done by introducing the independent variables transformation given by: $\xi = \lambda x$ and $t = \tau$, where $\lambda = L/(2\pi)$. By using the new dependent variable $u(x, t) = q(\lambda x, \tau)$, it is a simple matter to rewrite the problem (0.1) as:

$$(0.2)\qquad \begin{aligned} &u_t + f\left(u, \frac{1}{\lambda}u_x, \frac{1}{\lambda^2}u_{xx}, \frac{1}{\lambda^3}u_{xxx}\right) = 0 \ , \qquad x \in [0, 2\pi] \\ &u(x, 0) = q_0(\lambda x) \ , \qquad u(0, t) = u(2\pi, t) \ . \end{aligned}$$

From the model problem (0.1) we recover several problem of interest governed by well-known equations by setting particular functional forms of f. We can, for instance, recover problems governed by the KdV, the Burger's or the advection equations. The reformulation of the considered problems given by equation (0.2) can be used for all cases.

As a simple test problem, we consider the classical two solitons interaction discovered by Kruskal and Zabusky in the 1960's [1]. The problem to be solved is given by:

$$(0.3)\qquad \begin{aligned} &q_\tau + q_{\xi\xi\xi} + q q_\xi = 0 \ , \qquad \xi \in [0, L] \\ &q(\xi, 0) = q_0(\xi) \ , \qquad q(0, \tau) = q(L, \tau) \ , \end{aligned}$$

where $L = 50$ and the initial condition is

$$(0.4)\qquad q_0(\xi) = 12c_1^2 sech^2(c_1(\xi - 0.1L)) + 12c_2^2 sech^2(c_2(\xi - 0.4L)) \ ,$$

with $c_1 = 1$ and $c_2 = 0.5$.

A MATLAB code was used to implement the second order method defined above and to produce the numerical results reported in figure 0.1.

The extension of the considered implicit methods to two or more spatial dimension is straightforward. So that, at the end of this note we report some numerical results obtained by the first order spectral method (implicit Euler) applied to a 2D advection problem. This test problem is the test 9.2 used by LeVeque [2]

$$(0.5)\qquad \frac{\partial c}{\partial t} + \frac{\partial (u\ c)}{\partial x} + \frac{\partial (v\ c)}{\partial y} = 0 \ ,$$

defined in $\Omega = [0, 1] \times [0, 1]$, where $u = 1$ and $v = 1$ and initial and boundary conditions are

$$(0.6)\qquad \begin{aligned} &u(x, y, 0) = sin(2\pi x)\ sin(2\pi y) \\ &u(0, y, t) = u(1, y, t) \ , \qquad u(x, 0, t) = u(x, 1, t) \ . \end{aligned}$$

In figure (0.2) we display the initial condition and sample numerical solutions: at the intermediate time $t = 0.25$ and at the final time $t = 1$, when the solution is equal to the initial condition, computed with a MATLAB code.

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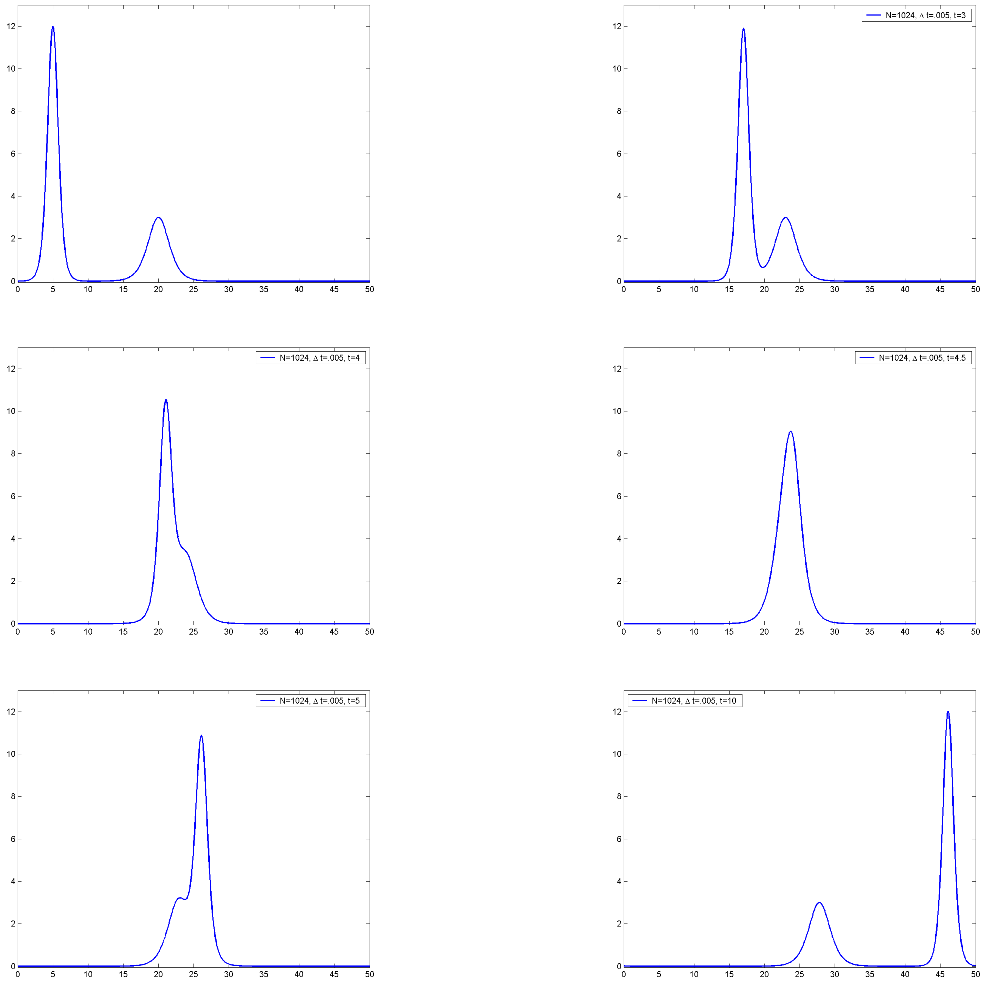

Figure 0.1: Interaction of two solitons for the KdV equation. Numerical solutions with 1024 mesh-points in the x variable and $\Delta t = 0.005$. Top-left: $t = 0.0$, top-right: $t = 3.0$, center-left: $t = 4.0$, center-right: $t = 4.5$, bottom-left: $t = 5.0$, and bottom-right: $t = 10.0$.

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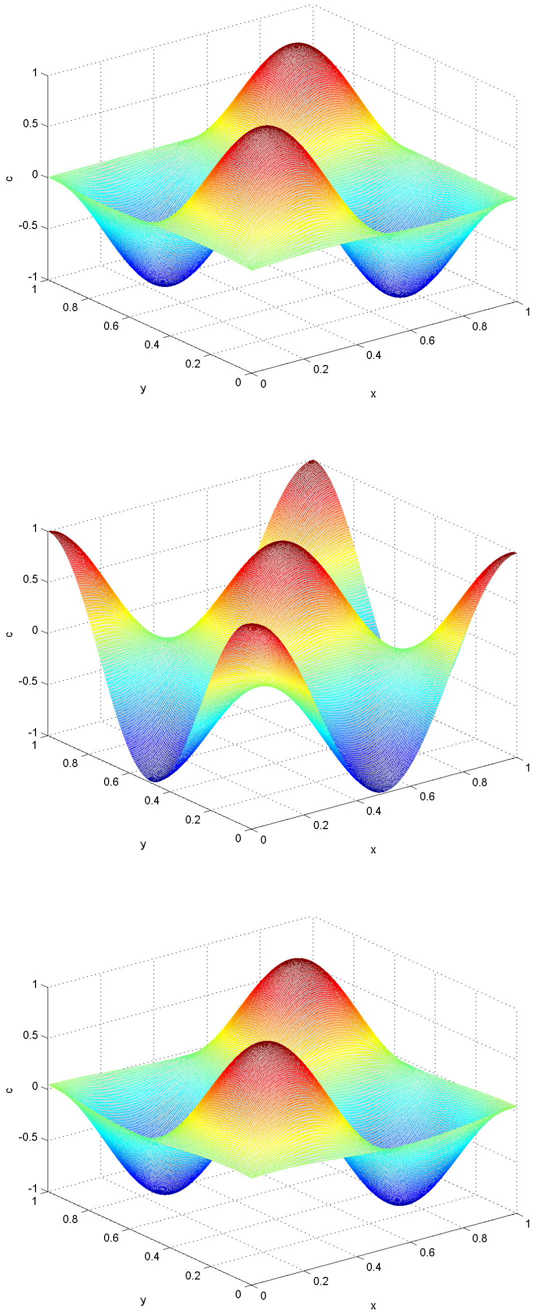

Figure 0.2: 2D test problem: with 256×256 mesh-points in the x and y variables and a time step $\Delta t = 0.001$. Top: initial condition. Middle: numerical solution at $t = 0.25$. Bottom: numerical solution at $t = 1$.