Note on Dispersive Effects in Quantum Kinetic Equations

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In the paper [1] we presented a new strategy for the well-posedness analysis of quantum kinetic problems that include a Hartree-type nonlinearity. There we focused on the 3-dimensional Wigner-Poisson-Fokker-Planck (WPFP) system, but we expect this new approach to be suitable for a broad range of quantum kinetic problems.

In order to describe the non-reversible interaction of a quantum system with its environment, a possible modification of the Wigner equation, [9], consists in introducing a Fokker-Planck type operator on the right hand side, [7] e.g.:

$$(0.1) \qquad \partial_t w + v \cdot \nabla_x w - \Theta[V]w = \beta \mathrm{div}_v(vw) + \sigma \Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha \Delta_x w$$

with $w = w(x, v, t)$ Wigner function, $V = V(x)$ potential, $\Theta[V]$ pseudo-differential operator defined by

$$(\Theta[V]w)(x, v, t) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \delta V(x, \eta) \mathcal{F}_{v \to \eta} w(x, \eta, t) e^{iv \cdot \eta} \, d\eta$$

where $\delta V(x, \eta) := V(x + \eta/2) - V(x - \eta/2)$ and $\mathcal{F}_{v \to \eta} w$ denotes the Fourier transform of w with respect to v. $\beta \geq 0$ is the friction parameter and the coefficients $\alpha, \gamma \geq 0$, $\sigma > 0$ constitute the phase-space diffusion matrix of the system. In the Fokker-Planck equation of classical mechanics one would have $\alpha = \gamma = 0$. For the WFP equation (0.1), the so-called Lindblad condition guarantees that the evolution of the system is "quantum mechanically correct" (i.e., it corresponds to a positive density matrix, [3]); however, for the mathematical analysis, it suffices that (0.1) is parabolic or degenerate parabolic. Thus, we shall only assume $\alpha\sigma \geq \gamma^2$ henceforth.

We consider the case when $V = V(x, t)$ models the mean-field interaction in the quantum system: how to define rigorously the Hartree-potential in a quantum kinetic framework is indeed one of the crucial points in the paper.

DEFINITION 1 (STANDARD DEFINITION OF MEAN-FIELD QUANTITIES). *To a Wigner function $w(t)$ is associated the position density $n := n(x, t) := \int w(x, v, t) \, dv$, $x \in \mathbb{R}^3$, $t > 0$, the potential $V = V(x, t) := -\frac{1}{4\pi|x|} * n(x, t)$, which solves the Poisson equation*

$$(0.2) \qquad -\Delta V(t) = n(t), \ x \in \mathbb{R}^3, \ t > 0,$$

and the field $E(x, t) := \nabla_x V(x, t)$. In classical kinetic theory the phase space density typically satisfies $f(., ., t) \in L^1(\mathbb{R}^6)$ which yields a position density $n(\cdot, t) = \int f \, dv \in L^1(\mathbb{R}^3)$.

In quantum kinetic theory, however, the natural framework is $w(.,.,t) \in L^2(\mathbb{R}^6)$, which makes Def. 1.1 meaningless. In order to establish well-posedness of the (WP or) WPFP (0.1),(0.2) systems, two strategies have been used so far. The first possibility is to reformulate the WP or WPFP systems either in terms of Schrödinger wave-function sequences, [6], or in terms of density matrices, [3]. In such a framework, all physical quantities are well-defined, in particular $n(t) \in L^1_+(\mathbb{R}^3)$ and the physical conservation laws for mass and energy play a crucial role in the analysis for large time. Alternatively, one can keep to the kinetic formulation and to kinetic tools, with the perspective of later tackling boundary-value problems, which are more reasonable models for real simulations. The literature related to the latter approach can be split into two groups: in several articles ([2], e.g.), a L^2-setting is chosen for $w(t)$, such that $w(t)$ satisfies at least the necessary condition to describe a quantum system, [9]. Then, v-weights are introduced in order to enforce integrability in the v-variable, so to give sense to Def. 1.1. In other articles ([5], e.g.) instead, a L^1-setting is chosen with the same motivation. However, in neither of the two above approaches physical conservation laws can be exploited directly, since both the mass and the kinetic energy are *not* positive functionals under the assumptions made at the kinetic level.

A third aspect that differentiates quantum from classical kinetic theory, is the lack of a maximum principle: $\|w(t)\|_{L^2(\mathbb{R}^6)}$ is the only conserved norm by the Wigner equation. Due to the described differences, the analytic approach used for classical kinetic models can *not* be adapted to quantum kinetic ones.

In order to achieve a global-in-time result for the WPFP system, in [2] we exploited dispersive effects of the free-streaming operator jointly with the parabolic regularization of the Fokker-Planck term, since this yields a-priori estimates for the solution $w(t)$ in a weighted L^2-space. Such dispersive techniques for kinetic equations were first developed for the VP system in [11]. In [2] these tools were extended to quantum kinetic theory.

In [1], we achieved as well a global-in-time well-posedness result for the WPFP system in the space $L^2(\mathbb{R}^6)$, *but without* introducing weights. This is possible thanks to an alternative strategy that relies first of all on an a-priori estimate for the field $\nabla_x V(t)$ in terms of the $\|w(t)\|_{L^2(\mathbb{R}^6)}$ only. This estimate was derived in [2] using dispersive effects of the free-streaming operator. It allow a *novel definition* of the macroscopic quantities, which, in contrast to the Definition 1, is now non-local in time. This way, no v-integrability of w is needed, and hence no moments in v either. Secondly, we shall use the (degenerate) parabolic regularization of the Fokker-Planck term. These techniques allow to overcome the described analytical difficulties and they yield –a-posteriori– some L^p-estimates on the density.

In conclusion, our purely kinetic L^2-analysis solves both main problems of quantum kinetic theory, namely the definition of the density (due to the missing v-integrability of w) and the lack of usable a-priori estimates on w (due to its non-definite sign). Finally, we point out that we expect that this approach could also be a crucial step towards developing a kinetic analysis for the Wigner-Poisson system, which has been an open problem for 15 years.

Let us present the assumptions on the initial data w_0 in order to quote the main result of the paper [1]:

$$w_0 \in L^2(\mathbb{R}^6), \quad \|n_0^\vartheta(t)\|_{L^\theta(\mathbb{R}^3_x)} \le C_T t^{-\omega_\theta}, \text{ for some } \omega_\theta \ge 0, \ \forall\, t \in (0,T] \quad (\mathbf{A})$$

where $n_0^\vartheta(x,t) := \int w_0(x - \vartheta(t)v, v)\, dv$, $\vartheta(t) := (1 - e^{-\beta t})/\beta$. Observe that this is consistent with Strichartz estimate for the (free) kinetic equation [8]

$$\|n_0^\vartheta(t)\|_{L^\theta(\mathbb{R}_x^3)} \leq Ct^{-\omega_S(\theta)}\|w_0\|_{L_x^1(L_v^\theta)}, \quad \forall\, t \in (0,T], \quad \omega_S(\theta) := 3(1 - 1/\theta).$$

Theorem 4.1 *Let* **(A)** *hold for some* $\theta \in I_\theta$ *and* $0 \leq \omega_\theta < \kappa(\theta)$. *Then, there exists a unique mild solution* $w \in \mathcal{C}([0,\infty); L^2(\mathbb{R}^6))$ *of the WPFP problem.*

A-posteriori, we obtain analogous regularity result to [4,10] for VPFP case:

Theorem 5.1 *Under the same assumptions,* $w \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^6))$,

$$\|D_x^l D_v^m w(t)\|_{L^2(\mathbb{R}^6)} \leq C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{-\frac{L}{2}} t^{-\frac{M}{2}}, \quad \forall\, t \in (0,T],$$

for all $T > 0$, *and all multiindices* $l, m \in \mathbb{N}_0^3$, *with* $|l| = L$, $|m| = M \in \mathbb{N}_0$, *where* $R(t) = \mathcal{O}(t)$, *if* $\alpha > 0$, $R(t) = \mathcal{O}(t^3)$, *if* $\alpha = 0$, $\sigma > 0$ *(hypo-elliptic case). Moreover,* $E, V, n \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$, *and for all* $T > 0$:

$$\|D_x^l E(t)\|_{L^2(\mathbb{R}^3)} \leq C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{2} - \frac{1}{\theta}\right) - \frac{L-1}{2}} t^{-\omega_\theta}, \ \forall\, t \in (0,T].$$

REFERENCES

1. A. Arnold, E. Dhamo and C. Manzini, Dispersive effects in quantum kinetic equations, submitted (2006).
   Preprint: http://mti.sns.it/cmanzini/disp.pdf

2. A. Arnold, E. Dhamo and C. Manzini, The Wigner-Poisson-Fokker-Planck system: global–in–time solution and dispersive effects, submitted (2005). Preprint: http://mti.sns.it/cmanzini/preprintADM04.pdf

3. A. Arnold and C. Sparber, Conservative quantum dynamical semigroups for mean-field quantum diffusion models, *Comm. Math. Phys.*, vol. 251(1), 2004, 179–207.

4. F. Bouchut, Smoothing effect for the non-linear Vlasov-Poisson-Fokker-Planck system, *J. Diff. Eq.*, vol. 122(2), 1995, 225–238.

5. J.A. Cañizo, J.L. López and J. Nieto, Global L^1-theory and regularity for the 3D nonlinear Wigner-Poisson-Fokker-Planck system, *J. Diff. Eq.*, vol. 198, 2004, 356–373.

6. F. Castella, L^2 solutions to the Schrödinger-Poisson system: existence, uniqueness, time behaviour, and smoothing effects, *Math. Models Methods Appl. Sci.*, vol. 7(8), 1997, 1051–1083.

7. F. Castella, L. Erdös, F. Frommlet and P.A. Markowich, Fokker-Planck equations as scaling limits of reversible quantum systems, *J. Stat. Physics*, vol. 100(3/4), 2000, 543–601.

8. F. Castella and B. Perthame, Strichartz estimates for kinetic transport equations, *C. R. Acad. Sci.*, vol. 332(1), 1996, 535–540.

9. P.L. Lions and T. Paul, Sur les mesures de Wigner, *Rev. Math. Iberoam.*, vol.9(3), 1993, 553–561.

10. K. Ono and W.A. Strauss, Regular solutions of the Vlasov-Poisson-Fokker-Planck system, *Discrete Contin. Dynam. Systems*, vol. 6, 2000, 751–772.

11. B. Perthame, Time decay, propagation of low moments and dispersive effects for kinetic equations, *Comm. P.D.E.*, vol. 21(1/2) 1996, 659–686.