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REGULARITY RESULTS FOR TIME-DEPENDENT VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES WITH APPLICATIONS TO DYNAMIC TRAFFIC NETWORKS

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Abstract

The aim of this paper is to consider time-dependent variational and quasi-variational inequalities and to study under which assumptions the continuity of solutions with respect to the time can be ensured. Making on appropriate use of the set convergence in Mosco's sense, we get the desiderate continuity results for strongly monotone variational and quasi-variational inequalities. The continuity results allow us to provide a discretization procedure for the calculation of the solution to the variational inequality which expresses the time-dependent traffic network equilibrium problem.

Key words: time-dependent variational and quasi-variational inequalities, continuity solution, dynamic traffic networks, projection method.

1 Introduction

The paper presents results regarding continuity solutions to evolutionary variational and quasi-variational inequalities, showing that the continuity results hold not only for linear evolutionary variational and quasi-variational inequalities but also for nonlinear evolutionary variational and quasi-variatio- nal inequalities.

These results reveal themselves very useful for the calculus to solutions to dynamic network equilibrium problems, because they allow to apply a discretization procedure which reduces the problem to the calculus of solutions to the finite variational inequality. Moreover, it is possible to solve other equilibrium problems, as for example the spatial equilibrium problems with either quantity or price formulations and a variety of financial equilibrium problems, because they have a similar formulation in terms of variational inequalities.

2 The dynamic model

Let us recall the model of equilibrium flows in a dynamic traffic network. It is represented by a graph $G = [N, L]$, where N is the set of nodes and L is the set of directed links between the nodes. Let R_r be a path consisting of a sequence of links which connect an Origin-Destination (O/D) pair of nodes. Let m be the number of the paths in the network. Let $\mathcal{W}$ denote the set of the O/D pairs with typical O/D pair w_j, $|\mathcal{W}| = l$ and $m > l$. The set of paths connecting the O/D pair w_j is represented by $\mathcal{R}_j$ and the entire set of paths in the network by $\mathcal{R}$. The topology of the network is described by the pair-link incidence matrix $\Phi = (\varphi_{j,r})$, where $\varphi_{j,r}$ is 1 if path $R_r \in \mathcal{R}_j$ and 0 otherwise. The flow vector is a time-dependent flow trajectory $F : [0, T] \to \mathbb{R}^m_+$, while the topology

remains fixed. Let $\lambda, \mu \in L^2([0,T], \mathbb{R}^m_+)$ be capacity constraints and let $\rho \in L^2([0,T], \mathbb{R}^m_+)$ be the travel demand function. Let $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be the cost trajectory.
The vector-function $H \in \mathbf{K}$, where

$$\mathbf{K} = \Big\{ F \in L^2([0,T], \mathbb{R}^m_+) : \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t), \text{ a.e. in } [0,T] \Big\}$$

is a user traffic equilibrium flow if $\forall w_j \in \mathcal{W}$, $\forall R_q, R_s \in \mathcal{R}_j$ and a.e. in $[0,T]$ it results:

$$C_q(t, H(t)) > C_s(t, H(t)) \Longrightarrow H_q(t) = \lambda_q(t) \quad \textit{or} \quad H_s(t) = \mu_s(t). \tag{2.1}$$

Such a user traffic equilibrium flow is characterized by the following evolutionary variational inequality:

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T]. \tag{2.2}$$

Now, let us introduce the time-dependent elastic problem which arose whenever travel demands are not only dependent on the time but also on the equilibrium distribution. Let $\rho : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^l_+$, let $D \subseteq L^2([0,T], \mathbb{R}^m_+)$ be a nonempty, compact and convex subset and let $\mathbf{K} : D \to 2^{L^2([0,T], \mathbb{R}^m_+)}$ be a set-valued mapping, defined by

$$\mathbf{K}(H) = \Bigg\{ F \in L^2([0,T], \mathbb{R}^m_+) : \; \lambda(t) \leq F(t) \leq \mu(t), \quad \text{a.e. in } [0,T],$$

$$\Phi F(t) = \frac{1}{T} \int_0^T \rho(t, H(\tau)) d\tau \quad \text{a.e. in } [0,T] \Bigg\}.$$

Then the quasi-variational inequality that models the traffic equilibrium problem in the elastic case is the following:

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0,T]. \tag{2.3}$$

3 Continuity results

In this section, we present results of continuity for solutions to evolutionary variational and quasi-variational inequalities associated to a linear operator, namely when $C(t, H(t)) = A(t)H(t) + B(t)$, and a nonlinear operator.

Theorem 3.1. ([1]) *Let $A \in C([0,T], \mathbb{R}^{m \times m}_+)$ be a positive defined matrix-function and let $B \in C([0,T], \mathbb{R}^m_+)$ be a vector function. Suppose that $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ and $\rho \in C([0,T], \mathbb{R}^l_+)$. Then, the linear evolutionary variational inequality admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m_+)$. Moreover, the estimate*

$$\|H_1 - H_2\|_{C([0,T], \mathbb{R}^m_+)} \leq \frac{1}{\nu} \|B_1 - B_2\|_{C([0,T], \mathbb{R}^m_+)}$$

holds, where ν is the constant of positive definition of matrix $A(t)$, for each $t \in [0,T]$.

Moreover, if the cost operator C is nonlinear, strongly monotone, belongs to $C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^m_+)$ and satisfies the following condition

$$\|C(t, F)\|_m \leq A(t) \|F(t)\|_m + B(t), \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T],$$

with $A, B \in C([0,T], \mathbb{R}^m_+)$, then the nonlinear evolutionary variational inequality (2.2) has a unique solution which is continuous (see [2]).

THEOREM 3.2. ([1]) *Let* $A \in C([0,T], \mathbb{R}^{m\times m}_+)$ *be a positive definite matrix-function and let* $B \in C([0,T], \mathbb{R}^m_+)$ *be a vector function. Let* $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ *be and let* $\rho \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ *be such that*

$$\exists \psi \in L^1([0,T], \mathbb{R}^m_+) : \; \|\rho(t,F)\|_l \leq \psi(t) + \|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+,$$

$$\exists \nu \in L^2([0,T], \mathbb{R}_+) : \; \|\rho(t,F_1) - \rho(t,F_2)\|_l \leq \nu(t)\|F_1 - F_2\|^2_m, \quad \forall F_1, F_2 \in \mathbb{R}^m_+,$$

a.e. in $[0,T]$*. Then, the quasi-variational inequality (2.3) admits a solution* $H \in \mathbf{K}(H)$ *such that* $H \in C([0,T], \mathbb{R}^m_+)$*.*

This theorem holds for the solutions to nonlinear time-dependent quasi-variational inequalities if the nonlinear operator C belongs to $C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^m_+)$, and satisfies the following conditions

$$\exists \gamma \in L^2([0,T], \mathbb{R}_+) : \; \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \text{ a.e. in } [0,T],$$

$$\exists \nu > 0 : \; \langle C(t,F), F \rangle \geq \nu \|F\|^2, \quad \forall F \in \mathbb{R}^m_+, \text{ a.e. in } [0,T].$$

4 Approximation method

We consider the evolutionary variational inequality (2.2) and we suppose that the assumptions above established are satisfied and hence the solution $H \in C([0,T], \mathbb{R}^m_+)$. As a consequence, (2.2) holds for each $t \in [0,T]$, namely

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall t \in [0,T].$$

We consider a partition of $[0,T]$, such that $0 = t_0 < \ldots < t_i < \ldots < t_N = T$. Then, for each value t_i, for $i = 0, \ldots, N$, we apply the projection method for solving the variational inequality

$$\langle C(t_i, H(t_i)), F(t_i) - H(t_i) \rangle \geq 0, \quad \forall F(t_i) \in \mathbf{K}(t_i), \tag{4.1}$$

where

$$\mathbf{K}(t_i) = \left\{ F(t_i) \in \mathbb{R}^m_+ : \; \lambda(t_i) \leq F(t_i) \leq \mu(t_i), \; \Phi F(t_i) = \rho(t_i) \right\}.$$

The algorithm, starting from any $H^0(t_i) \in \mathbf{K}(t_i)$ fixed, iteratively updates $H(t_i)$ according to the formula

$$H^{k+1}(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(t_i, H^k(t_i))),$$

for $k \in \mathbb{N}$, where $P_{\mathbf{K}(t_i)}(\cdot)$ denotes the orthogonal projection map onto $\mathbf{K}(t_i)$ and α is a judiciously chosen positive steplength. If C is strongly monotone (with constant ν) and Lipschitz continuous on $\mathbf{K}$ (with Lipschitz constant L), and if $\alpha \in (0, 2\nu/L^2)$, the projection method determines a sequence $\{H^k(t_i)\}_{k\in\mathbb{N}}$ convergent to the solution to (4.1). After iterative procedure, we can construct an approximated equilibrium solution by linear interpolation.

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