

## An S- linear State Preference Model

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We first recall the standard setting of the state preference model.

**Economic model.** We consider a market, and we observe it only two times, called the initial time and the final time. Assume that in the market there are  $n$  goods. For every  $j \in \underline{n}$ , each unit of the  $j$ -th good can assume  $m$ -possible values at the final time. These values depend on  $m$  states of the world. The value of a unit of the  $j$ -th good, in the  $i$ -th state of the world, is the real number  $a_{ij}$ .

**Definition.** A state preference model is a system  $(G, S, a)$ , where  $G$  is an ordered set with  $n$  elements, called the set of the goods of the model,  $S$  is an ordered set with  $m$  elements, called the set of the states of the world, and  $a$  is an  $(m, n)$ -matrix, called the values-matrix of the model. For every positive integer  $j \leq n$ , the vector  $C_j(a) = (a_{ij})_{i \in \underline{m}}$  (the  $j$ -th column of  $a$ ) is called the values-vector of the  $j$ -th good.

**Definition (of portfolio).** We say portfolio, of the considered market, every  $n$ -tuple  $x \in \mathbb{R}^n$ .

**Economic interpretation.** If  $j \in \underline{n}$ , the  $j$ -th component of  $x$  is the quantity of the  $j$ -th good, bought if it is positive, sold (at overdraft) if  $x_j$  is negative.

If we consider a portfolio  $x$  and we have to calculate the value of the portfolio in the  $i$ -th state of the world, we have simply to calculate the following number

$$v_i(x) = \sum_{j=1}^n a_{ij} x_j = R_i(a) \cdot x.$$

**Definition (of values-representation).** The vector of  $\mathbb{R}^m$  defined by

$$ax = (R_i(a) \cdot x)_{i=1}^m,$$

is called the  $a$ -representation of the portfolio  $x$ .

**Economic interpretation.** A portfolio  $x$  can be represented by the  $m$ -vector  $ax$ , its  $a$ -representation, whose components are the values that the portfolio  $x$  takes on the  $m$  states of the world. In these conditions,  $x$  is a vector of quantities,  $ax$  is a vector of values.

The matrix  $a$  generates in a natural way a preference relation.

**Definition (the preference relation generated by  $a$ ).** We say that a portfolio  $x$  is preferred or indifferent to  $x'$  with respect to  $a$ , and we write  $x \succeq_a x'$ , if  $ax \geq ax'$  (i.e.,  $a(x - x')$  is a vector with non negative components). In other words, for every state of the world  $s$ , the value of  $x$  in  $s$  is greater or equal to the value of  $x'$  in  $s$ .

**Definition (the price of a portfolio).** Let  $p$  be an  $n$ -vector, the price of a portfolio  $x$  relative to  $p$  is the product  $(p | x)_n = p \cdot x$ .

**Definition (no-arbitrage price vectors).** An  $n$ -vector  $p$  such that, for every  $n$ -portfolios  $x$  and  $x'$ , one has

$$x \succeq_a x' \Rightarrow px \geq px',$$

is said compatible with  $\succeq_a$  or a no-arbitrage price vector.

**Definition (the  $a$ -representation of a system of prices).** Let  $A$  be the linear operator canonically associated to the matrix  $a$ . Let  $p$  be an  $n$ -vector. An  $m$ -tuple  $q \in \mathbb{R}^m$ , such that, for every portfolio  $x$ , we have

$$(p | x)_{\mathbb{R}^n} = (q | Ax)_{\mathbb{R}^m},$$

is called an  $a$ -representation of  $p$  in  $\mathbb{R}^m$ .

**Theorem (characterization of the representations of a price-vector).** Let  $(G, S, a)$  be a state preference model, and let  $A$  be the operator canonically associated to  $a$ . Then, an  $m$ -vector  $q$  is a representation of  $p$  if and only if  $p = A^*u$ , where  $A^*$  is the euclidean-adjoint of  $A$ .

*Proof.* It is well known that, there exists an operator  $A^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that

$$(u | Ax)_{\mathbb{R}^m} = (A^*u | x)_{\mathbb{R}^n},$$

for every  $m$ -tuple  $u$  and every  $n$ -tuple  $x$ ,  $A^*$  is called the euclidean-adjoint of  $A$  ( $A^*$  is the operator canonically associated to  ${}^t a$ ). By definition  $q$  is a representation of  $p$  if and only if

$$(p | x)_{\mathbb{R}^n} = (q | Ax)_{\mathbb{R}^m},$$

for every portfolio  $x$ , or equivalently,

$$(p | x)_{\mathbb{R}^n} = (A^*q | x)_{\mathbb{R}^n}.$$

The last equality holds if and only if  $p = A^*u$ . As desired. ■

**Economic interpretation.** In the state preference model, every price vector  $p \in \text{lim func im } A^*$ , on the space of quantities  $\mathbb{R}^n$ , can be represented by the  $m$ -vectors  $q$  (price-vectors on the space of values  $\mathbb{R}^m$ ) such that  $p = A^*q$ . The price of  $x$  in  $p$  can be viewed as the price of its  $a$ -representation  $ax$  in such  $q$ .

Now we can find the  **$\mathcal{S}$ -linear state preference model**.

**Economic model.** We consider a situation in which there are  $n$ -goods and an  $m$ -dimensional continuous infinity of states of the world. Without loss of generality, we assume that, in our model, the set of the states of the world is  $\mathbb{R}^m$ , and the set of the goods is  $\underline{n} = \{k \in \mathbb{N} : k \leq n\}$ .

In these conditions, we give the following definition.

**Definition (of  $\mathcal{S}$ -linear state preference model).** We define  $\mathcal{S}$ -linear state preference model asystem  $(\underline{n}, \mathbb{R}^m, A)$ , where  $A : \mathbb{R}^n \rightarrow \mathcal{S}'_m$  is a linear operator. We call  $A$  the values-operator of the model, every  $n$ -vector  $x$  a portfolio of the model and, for every portfolio  $x$ , we call the tempered distribution  $A(x)$  the  $A$ -representation of  $x$ .

**Remark.** Note that, if  $x$  is a portfolio, then it is a linear combination of the canonical basis  $e$  of  $\mathbb{R}^n$ ,  $x = \sum x e$ . Then,  $A(x) = \sum x A(e)$ , and consequently  $\dim A(\mathbb{R}^n) \leq n$ .

**Definition (of regular portfolio).** If  $x$  is an  $n$ -portfolio, we say  $x$   $A$ -regular if its representation  $A(x)$  is a regular distribution. If  $s$  is a state of the world (in our model  $s$  is a real  $m$ -vector) and  $A(x)$  is a regular tempered distribution generated by a continuous function  $f_x$ , we say that  $f_x(s)$  is the value of the portfolio  $x$  in the state  $s$ .

**Economic interpretation.** The first goal is the presence, in our model, of the analogous of the Arrow-Debreu “contingent claims”. Using the canonical  $\mathcal{S}$ -basis  $\delta$ , we have  $A(x) = \int A(x)\delta$ . So we can argue that, for every portfolio  $x$ , the  $A$ -representation of  $x$  is an  $\mathcal{S}$ -linear combination of the “elementary securities” represented by the elements of the canonical  $\mathcal{S}$ -basis. This proves that the Dirac  $\mathcal{S}$ -basis represents the analogous of the family of the Arrow-Debreu contingent claims. In particular, for every state of the world  $s$ , the delta centered at  $s$ ,  $\delta_s$ , represents the elementary security whose value is 1 in the state of the world  $s$  and 0 in every other state of the world.

**Example.** Assume that the state of a portfolio  $x$  is the tempered distribution generated by  $\sin$ :  $A(x) = [\sin]$  then  $x$  has value 0 in every state  $s = k\pi$ , with  $k$  an integer.

**Example.** Let the values of a portfolio  $x$  be 3, 7.5 and 4.8 in three distinct states of the world  $s$ ,  $t$  and  $u$ , and let the value of  $x$  be 0 in every other state of the world. Then the values-representation of  $x$  is  $3\delta_s + 7.5\delta_t + 4.8\delta_u$ .

**Definition (of system of prices).** A system of prices in the space  $\mathbb{R}^n$  is an  $n$ -tuple. A system of prices in  $\mathcal{S}'_m$  is a smooth function of class  $\mathcal{S}$  defined on the set of the state of the world. If  $p$  is a system of prices in  $\mathbb{R}^n$ , we define the price of a portfolio  $x$  in  $p$ , as usual, as the product  $(p | x)_{\mathbb{R}^n} = \sum xp$ . If  $q$  is a system of prices in  $\mathcal{S}'_m$ , we define the price of a tempered distribution  $y$  as the following product

$$(q | y)_{\mathcal{S}'_m} := y(q).$$

**Remark (the price of a tempered distribution as superposition).** Concerning the preceding definition, note that  $y(q) = \int_{\mathbb{R}^m} qy dl_m$ . In fact, applying, first the definition of integral of an integrable distribution and then the definition of the product by a smooth function of a distribution we have

$$\int_{\mathbb{R}^m} qy dl_m = (qy)(1_{\mathbb{R}^n}) = y(1_{\mathbb{R}^n}q) = y(q).$$

But the value  $y(q)$  can be interpreted in a more impressive way: it is the superposition of the family  $q$  under the system of coefficients  $y$ , as defined by Carfi:

$$\int_{\mathbb{R}^m} yq := y(q).$$

**Remark (the system of prices as  $\mathcal{S}$ -linearfunctional).** Classically the price-systems in an infinite-dimensional vector space  $X$  are the linear functionals on  $X$ . Our definition of price-vector in  $\mathcal{S}'_m$ , although more adherent to the finite dimensional case, returns in the classical definition of system of prices. In fact, let  $q$  be a price-system, in our acceptance, we can associate the functional  $(q | \cdot)_{\mathcal{S}'_m}$ , it is linear by definition of addition and multiplication by scalar for tempered distributions. But we can say more: it is  $\mathcal{S}$ -linear. In the sense of the following definition.

**Definition (of  $\mathcal{S}$ -linear functional).** Let  $L : \mathcal{S}'_m \rightarrow \mathbb{R}$  be a functional. We say that  $L$  is an  $\mathcal{S}$ -linear functional if, for every tempered distribution  $a$  on  $\mathbb{R}^k$  and for every family  $v$  of tempered distributions on  $\mathbb{R}^m$  indexed by  $\mathbb{R}^k$ , we have that the family  $L(v) = (L(v_i))_{i \in \mathbb{R}^k}$  is of class  $\mathcal{S}$  (that is the function  $\mathbb{R}^k \rightarrow \mathbb{R} : i \mapsto L(v_i)$  is of class  $\mathcal{S}$ ) and moreover

$$L\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} aL(v).$$

**Theorem.** Let  $q \in \mathcal{S}_m$ . Then the functional  $(q | \cdot)_{\mathcal{S}'_m}$  is an  $\mathcal{S}$ -linear functional.

*Proof.* We have to prove that, for every family  $v$  of tempered distributions on  $\mathbb{R}^m$  indexed by  $\mathbb{R}^k$ , the family

$$(q | v)_{\mathcal{S}'_m} := ((q | v_i)_{\mathcal{S}'_m})_{i \in \mathbb{R}^k}$$

is a of class  $\mathcal{S}$  and moreover that

$$(q | \int_{\mathbb{R}^k} av)_{\mathcal{S}'_m} = \int_{\mathbb{R}^k} a(q | v)_{\mathcal{S}'_m}.$$

Concerning the first point, the function

$$\mathbb{R}^k \rightarrow \mathbb{R} : i \mapsto (q | v_i)_{\mathcal{S}'_m} = v_i(q),$$

is the function  $v(q)$ , that is of class  $\mathcal{S}$  since  $v$  is of class  $\mathcal{S}$  (and  $q$  is of class  $\mathcal{S}$ ). For the second point, we have

$$\begin{aligned} (q | \int_{\mathbb{R}^k} av)_{\mathcal{S}'_m} &= \left( \int_{\mathbb{R}^k} av \right) (q) = \\ &= a(\widehat{v}(q)) = \\ &= \int_{\mathbb{R}^k} a(v_i(q))_{i \in \mathbb{R}^k} = \\ &= \int_{\mathbb{R}^k} a((q | v_i)_{\mathcal{S}'_m})_{i \in \mathbb{R}^k} = \\ &= \int_{\mathbb{R}^k} a(q | v)_{\mathcal{S}'_m}. \blacksquare \end{aligned}$$