

Gröbner bases in economy

Marilena Crupi

Università di Messina, Dipartimento di Matematica

Contrada Papardo, Salita Sperone 31

98166 Sant'Agata (Me), Italy

`mcrupi@dipmat.unime.it`

Abstract

We analyse a special integer programming problem by computing a reduced Gröbner basis via graph theory. **Introduction.** The theory of Gröbner bases is a general method by which many fundamental problems in various branches of mathematics and engineering can be solved by structural algorithms. The applications of Gröbner bases to integer programming has its origin in [3]. Integer programming is one of the most attractive research areas lying between pure mathematics and applied mathematics. The integer programming problem is the problem of finding a solution in non-negative integers for a system of linear equations with integer coefficients which minimizes a linear cost function. In [3], the authors suggested a method which leads to the calculation of a suitable Gröbner basis by using the Buchberger algorithm. More precisely, if I_A denotes the toric ideal canonically associated with the matrix A of the integer programming problem, then one needs to find the reduced Gröbner basis of I_A relative to some suitable term order. It is possible to deal with integer programming problems also by using methods from graph theory.

1. Preliminaries and notations.

Definition 1.1 Let G be a graph and $C := v_0v_1 \dots v_n$ be a path of G . If e is an edge of C , we call parity of e relative to C the integer

$$p_C(e) := d - p,$$

where d is the number of times e occurs in C in odd position, and p is the number of times e occurs in C in even position.

Let $\underline{r} := \{1, \dots, r\}$, $\underline{s} := \{1, \dots, s\}$, $\underline{t} := \{1, \dots, t\}$ and $\underline{h} := \{1, \dots, h\}$, where r, s, t, h are positive integers. It is possible to construct a special bipartite graph related to these sets of integers [4].

Consider the two sets:

$$V_1 = \underline{r} \times \underline{s} \times \underline{t} \quad \text{and} \quad V_2 = \underline{r} \times \underline{s} \times \underline{h},$$

and denote by $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ the graph having $V = V_1 \cup V_2$ as set of vertices, and $E := \{e_{ijk} \mid (i, j, k, u) \in \underline{r} \times \underline{s} \times \underline{t} \times \underline{h}\}$ as set of edges, where $e_{ijk} = \{(i, j, k) \in V_1, (i, j, u) \in V_2\}$.

From the definition of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$, we have:

(i): $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ is a bipartite graph, with vertex sets V_1 and V_2 .

(ii): For every $(i, j) \in \underline{r} \times \underline{s}$, the subgraph $K_{t,h}^{(i,j)}$ of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ induced by the set of vertices

$$V'' := V_{(ij)_1} \cap V_{(ij)_2}$$

with $V_{(ij)_1} = \{(i, j)\} \times \underline{t}$ and $V_{(ij)_2} = \{(i, j)\} \times \underline{h}$ is isomorphic to the complete bipartite graph $K_{t,h}$.

(iii): $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}} = rsK_{t,h}$.

Definition 1.2 For every choice of $(i, j), (i', j') \in \underline{r} \times \underline{s}$, of k in \underline{t} and of u in \underline{h} , we say that the edges e_{ijk_u} and $e_{i'j'ku}$ are parallel.

Observe that if $\varphi^{((i,j),(i',j'))} : K_{t,h}^{(i,j)} \rightarrow K_{t,h}^{(i',j')}$ is the isomorphism defined by: $\varphi^{((i,j),(i',j'))}(i, j, k) := (i', j', k)$, $\varphi^{((i,j),(i',j'))}(i, j, u) := (i', j', u)$, $\forall k \in \underline{t}, \forall u \in \underline{h}$, then $e_{i'j'ku} = \varphi^{((i,j),(i',j'))}(e_{ijk_u})$.

It is possible to examine a special integer programming problem by using the graph $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ above defined.

Precisely, let A be the $(rs + rt + rh + st + th) \times (rsth)$ matrix whose columns are given by:

$$a_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}^{(ijk_u)} = \underline{e}_{ij} \oplus \underline{e}_{ik} \oplus \underline{e}_{iu} \oplus \underline{e}_{jk} \oplus \underline{e}_{ju} \oplus \underline{e}_{ku}, \quad (i, j, k, u) \in \underline{r} \times \underline{s} \times \underline{t} \times \underline{h},$$

where $\{\underline{e}_{ij}\}$ (resp. $\{\underline{e}_{ik}\}, \{\underline{e}_{iu}\}, \{\underline{e}_{jk}\}, \{\underline{e}_{ju}\}, \{\underline{e}_{ku}\}$) is the canonical basis of the \mathbb{Z} -module of $r \times s$ integer matrices $\mathbb{Z}^{\underline{r} \times \underline{s}}$ (resp. of the \mathbb{Z} -module of $r \times t$ integer matrices $\mathbb{Z}^{\underline{r} \times \underline{t}}$, of the \mathbb{Z} -module of $r \times h$ integer matrices $\mathbb{Z}^{\underline{r} \times \underline{h}}$, of the \mathbb{Z} -module of $s \times t$ integer matrices $\mathbb{Z}^{\underline{s} \times \underline{t}}$, of the \mathbb{Z} -module of $s \times h$ integer matrices $\mathbb{Z}^{\underline{s} \times \underline{h}}$, of the \mathbb{Z} -module of $t \times h$ integer matrices $\mathbb{Z}^{\underline{t} \times \underline{h}}$).

Consider the integer programming problem associated with A , the corresponding toric ideal to be studied is $I_A := \text{Ker}(\Pi_A)$, where

$$\Pi_A : K[x_{ijk_u}] \rightarrow K[u_{ij}, v_{ik}, w_{iu}, p_{jk}, q_{ju}, z_{ku}], \quad x_{ijk_u} \mapsto u_{ij}v_{ik}w_{iu}p_{jk}q_{ju}z_{ku}$$

where $(i, j, k, u) \in \underline{r} \times \underline{s} \times \underline{t} \times \underline{h}$, and K is any field. We think of A as the matrix of the morphism

$$\mathbb{Z}^{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}} \rightarrow \mathbb{Z}^{\underline{r} \times \underline{s}} \oplus \mathbb{Z}^{\underline{r} \times \underline{t}} \oplus \mathbb{Z}^{\underline{r} \times \underline{h}} \oplus \mathbb{Z}^{\underline{s} \times \underline{t}} \oplus \mathbb{Z}^{\underline{s} \times \underline{h}} \oplus \mathbb{Z}^{\underline{t} \times \underline{h}}, \quad \underline{y} \mapsto A\underline{y}.$$

It is well known that if $<$ is any term order on $K[\underline{x}] := K[x_{ijk_u}]$, then

$$\text{In}_{<}(I_A) = \text{In}_{<}(B)$$

where $B := \{\underline{x}^{u^+} - \underline{x}^{u^-} \mid \underline{u} \in \text{Ker}_{\mathbb{Z}}(A)\} \subseteq I_A$, and $\text{In}_{<}(B)$ is the ideal generated by all initial terms $\text{In}_{<}(f)$ with $f \in B$. A finite set $Gr \subseteq I_A$ is a Gröbner basis for I_A with respect to $<$, if and only if, $\text{In}_{<}(f) \in \text{In}_{<}(Gr)$ for every $f \in B$.

2. The graph $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ and integer programming problems.

Definition 2.1 Let $S := \{C_{ij}\}_{(i,j) \in \underline{r} \times \underline{s}}$ be an rs -uple satisfying the following properties:

- (1): For every $(i, j) \in \underline{r} \times \underline{s}$, either $C_{ij} = \emptyset$ or C_{ij} is a closed path of the subgraph $K_{t,h}^{(i,j)}$ of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$.
- (2): Whenever an edge e occurs in C_{ij} in odd (resp. even) position, there exists an edge e' parallel to e occurring in even (resp. odd) position in some $C_{i'j'}$.

We call S an admissible rs -uple of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$.

Definition 2.2 Let $S := \{C_{ij}\}_{(i,j) \in \underline{r} \times \underline{s}}$ be an admissible rs -uple of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$. For every $(i, j, k, u) \in \underline{r} \times \underline{s} \times \underline{t} \times \underline{h}$, we set

$$a_{ijk u} := \begin{cases} 0 & \text{if } e_{ijk u} \text{ does not occur in } C_{ij} \\ p(e_{ijk u}) & \text{if } e_{ijk u} \text{ occurs in } C_{ij} \end{cases}$$

where $p(e_{ijk u})$ is the parity of $e_{ijk u}$ relative to C_{ij} .

We say that $(a_{ijk u})$ is the sequence associated with S .

Theorem 2.3 ([4]) The following facts hold:

(a): Let $S := \{C_{ij}\}_{(i,j) \in \underline{r} \times \underline{s}}$ be an admissible rs -uple of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$, and let $\underline{a} = (a_{ijk u})$ be its associated sequence. Then $\underline{a} \in \text{Ker}_{\mathbb{Z}}(A)$.

(b): If $\underline{b} := (b_{ijk u})$ is in $\text{Ker}_{\mathbb{Z}}(A)$, then there exists at least one admissible rs -uple of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$, whose associated sequence coincides with \underline{b} .

Corollary 2.4 ([4]) Let us associate the variable $x_{ijk u}$ to the edge $e_{ijk u}$ of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$, and viceversa. Then, given $\underline{x}^{u^+} - \underline{x}^{u^-} \in B$, there exists an admissible rs -uple $S := \{C_{ij}\}_{(i,j) \in \underline{r} \times \underline{s}}$ of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ such that every edge of C_{ij} is always in odd (even) position, and such that u^+ is given by the parities of all odd edges of S , u^- is given by the parities of all even edges. Conversely, given an admissible rs -uple, a binomial $\underline{x}^{u^+} - \underline{x}^{u^-} \in B$ is obtained defining u^+ and u^- as above.

The above result clarifies the relationship between the graph $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$ and $B := \{\underline{x}^{u^+} - \underline{x}^{u^-} \mid \underline{u} \in \text{Ker}_{\mathbb{Z}}(A)\}$.

Definition 2.5 Let $S := \{C_{ij}\}_{(i,j) \in \underline{r} \times \underline{s}}$ be an admissible rs -uple of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$. Two paths C_{ij} and $C_{i'j'}$, $(i, j) \neq (i', j')$ are called anti-isomorphic if they only contain parallel edges with opposite parities.

Observe that if $S := \{C_{ij}\}_{(i,j) \in \underline{r} \times \underline{s}}$ is an admissible rs -uple of closed paths of $G_{\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}}$. Then either $C_{ij} = \emptyset$, for every (i, j) , or there are at least two nonempty paths. Moreover if there are exactly two nonempty paths, then they are anti-isomorphic.

Let us now introduce in the set $\underline{r} \times \underline{s} \times \underline{t} \times \underline{h}$ the lexicographic order $<_{lex}$ defined by: $(i, j, k, u) <_{lex} (i', j', k', u')$ if and only if the first non-zero component of the difference vector is negative.

Then $K[x_{ijk u}]$ is endowed with the pure lexicographic term order $<_{Lex}$ such that: $x_{ijk u} <_{Lex} x_{i'j'k'u'} \Leftrightarrow (i, j, k, u) <_{lex} (i', j', k', u')$.

Proposition 2.6([4]) Let $r = 2$, and s, t, h any integer ≥ 2 . The binomials associated with $2s$ -uple of anti-isomorphic cycles of $G_{2 \times \underline{s} \times \underline{t} \times \underline{h}}$ form a reduced Gröbner basis of I_A relative to $<_{Lex}$.

Remark 2.7 If one of the sets \underline{r} , \underline{s} , \underline{t} , \underline{h} is empty, then the integer programming problem associated to the matrix A is called 3-dimensional transportation problem and has been examined in [2].

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