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Vector Dynamic Optimization Problems: Preliminary Results And Applications

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Abstract

In the first part of this work we show optimality conditions for vector problems involving nonsmooth functionals. In the second part we apply these results to a class of vector dynamic optimization problems.

Keywords: Vector Optimization, Dynamic Optimization, Nonsmooth Optimization.

1 Introduction

In the following $(X, \|\|)$ will be a real vector space, $Y = \mathbb{R}^m$ and $J : X \to Y$ be a given vector functional on X. As usual we think Y be ordered by a pointed closed convex cone C, that is $a \leq_C b$ iff $b - a \in C$. In the first part of the paper we are interested in nonsmooth conditions for the problem

$$\min_{x \in E} J(x)$$

where $E \subset X$. In the second part of this work we apply these results to vector dynamic optimization problems as

$$\min_{x \in E} J(x) = \int_a^b f(t, x(t))dt$$

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where $x(t) \in E$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is a given vector function and $E \subset X = C([a, b])$ is given by

$$E = \{x(t) \in C([a, b]), x(a) = x_0, x(b) = x_1\}.$$

Some possible applications in economics are proposed too.

2 Generalized Derivatives For Locally Lipschitz Functionals

The notions of local minimum point and local weak minimum point are recalled in the following definitions.

Definition 2.1. *$x_0 \in X$ is called a local minimum point if there exists a neighborhood $N \cap X$ of x_0 such that no $x \in N \cap X$ satisfies $J(x_0) - J(x) \in C\backslash\{0\}$.*

Definition 2.2. *$x_0 \in X$ is called a local weak minimum point if there exists a neighborhood $N \cap X$ of x_0 such that no $x \in N \cap X$ satisfies $J(x_0) - J(x) \in \operatorname{int} C$.*

As usual, a function $J : X \to Y$ is said to be locally Lipschitz at $x_0 \in X$ if there exist a constant K_{x_0} and a neighbourhood U of x_0 such that $\|J(x_1) - J(x_2)\|_Y \leq K_{x_0}\|x_1 - x_2\|_X$, $\forall x_1, x_2 \in U$. Let $J : X \to Y$ be a given function and $x_0 \in X$. There are many papers in literature dealing with some extensions of classical derivatives for nonsmooth vector functions (see the references). Here we extend to this setting the notion of Dini and Peano derivatives under the only hypothesis that the involved data are locally Lipschitz. We consider the following definition of Dini generalized derivative $\overline{J}'_D$ at x_0 in the direction $d \in X$

$$J'_D(x_0; d) = \left\{ l : l = \lim_{k \to +\infty} \frac{J(x_0 + t_k d) - J(x_0)}{t_k}, t_k \downarrow 0 \right\}.$$

which is obviously nonempty when locally Lipschitz functions are considered. In fact if J is locally Lipschitz at x_0 then $J'(x_0; u)$ is a nonempty compact subset of Y. For the second order, we introduce the following definition of generalized Peano derivative

$$J''_P(x_0; d) = \left\{ s : s = \lim_{k \to +\infty} 2\frac{J(x_0 + t_k d) - J(x_0) - t_k l}{t_k^2}, t_k \downarrow 0, l \in J'_D(x_0; d) \right\}.$$

3 Convex Functions and Optimization

It is well know that the notion of convexity (and generalized convexity) is crucial in optimization; under this hypothesis necessary conditions become sufficient. The following definition extends to the vector case the classical notion of convexity for real functions. We say that J is C-convex if

$$tJ(x_1) + (1-t)J(x_2) - J(tx_1 + (1-t)x_2) \in C$$

for all $t \in (0,1)$ and $x_1, x_2 \in X$.

Theorem 3.1. *If $J : X \to Y$ is C-convex and locally Lipschitz at x_0 then*

$$J(x) - J(x_0) \in J'_D(x_0, x - x_0) + C$$

for all $x \in X$.

Corollary 3.1. *If $J : X \to Y$ is C-convex and differentiable at x_0 then*

$$J(x_0 + h) - J(x_0) \in A(x - x_0) + C$$

for all $x \in X$.

We are now interested in proving optimality conditions for the problem

$$\min_{x \in E} J(x).$$

where $E \subset X$. The following definition states some notions of local approximation of X at $x_0 \in \operatorname{cl} E$.

Definition 3.1. *The cone of weak feasible directions of E at x_0 is the set:*

$$WF(E, x_0) = \{d \in X : \exists\, t_k \downarrow 0 \ s.t.\ x_0 + t_k d \in X\}.$$

Theorem 3.2. *(Necessary condition) Let x_0 be a local weak minimum point. If J is locally Lipschtz at x_0 then, for all $d \in WF(E, x_0)$, we have*

$$J'_D(x_0; d) \cap (-\operatorname{int} C)^c \neq \emptyset.$$

Definition 3.2. *A subset $E \subset X$ is said to be star shaped at x_0 if $[x_0, x] \subset E$ for all $x \in E$ where*

$$[x_0, x] = \{y \in E : y = tx_0 + (1-t)x\}.$$

Theorem 3.3. *(Sufficient condition) Let E be a star shaped set at x_0. If J is C-convex and $J'_D(x_0; x - x_0) \subset (-\operatorname{int} C)^c$ for all $x \in E$, then x_0 is a weak minimum point.*

4 Dynamic Optimization

In this section we are interested in solving dynamic optimization problems as

$$\min J(x) = \int_a^b f(t, x(t))dt$$
$$x(t) \in E$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is a given vector function and $E \subset X = C([a, b])$ is given by

$$E = \{x(t) \in C([a, b]), x(a) = x_0, x(b) = x_1\}$$

The set X can be endowed with the usual d_∞ metric

$$d_\infty(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|.$$

We now can apply the results of the previous sections for proving the following necessary condition.

Theorem 4.1. *Suppose that f be locally Lipschitz in y and $x_0 \in X$ be a local weak efficient solution of (OP). Then*

$$\int_a^b f'_D((t, x_0(t)); (0, h(t))dt \cap (-\mathrm{int}\, C)^c \neq \emptyset$$

for all $h \in C([a, b])$ such that $h(a) = h(b) = 0$.

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