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Generalization of the alternative Vinen's equation describing the superfluid turbulence in rotating container

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Abstract.

In this work, a generalization for the alternative Vinen's equation in counterflow rotational superfluid turbulence is proposed. It is compared with the equation proposed in Phys. Rev. B, 69, 094513 (2004) and with the experimental results. We consider not only steady-states but also unsteady situations. From this analysis follows that the solutions of the alternative Vinen's equation tend significantly faster to the corresponding final steady state values than the solutions of the usual Vinen's equation.

Keywords: superfluid turbulence; vortex tangle; rotating counterflow turbulence.

1. Introduction

Quantum turbulence is described as a chaotic motion of quantized vortices in a disordered tangle.[1,2] The measurements of vortex lines are described in terms of a macroscopic average of the vortex line length per unit volume L (briefly called *vortex line density* and which has dimensions $length^{-2}$).

The evolution equation for L under constant values of the counterflow velocity $\mathbf{V}$ ($\mathbf{V} = \mathbf{v_n} - \mathbf{v_s}$, $\mathbf{v}_n$ and $\mathbf{v}_s$ being the velocities of the normal and superfluid components) and in absence of rotation was formulated by Vinen. Neglecting the influence of the walls, such an equation is:[3,2]

$$\frac{dL}{dt} = \alpha V L^{3/2} - \beta\kappa L^2, \tag{1}$$

with $V = |<\mathbf{v}_n - \mathbf{v}_s>|$ the absolute value of the average counterflow velocity, $\kappa = h/m$ the quantum of vorticity (m the mass of the ^{4}He atom and h Planck's constant, $\kappa \simeq 9.97\ 10^{-4}\mathrm{cm}^2/\mathrm{s}$) and α and β dimensionless parameters.

Note however that another version of (1) is the so-called alternative Vinen's equation, which is also admissible on dimensional grounds:[4−6]

$$\frac{dL}{dt} = A_1 \frac{V^2}{\kappa} L - \beta\kappa L^2. \tag{2}$$

The steady state solutions of (1) and (2) are $L = (\alpha V/\beta\kappa)^2$ and $L = A_1 V^2/\beta\kappa^2$ respectively, in agreement with the experimental results in completely developed turbulent regime, which lead to $L^{1/2} = \gamma V/\kappa$, with γ a dimensionless coefficient which depends on the temperature. Therefore, the difference between (1) and (2) must be searched in the dynamical aspects. This was carried out by Vinen himself (see sections 6 and 7 of Ref. 4) and in more detail by Nemirovskii *et al.*[6] without arriving to definite conclusions, because the predictions of (1) and (2) in the domain of available experimental results are very similar to each other. Here we will look for a more general situation where the difference between (1) and (2) becomes enhanced.

In recent years there has been growing attention in superfluid turbulence in rotating containers,[7−8] in which the formation of vortex lines is due both to the counterflow and the rotation, which has fostered the extension of Vinen's ideas to a wider range of situations.[9−10] First of all, we briefly recall the Vinen's basic ideas underlying the derivation of (1)–(2) which are, in summary, the following ones.[3,4] Vinen,[3] following a suggestion of Feynman, supposed that in homogeneous counterflow turbulence there is a balance between generation and decay processes, which leads to a steady state in the form of a self-maintained vortex tangle. He assumed that the evolution of L is described by

$$\frac{dL}{dt} = \left[\frac{dL}{dt}\right]_f - \left[\frac{dL}{dt}\right]_d, \tag{3}$$

the first term responsible for the growth of L, the second for its decay. He supposed that the growth of L depends on the instantaneous value of L, on the intensity V of the counterflow velocity and on the quantum of circulation κ; dimensional analysis leads to the equation:[3−6]

$$\left[\frac{dL}{dt}\right]_f = \kappa L^2 \phi_f \left[\frac{V}{\kappa L^{1/2}}\right] = A_n \kappa L^2 \left(\frac{V}{\kappa L^{1/2}}\right)^n, \tag{4}$$

where ϕ_f is a dimensionless function, A_n is constant and n is an integer.

The form of the $[dL/dt]_d$ term, responsible for the vortex decay, was determined by Vinen in analogy with classical turbulence, obtaining:

$$\left[\frac{dL}{dt}\right]_d = -\beta\kappa L^2, \tag{5}$$

being β a dimensionless constant.[3−5]

From the assumption $n = 1$ in (4) and from (5), Vinen's equation (1) follows immediately, while, using $n = 2$ for the production term in (4), one obtains the equation (2).

In Ref. 9, two of us have proposed for the evolution of L in the simultaneous presence of $\mathbf{V}$ and $\mathbf{\Omega}$ ($\mathbf{\Omega}$ being the angular velocity of the container) a phenomenological generalization of Vinen's equation (1.1). Here, we explore the extension of the form (2) of Vinen's equation to rotating counterflow turbulence in order to explore whether this more general situation may provide further arguments to decide which of both starting equations, (1) or (2), is more suitable to describe actual experimental results.

In Section 2 we write a new equation for the evolution of L in counterflow in rotating containers, through a modification of the Vinen's alternative equation. In Section 3 we solve it in steady situation and we compare it with the generalization of the usual Vinen's one, made in Ref. 9. In Section 4 the unsteady situations are studied in order to have a more complete comparison between the two extensions of the Vinen's equations.

2. New equation for the dynamics of L in rotating counterflow superfluid turbulence

There are not many experiments on counterflow in rotating containers. In the work of Swanson *et al*,[11] the counterflow velocity V was parallel to the rotation axis and the experimental observations consisted in measuring the attenuation of second sound, when it is propagated orthogonal to the rotation axis. They interpreted their results as measurements of the vortex line density L, and compared the observed line density with what would be expected if the two sources of vorticity (rotation and counterflow) simply added. Their results showed an interesting interplay between the ordered vortices of rotation and the disordered ones of counterflow. More precisely, they observed that the effects of V and Ω are not additive: in fact, for fast enough values of Ω, the total vortex line density is lower than $L_R + L_H$, L_R and L_H being the values of L in steady rotation and in steady counterflow superfluid turbulence respectively:

$$L = L_R = \frac{2\Omega}{\kappa}, \qquad L_H = \gamma^2 \frac{V^2}{\kappa^2}, \tag{1}$$

with γ a dimensionless coefficient.

They found two critical counterflow-rotation velocities V_{c1} and V_{c2}, which scale as $\Omega^{1/2}$ ($V_{c1} = C_1\sqrt{\Omega}$, $V_{c2} = C_2\sqrt{\Omega}$, with $C_1 = 0.053$ cm sec$^{-1/2}$, $C_2 = 0.118$ cm sec$^{-1/2}$). For $V \leq V_{c1}$, the length L per unit volume of the vortex lines is independent of V and agrees with the first expression in (1). For $V_{c1} \leq V \leq V_{c2}$, L is still independent of V and proportional to Ω, with a slightly different proportionality constant than in the previous situation; finally, for $V \geq V_{c2}$, L increases and becomes proportional to V^2 at high values of V.

Swanson *et al.*[11] interpreted the first transition as the Donnelly-Glaberson instability:[12,13] excitation of helical waves (Kelvin waves) by the counterflow on the vortex lines induced by rotation and the second as a transition to a turbulent disordered tangle. Tsubota *et al.*[8] also have paid attention to this experiment. They proposed that the regime $V_{c1} < V < V_{c2}$ is a state of polarized turbulence, while for $V > V_{c2}$ the polarization is decreased by the large number of reconnections.

In Ref. 9, in a first macroscopic study of this intricate behavior, two of us have considered the experiments of Swanson *et al.* in the regime of high rotation, when the influence of the walls on the formation and destruction of vortices is negligible. A further work[14] was devoted to the study of the regime of low values of V and Ω, where the effect of the walls becomes important.

In the regime of high rotation (0.2 Hz $\leq \Omega/2\pi \leq$ 1.0 Hz and $0 \leq V^2 \leq 0.2$ cm^2/s^2), a phenomenological modification of Vinen's equation has been proposed for the evolution of vortex line density L, modeling the destruction contribution, as Vinen, with equation

(5) and the production contribution with a function depending on V and Ω (as well as on κ and L):[9]

$$\left[\frac{dL}{dt}\right]_f = \kappa L^2 \phi_f \left[\frac{V}{\kappa L^{1/2}}, \left(\frac{\Omega}{\kappa L}\right)^{1/2}\right]. \tag{2}$$

Note that, as arguments of the function ϕ_f, a term in V and a term in $\Omega^{1/2}$ were used; this was motivated by the dependence of the steady-state values of $L^{1/2}$, in counterflow only and in rotation only, on V and on $\Omega^{1/2}$ (see equations (1)), and by the observation that the microscopic mechanism responsible for the growth of vortices (the mutual friction force) is the same in rotating helium II and in superfluid turbulence. The equation obtained was:

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha_1 V + \beta_2\sqrt{\kappa\Omega}\right] L^{3/2} - \left[\beta_1\Omega + \beta_4 \frac{V\sqrt{\Omega}}{\sqrt{\kappa}}\right] L, \tag{3}$$

where the coefficients β, α_1, β_1, β_2 and β_4 depend on the polarization of the tangle, which was supposed function of Ω and V.

The Eq. (3) describes, in good agreement with experimental results, some of the most relevant effects observed in the experiments of Ref. 11. However, as we have mentioned in the Introduction, the alternative Vinen's equation also describes well the experimental results in pure counterflow and therefore it is natural to ask how does it work when extended to incorporate rotation.

Here, we suggest a new evolution equation for the vortex line density L in rotating counterflow, starting from the alternative Vinen's equation and following the lines of thought outlined in Ref. 9. We consider only the case in which V and Ω are parallel to each other. A more general situation will be taken in consideration in a future work.

We model the destruction term, as in Ref. 9, with the term $-\beta\kappa L^2$, and in the same spirit as Vinen, we focus our attention on the production term assuming for it a general form analogous to (2), but reducing to (2) for vanishing rotation. As in the extension (3) of (1), we choose a quadratic dependence of the function ϕ on its variables, which are $\frac{V^2}{\kappa^2 L}$ and $\frac{\Omega}{\kappa L}$, obtaining the following equation for the evolution of L:

$$\frac{dL}{dt} = -\beta\kappa L^2 + A_1\left[L - \nu_1\frac{\Omega}{\kappa}\right]\frac{V^2}{\kappa} + B_1\left[L - \nu_2\frac{\Omega}{\kappa}\right]\Omega, \tag{4}$$

where the coefficients β, A_1, ν_1, B_1 and ν_2 depend on the polarization of the tangle, which is function of Ω and V.

We have chosen the linear term depending on V^2 and Ω as production term, because both rotation and counterflow favor the vortex formation, while the terms $-A_1\nu_1 V^2\Omega/\kappa^2$ and $-B_1\nu_2\Omega^2/\kappa$ describe the ordering tendency of the rotation, which tends to straighten out the otherwise irregular vortex lines of the tangle, thus shortening them and reducing L. Thus, despite we lack for the moment a microscopic interpretation of equation (4), the previous considerations furnish a simple physical interpretation of the several terms in it.

Another aspect especially worth of comment is the meaning of the destruction term in (4) -or (3). One could argue, indeed, that at steady pure rotation there is no vortex destruction. Indeed, in purely rotation situations the vortices are usually produced on the walls and they migrate to the bulk of the fluid in the cylinder; in this case, the term in L^2 would represent a repulsion force between the parallel vortices, putting an upper limit to the possible number of straight vortices in the vortex array. In a general situation, the destruction term will incorporate real destruction of vortices due to breaking recombination of nonparallel vortices, as repulsion forces between parallel segments of vortices in the presence of rotation. The superposition of these two different effects on the term in L^2 is one of the reasons that the coefficient in it depends on the polarization of the tangle.

3. The stationary solutions and their stability

The non zero stationary solutions of (4) are solutions of the following second-order algebraic equation in the unknown L:

$$\beta\kappa L^2 - \left[\frac{A_1}{\kappa}V^2 + B_1\Omega\right] L + \left[B_2\frac{\Omega^2}{\kappa} + \frac{B_3}{\kappa^2}V^2\Omega\right] = 0, \tag{1}$$

where we have put $B_2 = B_1\nu_2$ and $B_3 = A_1\nu_1$.

Looking at the experimental results of Ref. 11, we note that L is almost independent of V for $V < V_{c2}$, with a step change around V_{c1}, while there is a variation of the slope near V_{c2}. We will concentrate in this Section on the change near V_{c2}.

Reasoning as in Ref. 9, we observe that the hypothesis

$$\frac{B_2}{\beta} = \frac{B_3}{A_1}\left(\frac{B_1}{\beta} - \frac{B_3}{A_1}\right) \quad \Rightarrow \quad \frac{B_1}{\beta} = \frac{\nu_1^2}{\nu_1 - \nu_2} \quad \text{and} \quad \frac{B_2}{\beta} = \frac{\nu_1^2\nu_2}{\nu_1 - \nu_2}, \tag{2}$$

such that the solutions of equation (1) can be written:

$$L = L_1^A = \nu_1\frac{\Omega}{\kappa} \qquad \text{and} \qquad L = L_2^A = \frac{A_1}{\beta}\frac{V^2}{\kappa^2} + \left(\frac{B_1}{\beta} - \nu_1\right)\frac{\Omega}{\kappa}. \tag{3}$$

In the plane (V^2, L), (3a) and (3b) represent two families of straight lines plotted in Fig. 1, the first of them (equation (3a)) horizontal and the second one (equation (3b)) with the same slope which scale with Ω. We study the stability of these solutions, writing the evolution equation for the perturbation δL

$$\frac{d\delta L}{dt} = \left[-2\beta\kappa L + \left(\frac{A_1}{\kappa}V^2 + B_1\Omega\right)\right]\delta L \tag{4}$$

and substituting in it equations (3a) and (3b). One deduces that solution (3a) is stable if the counterflow velocity V is lower than:

$$V_{c2}^2 = \frac{\beta}{A_1}\left[2\frac{B_3}{A_1} - \frac{B_1}{\beta}\right]\Omega\kappa = \frac{\beta}{A_1}\frac{\nu_1^2 - 2\nu_1\nu_2}{\nu_1 - \nu_2}\Omega\kappa, \tag{5}$$

(corresponding to the point of interception of the two straight lines (3a) and (3b)), while, for values of V higher than V_{c2}, the solution (3b) is stable. Therefore V_{c2} represents the second critical counterflow-rotation velocity observed in the experiments of Ref. 11. As we see, this critical velocity scales as $\sqrt{\Omega}$, in agreement with experimental observations.

The experimental data on the steady states of L allow us to determine the values assumed by the dimensionless quantities appearing in equation (4). One obtains:

$$\frac{A_1}{\beta} = 0.0125, \quad \frac{B_1}{\beta} = 3.90, \quad \frac{B_2}{\beta} = 3.79, \quad \frac{B_3}{\beta} = 0.025, \tag{6}$$

from which we obtain $\nu_1 = 2.036$ and $\nu_2 = 0.97$.

The coefficient β, which controls the rate of evolution of L, cannot be determined from the knowledge of the stationary solutions. Using the obtained values of the dimensionless quantities, the steady stationary solutions L_1^A and L_2^A become

$$L_1^A = 2.036\frac{\Omega}{\kappa}, \qquad \text{and} \qquad L_2^A = 0.0125\frac{V^2}{\kappa^2} + 1.86\frac{\Omega}{\kappa}. \tag{7}$$

In Fig. 1 a comparison of such stationary solutions L_1^A and L_2^A with the experimental data of Swanson *et al.*[11] is shown. The conclusion of such a fit is that the stationary vortex line density L_1^A and L_2^A, solutions of the alternative Vinen's equation in the combined situation, are in good agreement with experimental data of Swanson *et al.*.[11]

In Ref. 9, the stationary solutions of equation (3) had the form

$$L_1^{1/2} = 1.427\sqrt{\frac{\Omega}{\kappa}} \qquad \text{and} \qquad L_2^{1/2} = 0.047\frac{V}{\kappa} + 1.25\sqrt{\frac{\Omega}{\kappa}} \tag{8}$$

and the comparison with the experimental data led also to the conclusion that (8) are in agreement with the experiments by Swanson *et al.*.[11] Note that (7) and (8) have a different mathematical form but, in the range of the available experimental data, both of them lead to reasonable results.

From such conclusions an interesting problem is to establish which equation, either the one based on the usual Vinen's equation studied in Ref. 9 or the other one based on the alternative Vinen's equation (explored in the present paper), fits better the experimental data obtained by Swanson, Donnelly and Barenghi.[11]

From a first comparison, the two stationary solutions (8a) and (7a) represent the same straight line in the plane (L, V^2) in the range $V_{c1}^2 < V^2 < V_{c2}^2$. So, an eventual difference between both equations could be found in the range $V^2 > V_{c2}^2$. To do that, we calculate the errors σ between L_2 and the corresponding experimental value $\overline{L}$, and σ^A between L_2^A and $\overline{L}$, respectively, in such a way that we can compare the accuracy of the two models.

To find these errors, we consider the experimental values V_i^2 and Ω_j of the experiments to which L_{2ij}, L_{2ij}^A and $\overline{L}_{ij}$ correspond, obtaining

$$\sigma = \sqrt{\frac{\sum_{i,j}\left(L_{2ij} - \overline{L}_{ij}\right)^2}{N}} = 963.07, \quad \sigma^A = \sqrt{\frac{\sum_{i,j}\left(L_{2ij}^A - \overline{L}_{ij}\right)^2}{N}} = 418.85, \tag{9}$$

where N is the number of experimental data, which is equal for both cases. From (9) we can establish that the stationary solution of the alternative Vinen's equation approaches better the experimental data (for $V^2 > V_{c2}^2$) than that of the usual Vinen's equation.

4. Non-stationary solutions

In this Section we study the non-stationary solutions of both extensions of the Vinen's equations. Though the lack of experiments about the evolution of the vortex line density L in this more general case (rotation and counterflow) does not allow us to compare directly our results with experimental data, however we can arrive at some interesting conclusions concerning the difference of behavior.

The equation (4) can be also written as

$$\frac{dL}{dt} = -\beta\kappa\left(L - L_1^A\right)\left(L - L_2^A\right), \tag{1}$$

whose solutions are

$$\beta\kappa t(L_2^A - L_1^A) = \ln\left|\frac{(L-L_1^A)(L_0-L_2^A)}{(L_0-L_1^A)(L-L_2^A)}\right|, \tag{2}$$

where L_0 is the initial value of L.

In an analogous way, the equation (3) can be written as

$$\frac{dL}{dt} = -\beta\kappa L\left(\sqrt{L}-\sqrt{L_1}\right)\left(\sqrt{L}-\sqrt{L_2}\right), \tag{3}$$

whose non-stationary solutions are

$$\begin{aligned} -\frac{\beta\kappa}{2}t = \frac{1}{\sqrt{L_1L_2}}\ln\left|\frac{\sqrt{L}}{\sqrt{L_0}}\right| + \frac{1}{\sqrt{L_1}(\sqrt{L_1}-\sqrt{L_2})}\ln\left|\frac{\sqrt{L}-\sqrt{L_1}}{\sqrt{L_0}-\sqrt{L_1}}\right| \\ +\frac{1}{\sqrt{L_2}(\sqrt{L_2}-\sqrt{L_1})}\ln\left|\frac{\sqrt{L}-\sqrt{L_2}}{\sqrt{L_0}-\sqrt{L_2}}\right|, \end{aligned} \tag{4}$$

where L_0 is the initial value of L.

In order to compare the unsteady solutions (2) and (4) of both equations, a value for the coefficient β must be chosen. As already said before, this coefficient β may depend on the anisotropy and polarization of the tangle, which in turn depend on the values of Ω and V, therefore it may have a different value with respect to the one in pure counterflow situation. However, since this dependence is not known in this section, to perform this comparison we choose the value $\beta = 1/2\pi$, which is the value of β in pure counterflow.

Now, we choose some values for V^2 and Ω in order to plot the solutions of the two models.

First of all we consider the case $V^2 < V_{c2}^2$, and in particular the values $V^2 = 0.0072$ and $\Omega/2\pi = 0.4$ to which the following values of the stationary solutions correspond: $L_1 = L_1^A = 5132.42$, $L_2 = 4455.3$, $L_2^A = 4779.3$.

For the initial value L_0 we choose $L_0 = L_R = 2\Omega/\kappa$. Here, all the values for L, V^2 and Ω will be expressed in cm^{-2}, cm^2 s^{-2} and rad s^{-1}, respectively.

From the analysis of the Section 2 and from that of Ref. 9 we already know that in this range the stationary solution L_1 is stable. The same conclusion is reached by looking at the plot of the non-stationary solutions (2) and (4) of the two models in Fig. 2. Further, we note that the values of L_2 and L_2^A are smaller than L_1 and that the non-stationary solutions approach to the stable stationary one, L_1, in relatively similar times.

Then, following the same process as above and setting the same value for Ω and a value $V^2 = 0.0626$ slightly higher than V_{c2}^2, we find the following values for the stationary solutions: $L_1 = L_1^A = 5132.42$, $L_2 = 5553.13$, $L_2^A = 5475.97$, and for L_0 two different values $L_{2_{|V^2=0.0482}}$ and $L^A_{2_{|V^2=0.0482}}$ are chosen respectively for the two solutions (2) and (4) (see Fig. 3)). Note that in this case the value of L_1 is smaller than L_2 and L_2^A. As we know from previous studies, in this range the stationary solutions L_2 and L_2^A for the Vinen's equation and alternative Vinen's one are stable. This is confirmed by the graphics in Fig. 3, where the evolution of vortex line densities L (2) and (4) are plotted.

In Fig. 3 we also note a different behavior with respect to that in Fig. 2; in fact, the two non-stationary solutions L approach the corresponding stationary values L_2^A and L_2 in rather different times with a ratio of about 1:3, respectively. So, the solution of the alternative Vinen's equation is faster than that of the Vinen's equation.

Furthermore, if we plot the non-stationary solutions for a value of V^2 much higher than V_{c2}^2, we note that the ratio between the temporal scales is yet bigger than the factor 3. In fact, by setting the same value of Ω and taking $V^2 = 0.1878$, the corresponding values of the stationary solutions become: $L_1 = L_1^A = 5132.42$, $L_2 = 6910.23$, $L_2^A = 7050.4$. The graphics of the solutions (2) and (4) are shown in Fig. 4. As initial data, we have chosen $L_0 = L_{2_{|V^2=0.1626}}$ and $L_0 = L^A_{2_{|V^2=0.1626}}$ for the Vinen's equation and alternative Vinen's one, respectively. Looking at these unsteady solutions, we note that the solution of the alternative Vinen's equation approaches to L_2^A in a much shorter time than the other solution requires to approach L_2, by a ratio of about 1:5.

Note that the time scales in Fig. 3 [100–300 seconds] are much longer than those in Fig. 4 [15–75 seconds]. This is not surprising because Fig. 3 corresponds to a situation which is much closer to the critical velocity V_{c2} than that corresponding to Fig. 4. Indeed, it is known that the dynamics near critical points and phase transitions is much slower than in situations far from them. For instance, in the stability analysis, as that shown in equation (4), the critical velocity corresponds to the vanishing of the growth rate of perturbation and, therefore, to a very slow dynamics.

5. Conclusions

The possibility of at least two reasonable evolution equations for the vortex line density L, namely (1) and (2), was known since the early days in which Vinen proposed them. However, detailed comparisons for them are very scarce.[4,6] This was due, in part, to the fact that both of them lead to the same form for the steady state results, namely $L \sim V^2$, and that their unsteady solutions are not sufficiently different to reach a definitive conclusion on their relative merit. Here, we have carried out a detailed comparison of an extension of both equations (1) and (2) to the simultaneous presence of counterflow and rotation. The extension of (1) was already studied in Ref. 9. Here we have studied the analogous extension of (2). We have seen that in steady states the solutions of both equations, namely (7) and (8), have a different form but in the range of available experimental results both of them yield a satisfactory approximate description of the experimental data. However, a deeper comparison of the experimental errors, in (9), shows that the description based on the alternative Vinen's equation is slightly better than the one based on the most well-known Vinen's equation.

A new aspect we have explored is the unsteady behavior of the solutions of these equations. Here, both equations exhibit remarkable differences, and we show that the solutions of the alternative Vinen's tend much faster to their steady-state values. In fact, this difference depends on the value of the counterflow velocity. For $V^2 = 0.0626$, slightly higher than the critical velocity V_{c2}^2, the time required to reach the steady state solutions is 3 times shorter in the alternative Vinen's equation than in the usual Vinen's equation, whereas for $V^2 = 0.1878$ the difference is still more remarkable, the time scale of the alternative Vinen's equation being 5 times shorter than that for the usual one. Though we lack detailed experimental data on this unsteady behavior, we know that the time required to reach the steady state was less than 10 minutes according to Swanson *et al.*, when the counterflow velocity V is slightly above the critical velocity V_{c2} and it increases between two consecutive experimental values (see pag. 191 of Ref. 11). According to the results of the Fig. 3, the temporal scale of the solution of the usual Vinen's equation is closer to the observations than the temporal scale corresponding to the alternative equation, which tends too fast to the final result. Thus, it seems that the usual equation is preferable on these grounds.

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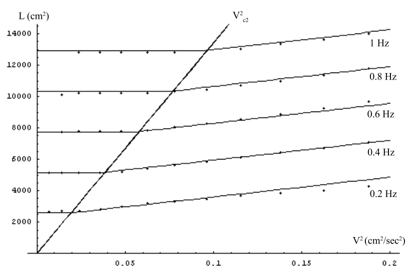


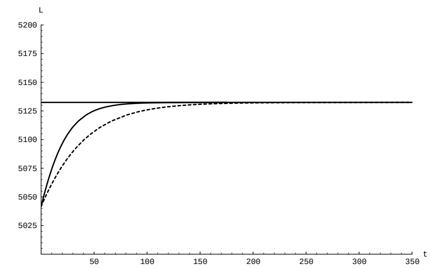
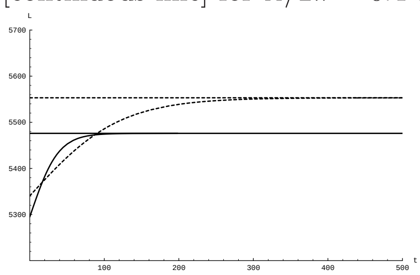


Fig. 1. Comparison of the solutions (7) (continuous line) with the experimental data by Swanson *et al.*.[11]

Fig. 2. Evolution of the vortex line density L towards its steady state value for the generalizations of the usual Vinen's equation (3) [dotted line] and the alternative Vinen's equation (4) [continuous line] for $\Omega/2\pi = 0.4$ and $V^2 = 0.0072$, lower than the critical value V_{c2}.



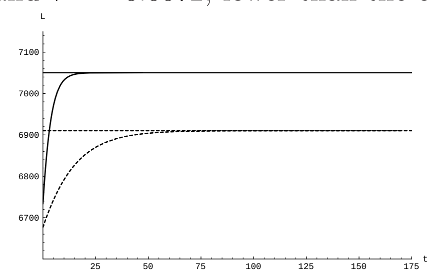


Fig. 3. As in Fig. 2, but for $\Omega/2\pi = 0.4$ and $V^2 = 0.0626$ slightly above the critical value V_{c2}. Note that the steady solutions differ only in a 0.75%, whereas the difference in the time necessary to reach the steady state differs in more than 300%. The values of L_0 are the unsteady solutions L_2 and L_2^A at the same Ω and $V^2 = 0.0482$.

Fig. 4. As in Fig. 2, but for $\Omega/2\pi = 0.4$ and $V^2 = 0.1878$ much higher than the critical value V_{c2}. The times necessary to reach the steady state differ in a 500% whereas the steady state values differ only in a 12.5%. The values of L_0 are the unsteady solutions L_2 and L_2^A at the same Ω and $V^2 = 0.1626$.