

## A thermodynamical model of inhomogeneous superfluid turbulence

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### Abstract.

In this paper we perform a thermodynamical derivation of a nonlinear hydrodynamical model of inhomogeneous superfluid turbulence. The theory chooses as fundamental fields the density, the velocity, the energy density, the heat flux and the averaged vortex line length per unit volume. The restrictions on the constitutive quantities are derived from the entropy principle, using the Liu method of Lagrange multipliers. The mathematical and physical consequences deduced by the theory are analyzed both in the linear and in the nonlinear regime. Field equations are written and the wave propagation is studied with the aim to describe the mutual interactions between the second sound and the vortex tangle.

*Keywords:* superfluid turbulence; nonequilibrium thermodynamics.

### 1. Introduction

Due to its quantum nature, the behavior of superfluid helium II is very different from that of ordinary fluids: it has an extremely low viscosity and temperature waves (second sound) propagate in it. An example of non classical behavior is heat transfer in counterflow experiments, characterized by no matter flow but only heat transport. Consider a channel with a heater at a closed end and open to the helium bath at the other end. Using an ordinary classical fluid, such as helium I, a temperature gradient can be measured along the channel, indicating a finite thermal conductivity. If helium II is used, and the heat flux inside the channel is lower than a critical value  $q_c$ , the temperature gradient is so small that it cannot be measured, so indicating that the liquid has an extremely high thermal conductivity. If the heat flux exceeds the critical value  $q_c$ , one observes an extra attenuation of second sound, which grows with the square of the heat flux. This phenomenon is known as "quantum turbulence" or "superfluid turbulence".<sup>1,2</sup> The damping force, known as "mutual friction", finds its origin in the interaction between the flow of excitations and a chaotic tangle of quantized vortices of equal circulation  $\kappa$ .

In many situations, the vortex tangle is assumed to be isotropic and may be described by introducing a scalar quantity  $L$ , the average vortex line length per unit volume (briefly

called vortex line density). The evolution equation for  $L$  in counterflow superfluid turbulence has been formulated by Vinen.<sup>3</sup> Neglecting the influence of the walls, such an equation can be written as:<sup>4</sup>

$$(1) \quad \frac{dL}{dt} = AqL^{3/2} - BL^2,$$

with  $A$  and  $B$  coefficients dependent on the temperature.

This equation assumes homogeneous turbulence, i.e. that the value of  $L$  is the same everywhere in the system. However, homogeneity may be expected if the average distance between the vortex filaments is much smaller than the size of the system, but it will be not so for dilute vortex tangles.

The aim of this paper is to describe the coupling between the heat flux and the inhomogeneities in the vortex line density, both in the linear and in the nonlinear regimes. In fact, nonlinear phenomena are important in the study of superfluid turbulence, because the vortices can be formed when nonlinear second sound and shock waves are propagated. Therefore we formulate, using Extended Thermodynamics,<sup>5,6</sup> a nonlinear hydrodynamical model of quantum turbulence, in which the role of inhomogeneities is explicitly taken into account. This is important because second sound provides the standard methods of measuring the vortex line density  $L$ , and the dynamical mutual interplay between second sound and vortex lines may modify the standard results.

We will choose as fundamental fields the density  $\rho$ , the velocity  $\mathbf{v}$ , the internal energy density  $E$ , the heat flux  $\mathbf{q}$ , and the averaged vortex line density  $L$ . The relations which constrain the constitutive quantities are deduced from the entropy principle, using the Liu method of Lagrange multipliers.<sup>7</sup> The mathematical and physical consequences deduced by the theory are analyzed both in the linear and in the nonlinear regime. The vortex diffusion and the propagation of second sound and its interaction with vortex waves will be also considered.

## 2. Balance equations and Constitutive Theory

Extended Thermodynamics (E.T.) offers a natural framework for the macroscopic description of liquid helium II. Indeed, in analogy with heat transport problem, using E.T., the relative motion of the excitations is well described by the dynamics of the heat flux. In Ref. 8, E.T. was applied to formulate a non-standard one-fluid model of liquid helium II, for laminar flows. Successively, a first thermodynamic study of turbulent flows was made in Ref. 9, where the presence of the vortex tangle was modelled through a constitutive relation. Here, we want to build up an hydrodynamical model of superfluid turbulence, which can describe also inhomogeneities and nonlinear phenomena.

### 2.1. Balance equations

We consider, as a starting point the formulation of E.T. which uses the method of Lagrange multipliers accounting for the restrictions set by the balance equations<sup>6,7</sup>. For the fields  $\rho, \rho\mathbf{v}, \rho\epsilon + \frac{1}{2}\rho v^2, \mathbf{q}$  and  $L$  general balance equations are chosen, which in terms

of non convective quantities can be written:

$$\begin{aligned}
(1) \quad & \dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0, \\
(2) \quad & \rho \dot{\mathbf{v}} + \nabla \cdot \mathbf{J}^{\mathbf{v}} = 0, \\
(3) \quad & \dot{E} + E \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{q} + \mathbf{J}^{\mathbf{v}} \cdot \nabla \mathbf{v} = 0, \\
(4) \quad & \dot{\mathbf{q}} + \mathbf{q} \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{J}^{\mathbf{q}} = \sigma^{\mathbf{q}}, \\
(5) \quad & \dot{L} + L \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{J}^L = \sigma^L.
\end{aligned}$$

Here  $E = \rho \epsilon$  is the specific energy per unit volume,  $\mathbf{J}^{\mathbf{v}}$  the stress tensor,  $\mathbf{J}^{\mathbf{q}}$  the intrinsic part of the flux of the heat flux and  $\mathbf{J}^L$  the flux of vortex lines;  $\sigma^{\mathbf{q}}$  and  $\sigma^L$  are terms describing the net production of heat flux and vortices. In this system an upper dot denotes the material time derivative.

## 2.2. Constitutive Theory

Constitutive equations for the fluxes  $\mathbf{J}^{\mathbf{v}}$ ,  $\mathbf{J}^{\mathbf{q}}$  and  $\mathbf{J}^L$  and the productions  $\sigma^{\mathbf{q}}$  and  $\sigma^L$  are necessary to close the set of equations (2)-(6). To describe nonlinear phenomena, we choose for the fluxes the following general constitutive equations:

$$\begin{aligned}
(6) \quad & \mathbf{J}^{\mathbf{v}} = p_0(\rho, E, L, q^2) \mathbf{U} + a(\rho, E, q^2, L) \mathbf{q} \mathbf{q}, \\
(7) \quad & \mathbf{J}^{\mathbf{q}} = \beta_0(\rho, E, L, q^2) \mathbf{U} + \gamma(\rho, E, q^2, L) \mathbf{q} \mathbf{q}, \\
(8) \quad & \mathbf{J}^L = \nu_0(\rho, E, L, q^2) \mathbf{q}.
\end{aligned}$$

For the production term in the equation of the heat flux we will take the simple expression  $\sigma_{\mathbf{q}} = -KL\mathbf{q}$  ( $K > 0$ ),<sup>4</sup> and for the one in the equation for the line density  $L$  we will choose Vinen's production and destruction terms (1).

Further restrictions on the constitutive relations are obtained imposing the validity of the entropy principle, applying the Liu procedure.<sup>7</sup> This method requires the existence of a scalar function  $S = S(\rho, E, q^2, L)$  and a vector function  $\mathbf{J}^S = \phi(\rho, E, q^2, L) \mathbf{q}$  of the fundamental fields, namely the entropy density and the entropy flux density respectively, such that the following inequality:

$$\begin{aligned}
(9) \quad & \dot{S} + S \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{J}^S - \Lambda^\rho [\dot{\rho} + \rho \nabla \cdot \mathbf{v}] - \Lambda^{\mathbf{v}} \cdot [\rho \dot{\mathbf{v}} + \nabla \cdot \mathbf{J}^{\mathbf{v}}] \\
& - \Lambda^E \left[ \dot{E} + E \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{q} + \mathbf{J}^{\mathbf{v}} \cdot \nabla \mathbf{v} \right] \\
& - \Lambda^{\mathbf{q}} \cdot [\dot{\mathbf{q}} + \mathbf{q} \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{J}^{\mathbf{q}} - \sigma^{\mathbf{q}}] \\
& - \Lambda^L \left[ \dot{L} + L \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{J}^L - \sigma^L \right] \geq 0,
\end{aligned}$$

is satisfied for arbitrary fields  $\rho$ ,  $\mathbf{v}$ ,  $E$ ,  $\mathbf{q}$  and  $L$ . This inequality expresses the restrictions coming from the second law of thermodynamics. The quantities  $\Lambda^\rho = \Lambda^\rho(\rho, E, L, q^2)$ ,  $\Lambda^{\mathbf{v}} = \Lambda^{\mathbf{v}}(\rho, E, L, q^2) \mathbf{q}$ ,  $\Lambda^E = \Lambda^E(\rho, E, L, q^2)$ ,  $\Lambda^{\mathbf{q}} = \Lambda^{\mathbf{q}}(\rho, E, L, q^2) \mathbf{q}$  and  $\Lambda^L = \Lambda^L(\rho, E, L, q^2)$  are Lagrange multipliers, which are also objective functions of  $\rho, E, \mathbf{q}$  and  $L$ .

The constitutive theory is obtained substituting (7)-(9) in (10) and imposing that the coefficients of all derivatives must vanish. After some lengthy calculations, we obtain  $\Lambda_v = 0$ ,  $a = 0$  and

$$\begin{aligned}
(10) \quad & dS = \Lambda_\rho d\rho + \Lambda_E dE + \lambda_q q_i dq_i + \Lambda_L dL, \\
(11) \quad & -\rho \Lambda_\rho - \Lambda_E (E + p) - \lambda_q q^2 - \Lambda_L L = 0, \\
(12) \quad & \phi = \Lambda_E + \lambda_q \gamma q^2 + \Lambda_L \nu, \\
(13) \quad & d\phi = \lambda_q (d\beta + (1/2)\gamma dq^2 + q^2 d\gamma) + \Lambda_L d\nu.
\end{aligned}$$

It remains the following residual inequality for the entropy production:

$$(14) \quad \sigma^S = \mathbf{\Lambda}^{\mathbf{q}} \cdot \sigma^{\mathbf{q}} + \Lambda^L \sigma^L \geq 0.$$

In the following Sections the coefficients introduced in (7)-(9) will be examined in depth, and related to specific situations specially suitable to stress their physical meaning.

### 2.3. *Physical Interpretation of the Constitutive Relations*

In order to single out the physical meaning and relevance of the constitutive quantities and of the Lagrange multipliers, we analyze now in detail the relations obtained in the previous section. Denoting with  $\Upsilon$  any of the scalar quantities  $S$ ,  $\phi$ ,  $p$ ,  $\beta$ ,  $\gamma$ ,  $\nu$ ,  $\Lambda_\rho$ ,  $\Lambda_E$ ,  $\lambda_q$ ,  $\Lambda_L$  we put  $\Upsilon_0(\rho, E, L) = \Upsilon(\rho, E, 0, L)$ , obtaining, for the first-order coefficients, the following relations:

$$(15) \quad dS_0 = \Lambda_0^\rho d\rho + \Lambda_0^E dE + \Lambda_0^L dL,$$

$$(16) \quad S_0 - \rho\Lambda_0^\rho - \Lambda_0^E(E + p_0) - \Lambda_0^L L = 0,$$

$$(17) \quad \phi_0 = \Lambda_0^E + \Lambda_0^L \nu_0,$$

$$(18) \quad d\phi_0 = \lambda_0^q d\beta_0 + \Lambda_0^L d\nu_0.$$

We first introduce a "generalized temperature" as the reciprocal of the first-order part of the Lagrange multiplier of the energy:

$$(19) \quad \Lambda_0^E = \left[ \frac{\partial S_0}{\partial E} \right]_{\rho, L} = \frac{1}{T}.$$

In the laminar regime (when  $L = 0$ ),  $\Lambda_0^E$  reduces to the absolute temperature of thermostatics. In the presence of a vortex tangle the "generalized temperature" depends also on the line density  $L$ .

If we write now equations (16) and (17) in the following way:

$$(20) \quad dE = T dS_0 - T \Lambda_0^\rho d\rho - T \Lambda_0^L dL,$$

$$(21) \quad -T \Lambda_0^\rho = \frac{E}{\rho} - T \frac{S_0}{\rho} + \frac{p_0 + L T \Lambda_0^L}{\rho},$$

we can define the "mass chemical potential"  $\mu_0^\rho$  in turbulent superfluid as:

$$(22) \quad \mu_0^\rho = -T \Lambda_0^\rho = -T \left[ \frac{\partial S_0}{\partial \rho} \right]_{E, L},$$

and the "chemical potential of vortex lines"  $\mu_0^L$  as:

$$(23) \quad \mu_0^L = -T \Lambda_0^L = -T \left[ \frac{\partial S_0}{\partial L} \right]_{\rho, L}.$$

Indeed, in absence of vortices ( $L = 0$ ), equation (21) is just Gibbs equation of thermostatics and the quantity (22) is the equilibrium chemical potential. The presence of vortices modifies the energy and the chemical potentials. For the chemical potential of vortex lines we will take the expression:<sup>4</sup>

$$(24) \quad \mu_0^L = \epsilon_V \ln(L/L^*),$$

where  $\epsilon_V$  is the energy per unit length of the vortex lines<sup>1,4</sup>, which depends essentially on  $T$ . In (25)  $L^*$  is a reference vortex line density, defined as the average length  $\langle l \rangle$  of the vortex loops composing the tangle, divided by the volume of the system, namely  $L^* \equiv \langle l \rangle / V$ .

Consider now equations (18) and (19) which concern the expressions of the fluxes. Using definitions (20) and (24), we get:

$$(25) \quad \lambda_0^q d\beta_0 = d\left(\frac{1}{T}\right) - \nu_0 d\left(\frac{\mu_0^L}{T}\right).$$

From this equation, recalling that  $\mu_0^L$  depends only on  $T$  and  $L$ , we obtain  $\partial\beta_0/\partial\rho = 0$ , and we put

$$(26) \quad \zeta_0 = \frac{\partial\beta_0}{\partial T} = -\frac{1}{T^2\lambda_0^q} \left[ 1 + \nu_0 T^2 \frac{\partial}{\partial T} \left( \frac{\mu_0^L}{T} \right) \right],$$

$$(27) \quad \chi_0 = \frac{\partial\beta_0}{\partial L} = -\frac{\nu_0}{T\lambda_0^q} \frac{\partial\mu_0^L}{\partial L} = -\frac{\nu_0}{T\lambda_0^q} \frac{\epsilon_V}{L}.$$

In Ref. 4 we have shown that it results  $\lambda_0^q < 0$ ,  $\zeta_0 \geq 0$ ,  $\nu_0 \leq 0$  and  $\chi_0 \leq 0$ .

#### 2.4. The Constitutive Relations far from Equilibrium

We analyze now the complete mathematical expressions far from equilibrium of the Lagrange multipliers, in order to shed some light on their physical meaning. Anyway, this is still an open topic for research, and we will only outline the general ideas.

First, we introduce the following quantity:

$$(28) \quad \theta = \frac{1}{\Lambda_E(\rho, E, L, q^2)},$$

which, near equilibrium ( $L = 0$ ,  $\mathbf{q} = 0$ ) can be identified with the local equilibrium absolute temperature  $T$ . In the following we will choose as fundamental field the quantity  $\theta$ , instead of the internal energy density  $E$ . In accord with Ref. 5, we will call  $\theta$  "non-equilibrium temperature", a topic which is receiving much attention in current non-equilibrium Thermodynamics.<sup>10</sup>

We have shown that at equilibrium the quantities  $-\Lambda_\rho/\Lambda_E$  and  $-\Lambda_L/\Lambda_E$  can be interpreted as the equilibrium mass chemical potential (eq. (23)) and the equilibrium vortex line density chemical potential (eq. (24)). We define, consequently, as nonequilibrium chemical potentials the quantities:

$$(29) \quad \mu = -\frac{\Lambda_\rho}{\Lambda_E}, \quad \text{and} \quad \mu_L = -\frac{\Lambda_L}{\Lambda_E}.$$

Using these quantities, relations (11) and (12) can be written:

$$(30) \quad \theta dS = dE - \mu d\rho - \mu_L dL + \theta \lambda_q \mathbf{q} \cdot d\mathbf{q},$$

$$(31) \quad \theta S = E + p - \rho\mu_\rho - L\mu_L + \theta \lambda_q q^2.$$

In particular, denoting with  $\epsilon$  the specific energy and with  $s$  the nonequilibrium specific entropy, we see that the non equilibrium chemical potentials  $\mu_\rho$  and  $\mu_L$  must satisfy the relation:

$$(32) \quad \mu_\rho + \frac{L}{\rho} \mu_L = \epsilon - \theta s + \frac{p}{\rho} + \frac{\theta}{\rho} \lambda_q q^2.$$

In the absence of vortices ( $\mu_L = 0$ ), this equation furnishes a generalized expression for the mass chemical potential in the laminar flow of superfluids (see also Ref. 11).

The theory developed here furnishes also the complete nonequilibrium expression of the entropy flux  $\mathbf{J}^s$ . Indeed, we can write:

$$(33) \quad \mathbf{J}^s = \left( \frac{1}{\theta} + \nu\Lambda_L + \gamma\lambda_q q^2 \right) \mathbf{q} = \frac{1}{\theta} (\mathbf{q} - \mu_L \mathbf{J}^L + \theta\gamma\lambda_q q^2 \mathbf{q}).$$

This equation shows that, in a nonlinear theory of Superfluid Turbulence, the entropy flux is different from the product of the reciprocal non-equilibrium temperature and the heat flux, but it contains additional terms depending on the flux of heat flux and on the flux of line density.

This concludes, for the moment, our analysis of the thermodynamic restrictions on the coefficients of the constitutive relations. Further consequences of the nonlinear constitutive theory will be presented in a following paper. In the next section, we will explore two simple but physically relevant situations where the terms related to vortex density inhomogeneities play an especially explicit role.

### 3. Fields equations

Substituting the constitutive equations (7)-(9) and the restriction  $a = 0$  in system (2)-(6), the following system of field equations is obtained:

$$\begin{aligned} (1) \quad & \dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0, \\ (2) \quad & \rho \dot{\mathbf{v}} + \nabla p = 0 \\ (3) \quad & \rho \dot{\epsilon} + \nabla \cdot \mathbf{q} + p \nabla \cdot \mathbf{v} = 0, \\ (4) \quad & \dot{\mathbf{q}} + \mathbf{q} \nabla \cdot \mathbf{v} + \nabla \beta + \nabla \cdot (\gamma \mathbf{q} \mathbf{q}) = \sigma^{\mathbf{q}}, \\ (5) \quad & \dot{L} + L \nabla \cdot \mathbf{v} + \nabla \cdot (\nu \mathbf{q}) = \sigma^L, \end{aligned}$$

Observe that in these equations there are the unknown quantities  $p$ ,  $\epsilon$ ,  $\beta$ ,  $\gamma$  and  $\nu$  and the productions  $\sigma^{\mathbf{q}}$  and  $\sigma^L$ . Concerning the fluxes, one observes that the five quantities  $p$ ,  $\epsilon$ ,  $\beta$ ,  $\gamma$  and  $\nu$  cannot be totally independent, because they must satisfy relations (11)-(14).

In the following we pay a special attention to the two last equations, which contain the new effects on which we are focusing our attention.

#### 3.1. The drift velocity of the tangle

In a hydrodynamical model of turbulent superfluids, the line density  $L$  acquires field properties: it depends on the coordinates, it has a drift velocity  $\mathbf{v}^L$ , and its rate of change must obey a balance equation of the general form:

$$(6) \quad \partial_t L + \nabla \cdot (L \mathbf{v}^L) = \sigma^L,$$

with  $\mathbf{v}^L$  the drift velocity of the tangle and  $\partial_t$  stands for  $\partial/\partial p$ . If we now observe that the equation (39) can be written:

$$(7) \quad \partial_t L + \nabla \cdot (L \mathbf{v} + \nu \mathbf{q}) = \sigma^L,$$

comparing (40) and (41), we conclude that the drift velocity of the tangle, with respect to the container, is given by:

$$(8) \quad \mathbf{v}^L = \mathbf{v} + \frac{\nu}{L} \mathbf{q}.$$

Note that the velocity  $\mathbf{v}^L$  does not coincide with the microscopic velocity of the vortex line element, but represents an averaged macroscopic velocity of this quantity. It is to make attention to the fact that often in the literature the microscopic velocity  $\dot{\mathbf{s}}$  is denoted with  $\mathbf{v}_L$ .

Another possibility is to interpret  $\nu \mathbf{q} = \mathbf{J}^L$  as the diffusion flux of vortices. Note that, in this model, if  $\mathbf{q} = 0$ ,  $\mathbf{J}^L$  also is zero.

### 3.2. Vortex diffusion

Here we will apply the general set of equations derived up to here to describe vortex diffusion. We will study this phenomenon neglecting non linear terms in the heat flux. In this case, equations (38) and (39) simplify as:

$$(9) \quad \dot{\mathbf{q}} + \zeta_0 \nabla T + \chi_0 \nabla L = \sigma^{\mathbf{q}} = -KL\mathbf{q},$$

$$(10) \quad \dot{L} + L \nabla \cdot \mathbf{v} + \nu_0 \nabla \cdot \mathbf{q} = \sigma^L = -BL^2 + AqL^{3/2},$$

with  $\zeta_0$  and  $\chi_0$  defined by (27)-(28).

Assume, for the sake of simplicity, that  $T = \text{constant}$  and that  $\mathbf{q}$  varies very slowly, in such a way that  $\dot{\mathbf{q}}$  may be neglected. We find from (43) that  $\chi_0 \nabla L = -KL\mathbf{q}$ . Then, we may write

$$(11) \quad \mathbf{q} = -\frac{\chi_0}{KL} \nabla L.$$

Introducing this expression in equation (44), we find:

$$(12) \quad \dot{L} + L \nabla \cdot \mathbf{v} - \frac{\nu_0 \chi_0}{KL} \Delta L + \frac{\nu_0 \chi_0}{KL^2} (\nabla L)^2 = \sigma^L.$$

Then, we have for  $L$  a reaction-diffusion equation, which generalizes the usual Vinen's equation (1) to inhomogeneous situations. The diffusivity coefficient is found to be

$$(13) \quad D = \frac{\nu_0 \chi_0}{KL}.$$

Since  $\lambda_0^q < 0$  and  $K > 0$ , it turns out that  $D > 0$ , as it is expected. Thus, the vortices will diffuse from regions of higher  $L$  to those of lower  $L$ .

If  $\mathbf{v}$  vanishes, or if its divergence vanishes, equation (46), neglecting also the term in  $(\nabla L)^2$ , yields:

$$(14) \quad \dot{L} = -BL^2 + AqL^{3/2} + D\Delta L.$$

Equation (48) indicates two temporal scales for the evolution of  $L$ : one of them is due to the production-destruction term ( $\tau_{decay} \approx [BL - AqL^{1/2}]^{-1}$ ) and another one to the diffusion:  $\tau_{diff} \approx \frac{X^2}{D}$ , where  $X$  is the size of the system. For large values of  $L$ ,  $\tau_{decay}$  will be much shorter and the production-destruction dynamics will dominate over diffusion; for small  $L$ , instead, diffusion processes may be dominant. This may be also understood from a microscopic perspective because the mean free path of vortex motion is of the order of intervortex spacing, of the order of  $L^{-1/2}$ , and therefore it increases for low values of  $L$ .

A more general situation for the vortex diffusion flux is to keep the temperature gradient in (43). In this more general case,  $\mathbf{q}$  is not more parallel to  $\nabla L$  but results

$$(15) \quad \mathbf{q} = -\frac{\chi_0}{KL} \nabla L - \frac{\zeta_0}{KL} \nabla T$$

in which case, it would become

$$(16) \quad \mathbf{J}^L = \nu_0 \mathbf{q} = -D \nabla L - D \frac{\zeta_0}{\chi_0} \nabla T.$$

The second term in (50) plays a role analogous to thermal diffusion -or Soret effect- in usual diffusion of particles. This kind of situations have not been studied enough in the context of vortex tangles, but they would arise in a natural way when trying to understand the behavior of quantum turbulence in the presence of a temperature gradient.

### 3.3. Wave propagation in counterflow vortex tangles

We briefly recall here the results on the propagation of second sound harmonic plane waves obtained in Ref. 4. Experiments show that in this case the velocity  $\mathbf{v}$  is zero, and only the fields  $T$ ,  $\mathbf{q}$  and  $L$  are involved. The equations for these fields, under these hypotheses, expressing the energy in terms of  $T$  and  $L$ , are simply:

$$(17) \quad \rho c_V \dot{T} + \rho \epsilon_L \dot{L} + \nabla \cdot \mathbf{q} = 0,$$

$$(18) \quad \dot{\mathbf{q}} + \zeta_0 \nabla T + \chi_0 \nabla L = -KL\mathbf{q}$$

$$(19) \quad \dot{L} + \nu_0 \nabla \cdot \mathbf{q} = -BL^2 + AqL^{3/2},$$

where  $c_V = \partial\epsilon/\partial T$  is the specific heat at constant volume and  $\epsilon_L = \partial\epsilon/\partial L \simeq \epsilon_V$ .

These equations are enough for the discussion of the physical effects of the coupling of second-sound and the distortion of the vortex tangle (represented by the inhomogeneities in  $L$ ), which must be taken into account in an analysis of the vortex tangle by means of second sound. In fact, some of the previous hydrodynamical analyses of turbulent superfluids had this problem as one of their main motivations.<sup>12</sup>

A stationary solution of system (43)-(45) is given by

$$(20) \quad \mathbf{q} = \mathbf{q}_0 = (q_{10}, 0, 0), \quad L = L_0 = \frac{A^2}{B^2} [q_{10}]^2, \quad T = T_0(\mathbf{x}) = T^* - \frac{KL_0 q_{10}}{\zeta_0} x_1,$$

with  $q_{10} > 0$ .

Let  $\mathbf{n}$  the unit vector in the direction of wave propagation. We assume heat flux parallel to  $x$  axis. It is seen that when the temperature wave is propagated along  $x$  axis, one obtains the following dispersion relation

$$(21) \quad \begin{aligned} \omega^2 &= k^2 [V_2^2 (1 - \rho \epsilon_L \nu_0) + \nu_0 \chi_0] + N_1 N_2 - i\omega (N_1 + N_2) + i \frac{k^2}{\omega} V_2^2 N_2 \\ &- ik [(\chi_0 + V_2^2 \rho \epsilon_L) N_4 - \nu_0 N_3] \end{aligned}$$

while, when the wave is orthogonal to the heat flux  $\mathbf{q}$ , one obtains

$$(22) \quad \begin{aligned} \omega^2 &= k^2 [V_2^2 (1 - \rho \epsilon_L \nu_0) + \nu_0 \chi_0] + N_1 N_2 - i\omega (N_1 + N_2) \\ &+ i \frac{k^2}{\omega} V_2^2 \left( N_2 - \frac{N_3 N_4}{\omega + i N_1} \right), \end{aligned}$$

where  $k = k_r + ik_s$  is the complex wave number and  $\omega$  the real frequency, and where we have put  $N_1 = KL_0$ ,  $N_2 = 2BL_0 - (3/2)AL_0^{1/2} q_{10}$ ,  $N_3 = Kq_{10}$  and  $N_4 = Aq_{10}L_0^{3/2}$ . In both equations (55) and (56), we have denoted with  $V_2$  the quantity:

$$(23) \quad V_2^2 = \frac{\zeta_0}{\rho c_V},$$

which, in the absence of vortices, coincides with the usual velocity of the second sound,<sup>8,11</sup>. As shown in Ref. 4, in the presence of vortices, this coefficient includes a positive contribution proportional to  $L$ , which shows that the speed of the waves increases when  $L$  increases.

We compare the result (56) with the result obtained in Ref. 9, where we supposed  $L$  a fixed quantity, and the term  $\nu_0$  was assumed to vanish, eliminating in this way the effects of the oscillations of  $\mathbf{q}$  on the vortex line density  $L$  of the tangle. In that work, the dispersion relation for the second sound was:

$$(24) \quad \omega^2 = V_2^2 k^2 - i\omega K L_0.$$

Comparison of (56) with (58) shows that the distortion of the vortex tangle under the action of the heat wave, and its corresponding back reaction on the latter, implies remarkable changes in the velocity and the attenuation of the second sound, the latter effect depending on the relative direction between  $\mathbf{q}_0$  and  $\mathbf{n}$ . Thus, if one uses (58) instead of (56) one obtains erroneous values for the average vortex line density  $L_0$  and the friction coefficient, leading to an incorrect interpretation of the physical results.

#### 4. Conclusions

The study of quantum turbulence in superfluids often assumes homogeneity of the vortex tangle line density  $L$ . In several situations this homogeneity will not hold, and the vortex lines will diffuse from the most concentrated to the less concentrated zones. For instance, vortex lines could be produced near the walls and migrate by diffusion to the bulk of the container until a homogeneous situation is reached. Furthermore, if vortex lines are flexible, they will be bent and their density will be compressed and rarefied by second-sound waves and this will produce an inhomogeneity in  $L$ , which, on its turn, will influence the propagation of second sound. This may be relevant in the interpretation of the experimental results on the speed and the attenuation of the second sound in terms of the average vortex line density of the system.

To incorporate these effects has been the main motivation of this paper. We have not limited ourselves to add a more general evolution equation for  $L$ , but we have tried to insure thermodynamical consistency of the mutual coupling of this equation and the equations considered previously for the other fields. We have worked in a macroscopic thermodynamic framework, which yields the several consequences of incorporating the additional terms to the evolution equations for the heat flux and the vortex line density. The thermodynamic consequences are shown as restrictions on the coefficients of the new terms. We have analyzed both in the linear and in the nonlinear regime the mathematical and physical consequences deduced by the theory. We have obtained in this way field equations for the relevant quantities, which have been applied to describe vortex diffusion and the propagation of harmonic plane waves. The study of nonlinear phenomena, as the propagation of non linear second sound and shock waves will be made in a following paper.

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## References

- <sup>1</sup>R.J. Donnelly, *Quantized vortices in helium II* (Cambridge University Press, Cambridge, U.K., 1991).
- <sup>2</sup>*Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C.F. Barenghi, R.J. Donnelly and W.F. Vinen (Springer, Berlin, 2001).
- <sup>3</sup>W.F. Vinen, Proc. Roy. Soc. London **A240**, 493 (1957).
- <sup>4</sup>M.S. Mongiovì and D. Jou, Phys. Rev. B, (2006) in print.
- <sup>5</sup>D. Jou, J. Casas-Vázquez and G. Lebon, *Extended Irreversible Thermodynamics* (Springer-Verlag, Berlin Heidelberg, 2001).
- <sup>6</sup>I. Müller and T. Ruggeri, *Rational Extended Thermodynamics* (Springer-Verlag, New York, 1998).
- <sup>7</sup>I. Liu, Arch. Rat. Mech. Anal. **46**, 131 (1972).
- <sup>8</sup>M.S. Mongiovì, Phys. Rev. B **48**, 6276 (1993).
- <sup>9</sup>D. Jou, G. Lebon and M.S. Mongiovì, Phys. Rev. B **66**, 224509 (2002).
- <sup>10</sup>J. Casas-Vázquez and D. Jou, Rep. Prog. Phys. **66**, 1937 (2003).
- <sup>11</sup>M.S. Mongiovì, J. Non-Equil. Thermody. **25**, 31 (2000).
- <sup>12</sup>M. Tsubota, T. Araki and W.F. Vinen, Physica B **329-333**, 224 (2003).