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LINEAR ORDERS AND SPECIFIC GRÖBNER BASES

G. D'AGUÌ *, G. FAILLA **

Department of Mathematics, University of Messina
Messina,98168, Italy
**E-mail: gdagui@dipmat.unime.it*
***E-mail: gfailla@dipmat.unime.it*
www.unime.it

Abstract.

We introduce linear orders on the set of variables of a polynomial ring with the aim to obtain quadratic lexicographical(or reverse lexicographical) Gröbner bases for toric ideals. The case of the Segre product of two polynomial rings is studied and discussed for its interest in the transportation problem of manufacturing.

Keywords: Polynomial ideals; Gröbner bases.

Introduction

Let K be a field and let $R = K[x_1, x_2, \ldots, x_s]$ be the polynomial ring in s variables $x_1, x_2, \ldots, x_s$. The introduction in R of a total order on the set Π of the monomials of R is subordinated to the introduction of an order on the variables $x_1, x_2, \ldots, x_s$. In the last years interesting results were obtained in the computational field of commutative algebra and algebraic geometry by considering orders of variables different from $x_1 > x_2 > \ldots > x_s$, for example for $x_s > \ldots > x_1$. Since we have $s!$ orders of variables it is clear that we can choose orders linked to the resolution of our problem. In Ref. 9 Villareal introduces linear orders on the set of variables $\mathbf{T} = \{T_{i_1 i_2 \ldots i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$, s and k two integers such that $4 \leq 2k \leq s$ "via" the lexicographic order. These linear orders are defined by special $k \times k$ non singular matrices with entries in $\mathbb{R}$ and they constitute a proper subset of the set of all total orders on $\mathbf{T}$. The variables $T_{i_1 i_2 \ldots i_k}$ appear in the presentation of the k-square free Veronese subring $K[V_k]$, where V_k is the set of all square free monomials of degree k in R.
In this paper, we define first the reverse linear orders on the variables $T_{i_1 i_2 \ldots i_k}$, "via" the reverse lexicographic order. Our intention is to extend the linear order (reverse linear order) to the set of variables that come from the Segre product of two square-free Veronese subrings $K[V_k]$ and $K[V_l]$, where V_k is the set of all square free monomials of degree k in R , V_l is the set of all square free monomials of degree l in $S = K[y_1, \ldots, y_t]$, $4 \leq 2l \leq t$. Let $<_1$ and $<_2$ be two linear orders (resp. inverse linear orders) respectively defined by the $k \times k$ matrix $\mathcal{A}_1$ and by the $l \times l$ matrix $\mathcal{A}_2$. The linear order $<_1 \star_S <_2$ for the variables $T_{i_1 \ldots i_k; j_1 \ldots j_l}$, of the polynomial ring in the presentation of the Segre product $K[V_k] \star_S K[V_l]$ is defined by the $l+k-$matrix $\begin{pmatrix} \mathcal{A}_1 & \Omega_1 \\ \Omega_2 & \mathcal{A}_2 \end{pmatrix}$ where Ω_1 and Ω_2 are the null matrices. We recall

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the following conjecture, contained in Ref. 9:
Conjecture: *If I is the toric ideal of $K[V_k]$ where $4 \leq 2k \leq n$, there exists a linear order of the variables $T_{i_1 \dots i_k}$ so that the reduced Gröbner basis of I, with respect to the lexicographical order, is generated by quadratics.*
We extend the conjecture in the following way:
Conjecture: *If I is the toric ideal of $K[V_k] \star_S K[V_l]$, there exists a linear order $<_1 \star_S <_2$ of the variables $T_{i_1 \dots i_k; j_1 \dots j_l}$ so that the reduced Gröbner basis of I, with respect to the lexicographical order (resp. reverse lexicographical order), is generated by quadratics.*
The problem to have quadratic Gröbner basis for the toric ideals is very important from the algebraic and geometric point of view. Algebraically speaking, furthermore, the property is related to Gröbner fan and state polytope. Geometrically speaking the problem is mainly linked to have varieties as intersection of quadrics (Ref. 7). In our context, linear orders on the variables T_{ij}, that appear in the presentation of the Segre product of two polynomial rings, could have interest in applications and problems about transportation and manufacturing . For instance we consider the classical problem of manufacturing.
Given r factories F_i and t stores S_j, we represent by $F_i S_j$ the exchange of units of indivisible goods among F_i and S_j: $F_1 \longrightarrow S_1, \dots F_r \longrightarrow S_t$. The problem is modelled by the Segre Product :

$$K[x_1, \dots, x_r] *_S K[y_1, \dots, y_t]$$

$$F_i S_j \longrightarrow x_i y_j$$

In general, the exchanges between r factories and t stores are supported without an order of exchanges. But, there are many situations where we need to set an order, for example, the production of goods from $F_1, \dots, F_r$ is not simultaneous, then the director of factory must decide which store has to serve first, the production is simultaneous but the deliveries are not contemporary, the production is simultaneous but the deliveries depend on the way of payment, the factory produces goods that are employed following an order, in the sense that the goods of the factory F_i are requested before the goods of factory F_j. Finally, we describe the contents of the paper . In N.1 we give the definition of reverse-lexicographical linear orders for the variables $T_{i_1 \dots i_k}$ and we introduce the order $<_1 \star_S <_2$. For the variables that come from the Segre product of two polynomial rings $K[x_1, x_2, \dots, x_s]$ and $K[y_1, ..., y_t]$ we obtain four linear lexicographical orders L_i, and four linear reverse lexicographical orders L'_i, $i = 1, ..., 4$.
In N.2 we prove two main theorems by using the semigroup rings theory, linked to Segre products of two polynomial rings, and results contained in Ref. 9 and 6. We obtain that for the toric ideal of $K[x_1, x_2, \dots, x_s] \star_S K[y_1, ..., y_t]$, all orders L_i induce lexicographical quadratic square free Gröbner bases and all orders L'_i induce reverse lexicographical quadratic square free Gröbner bases.

1. Linear Orders

Let K be a field (that will be $\mathbb{Q}$ or $\mathbb{R}$) and let $R = K[x_1, x_2, \dots, x_s]$ the polynomial ring in s variables $x_1, x_2, \dots, x_s$.
The set of the power products of R is denoted by:

$$T^s = \{x_1^{a_1} \cdots x_s^{a_s} \,|\, a_i \in \mathbb{N}, i = 1, \dots, s\}$$

where $x_1^{a_1} \cdots x_n^{a_s} = \mathbf{x}^{\mathbf{a}}, \qquad \mathbf{a} \in \mathbb{N}^s, \ \mathbf{a} = (a_1, a_2, \dots, a_s)$.

Definition 1.1. By term order on T^s we mean a total order $<$ on T^s satisfying two condition:

- $1 < \mathbf{x^b}$ for all $\mathbf{x^b} \in T^s$, $\mathbf{x^b} \neq 1$;
- If $\mathbf{x^a} < \mathbf{x^b}$, then
  $\mathbf{x^a x^c} < \mathbf{x^b x^c}$,for all $\mathbf{x^c} \in T^s$.

Definition 1.2. (See Ref. 1)
We define the lexicographical order on T^s with $x_1 > x_2 > \ldots > x_s$ as follows:
for $\mathbf{a} = (a_1, a_2, \ldots, a_s)$, $\mathbf{b} = (b_1, b_2, \ldots, b_s)$ we define
$\mathbf{x^a} < \mathbf{x^b} \Longleftrightarrow$ there exists k such that $a_1 = b_1, \ldots, a_k = b_k$, $a_{k+1} < b_{k+1}$, $k = 0, \ldots, s-1$.

Definition 1.3. Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ be a set of polynomials. G is an universal Gröbner basis for $(f_1, \ldots, f_n)$ if G is a Gröbner basis with respect to all term orders on the monomials of R.

Example 1.1. (See Ref. 7)
Let R be a polynomial ring in $2m$ variables

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \end{pmatrix}.$$

Let I be the ideal of R generated by the $\binom{m}{2}$ polynomials $D_{ij} = x_{1i}x_{2j} - x_{1j}x_{2i}$ for $1 \leq i < j \leq m$. This set of the $2 \times 2-$minors D_{ij} is a universal Gröbner basis.

Definition 1.4. Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ be a set of polynomials. G is a universal lexicographical Gröbner basis for $(f_1, \ldots, f_n)$ if G is a Gröbner basis with respect to the lexicographical order on the monomials of R and with respect to the $s!$ orders of the variables $x_1, x_2, \ldots, x_s$.

Definition 1.5. Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ be a set of polynomials. G is a universal reverse lexicographical Gröbner basis for $(f_1, \ldots, f_n)$ if G is a Gröbner basis with respect to the reverse lexicographical order on the monomials of R and with respect to the $s!$ orders of the variables $x_1, x_2, \ldots, x_s$.

Definition 1.6. Let $\Gamma = \{<_i\}$ be a set of orders on the variables $x_1, \ldots, x_s$ with $|\Gamma| < s!$

Definition 1.7. Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ be a set of polynomials. We say that G is a universal lexicographical (resp. reverse) Gröbner basis for $(f_1, \ldots, f_n)$ with respect to the orders of Γ if G is a Gröbner basis with respect to the lexicographical (resp. reverse) order on the monomials of R and with respect to the orders of Γ on the variables $x_1, x_2, \ldots, x_s$.

Now we recall linear orders for variables that appear in the Veronese $k-$subrings of polynomial rings, $K[V_k]$, where V_k is the set of square-free monomials of R of degree k.

Given two integers n and k such that $4 \leq 2k \leq n$ we consider the set of variables: $\mathbf{T} = \{T_{i_1 i_2 \ldots i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$. Let $A = (a_{ij})$ be a non singular matrix of order k with entries in $\mathbb{R}$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}.$$

This matrix defines linear orders of the variables in the following way:

Definition 1.8. (Linear lexicographical order in $\mathbf{T}$)
Let be $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k$ and $\mathbf{j} = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^k$ then

$$\mathbf{i} >_{A(lex)} \mathbf{j} \quad \text{iff} \quad \mathbf{i}A >_{lex} \mathbf{j}A.$$

More precisely:

$$\begin{pmatrix} i_1 \; i_2 \cdots i_k \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} = \begin{pmatrix} b_1 \; b_2 \cdots b_k \end{pmatrix}$$

where $b_t = i_1 a_{1t} + i_2 a_{2t} + \cdots + i_k a_{kt}$, $\mathbf{i}A = \begin{pmatrix} b_1 \; b_2 \cdots b_k \end{pmatrix}$ and $\mathbf{j}A = \begin{pmatrix} c_1 \; c_2 \cdots c_k \end{pmatrix}$.
So $\mathbf{i}A >_{lex} \mathbf{j}A$ iff $(b_1 \; b_2 \; \cdots b_k) >_{lex} (c_1 \; c_2 \; \cdots c_k)$ means $b_1 = c_1, \; b_2 = c_2, \ldots, b_s > c_s$ for some $s = 1, \ldots, k$. Thus, we can define the linear lexicographical order of variables by:

$$T_{i_1 i_2 \ldots i_k} >_{A(lex)} T_{j_1 j_2 \ldots j_k} \qquad \text{iff} \qquad (i_1, i_2, ..., i_k) >_{A(lex)} (j_1, j_2, ...j_k).$$

Definition 1.9. (Linear reverse lexicographical order in $\mathbf{T}$)
Let be $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k$ and $\mathbf{j} = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^k$ then

$$\mathbf{i} >_{A(revlex)} \mathbf{j} \quad \text{iff} \quad \mathbf{i}A >_{revlex} \mathbf{j}A.$$

More precisely:

$$\begin{pmatrix} i_1 \; i_2 \cdots i_k \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} = \begin{pmatrix} b_1 \; b_2 \cdots b_k \end{pmatrix}$$

where $b_t = i_1 a_{1t} + i_2 a_{2t} + \cdots + i_k a_{kt}$, $\mathbf{i}A = \begin{pmatrix} b_1 \; b_2 \cdots b_k \end{pmatrix}$ and $\mathbf{j}A = \begin{pmatrix} c_1 \; c_2 \cdots c_k \end{pmatrix}$.
So $\mathbf{i}A >_{revlex} \mathbf{j}A$ iff $(b_1 \; b_2 \; \cdots b_k) >_{revlex} (c_1 \; c_2 \; \cdots c_k)$ means $b_k = c_k, \ldots b_{s+1} = c_{s+1}$, $b_s < c_s$ for some $s = 1, \ldots, k$
Thus we can define the linear lexicographical order of variables by:

$$T_{i_1 i_2 \ldots i_k} >_{A(revlex)} T_{j_1 j_2 \ldots j_k} \qquad \text{iff} \quad (i_1, i_2, ..., i_k) >_{A(revlex)} (j_1, j_2, ...j_k).$$

Now we consider the set $\mathcal{A}_k$ of matrices of order k of the form

$$\begin{pmatrix} \pm e_{i_1} \\ \vdots \\ \pm e_{i_j} \\ \vdots \\ \pm e_{i_k} \end{pmatrix}$$

so that the set $\{e_{i_1}, \ldots, e_{i_k}\}$ is a permutation of the set of canonical vectors $\{e_1, \ldots, e_k\} \in \mathbb{R}^k$. $\mathcal{A}_k$ is a group of order $2^k k!$ (See Ref. 8)

Example 1.2. If $k = 2$ $\mathcal{A}_2$ has 8 matrices :

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ A_6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ A_7 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ A_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Let K a field and $R = K[x_1, \ldots, x_n]$ a polynomial ring. Let V_k be the set of square-free monomials of degree k in R. Consider the presentation :

$$\varphi : K[\mathbf{T}] \longrightarrow K[V_k]$$

induced by $T_{i_1,\ldots,i_k} \longmapsto x_{i_1} \cdots x_{i_k}$, where $P = ker(\varphi)$ is the toric ideal of $K[V_k]$.

Example 1.3. If $n = 4$ and $k = 2$, let $\mathbf{T} = \{T_{12}, T_{13}, T_{14}, T_{23}, T_{24}, T_{34}\}$ be a set of variables of $K[V_2] \subseteq K[x_1, x_2, x_3, x_4]$
The linear lexicographical orders on the variables induced by the matrices of $\mathcal{A}_2$ are respectively:

by A_1:	$T_{12} < T_{13} < T_{14} < T_{23} < T_{24} < T_{34}$
by A_2:	$T_{34} < T_{23} < T_{24} < T_{12} < T_{13} < T_{14}$
by A_3:	$T_{14} < T_{13} < T_{12} < T_{24} < T_{23} < T_{34}$
by A_4:	$T_{34} < T_{24} < T_{23} < T_{14} < T_{13} < T_{12}$
by A_5:	$T_{12} < T_{13} < T_{23} < T_{14} < T_{24} < T_{34}$
by A_6:	$T_{14} < T_{13} < T_{12} < T_{24} < T_{23} < T_{34}$
by A_7:	$T_{12} < T_{23} < T_{13} < T_{34} < T_{24} < T_{14}$
by A_8:	$T_{34} < T_{24} < T_{14} < T_{23} < T_{13} < T_{12}$.

Remark 1.1. If $k = 2$ and $n = 4$, the linear orders, induced by the matrices in $\mathcal{A}_2$, induce quadratic Gröbner bases for the toric ideal of $K[V_2]$ with respect to lexicographic and reverse lexicographic orders.

In Ref. 9 and 8 we have the complete list of $k \times k$ matrices of $\mathcal{O}_k(\mathbb{R})$ that induce a quadratic lexicographic Groebner basis of the toric ideal of $K[V_k]$ in the cases:
1)$k = 2, 3$ and $s = 6$ 2)$k = 4$ and $s = 8$.
The research can be done by Macaulay or COCOA.

Given four integers s, t and k, l such that $s \geq k \geq 1$ and $t \geq l \geq 1$ we consider two "label sets"

$$\mathcal{I}_k^s = \{[i_1, ..., i_k], 1 \leq i_1 < ... < i_k \leq s\} \qquad \mathcal{I}_l^t = \{[j_1, ..., j_l], 1 \leq j_1 < ... < j_l \leq t\}$$

and the set of variables

$$T = \{T_{i_1 \ldots i_k, j_1 \ldots j_l :}(i_1, ..., i_k, j_1, ..., j_l) \in \mathcal{I}_k^s \times \mathcal{I}_l^t\}.$$

Let $\mathcal{A}$ be a $(k + l) \times (k + l)$ matrix of the type

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \Omega_1 \\ \Omega_2 & \mathcal{A}_2 \end{pmatrix}$$

where $\mathcal{A}_1$ is a matrix of order k, $\mathcal{A}_2$ is a matrix of order l of the form:

$$\mathcal{A}_1 = \begin{pmatrix} \pm e_{i_1} \\ \vdots \\ \pm e_{i_k} \end{pmatrix} \qquad and \qquad \mathcal{A}_2 = \begin{pmatrix} \pm e'_{j_1} \\ \vdots \\ \pm e'_{j_l} \end{pmatrix}$$

and Ω_1 is the $l \times k$ null matrix and Ω_2 is the $k \times l$ null matrix, where $\{e_{i_1}, ..., e_{i_k}\}$ and $\{e'_{j_1}, ..., e'_{j_l}\}$ are a permutation of the set of canonical vectors $\{e_1, ..., e_k\}$ in $\mathbb{R}^k$ and $\mathbb{R}^l$. $\mathcal{A}_1$ and $\mathcal{A}_2$ are two groups of order $2^k k!$ and they are subgroups of the orthogonal group $\mathcal{O}_k(\mathbb{R})$.

Definition 1.10. The matrix $\mathcal{A}$ defines a linear ordering in $\mathbb{Z}^{k+l}$ given by

$$(x_1, ..., x_k; x_1^{'}, ..., x_l') >_{lex} (y_1, ..., y_k; y_1^{'}, ..., y_l') \qquad \textit{iff}$$

$$(x_1, ..., x_k; x_1^{'}, ..., x_l')\mathcal{A} >_{lex} (y_1, ..., y_k; y_1^{'}, ..., y_l')\mathcal{A}.$$

Definition 1.11. The matrix $\mathcal{A}$ defines an inverse linear ordering in $\mathbb{Z}^{k+l}$ given by

$$(x_1, ..., x_k; x_1^{'}, ..., x_l') >_{revlex} (y_1, ..., y_k; y_1^{'}, ..., y_l') \qquad \textit{iff}$$

$$(x_1, ..., x_k; x_1^{'}, ..., x_l')\mathcal{A} >_{revlex} (y_1, ..., y_k; y_1^{'}, ..., y_l')\mathcal{A}.$$

Proposition 1.1. *If $k = l = 1$, then $\mathcal{A} = \begin{pmatrix} \pm e_1 & 0 \\ 0 & \pm e_1' \end{pmatrix}$ and the set of matrices is $A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.*

Given two polynomial rings $K[x_1, ..., x_s]$, $K[y_1, ..., y_t]$, the Segre product is $K[x_1, ..., x_s] \otimes_S K[y_1, ..., y_t] \cong K[x_1y_1, ..., x_sy_t]$.
Consider the map

$$\widehat{\pi} : K[T_{1,1}, T_{1,2}, ..., T_{s,t}] \to K[x_1y_1, ..., x_sy_t]$$

$$T_{i,j} \mapsto x_iy_j.$$

The new set of variables is $\{T_{1,1}, T_{1,2}, ..., T_{s,t}\}$.
For this set, the matrices in A define four lex linear orders in $\mathbb{Z}^{k+l}$:

$$L_1 : T_{s,t} > T_{s,t-1} > ... > T_{s,1} > ... > T_{1,t} > ... > T_{1,1}$$

$$L_2 : T_{1,t} > T_{1,t-1} > ... > T_{1,1} > ... > T_{s,t} > ... > T_{s,1}$$

$$L_3 : T_{s,1} > T_{s,t-1} > ... > T_{s,t} > ... > T_{1,1} > ... > T_{1,t}$$

$$L_4 : T_{1,1} > T_{1,2} > ... > T_{1,t} > ... > T_{s,1} > ... > T_{s,t}$$

Proposition 1.2. *For $s = t = 2$, $\operatorname{Ker}\hat{\pi}$ admits an universal Gröbner basis. In particular it 's a quadratic Gröbner basis with respect to the lexicographic (resp. reverse) order on the monomials, and with the linear orders on the variables $L_i, i = 1, ..., 4$.*

Proof. In our case $\hat{\pi} : K[T_{1,1}, T_{1,2}, ..., T_{s,t}] \to K[x_1y_1, ..., x_sy_t]$ and by the mapping:

$$T_{1,1} \to x_1y_1, \qquad T_{1,2} \to x_1y_2, \qquad T_{2,1} \to x_2y_1, \qquad T_{2,2} \to x_2y_2.$$

We have $\hat{\pi}(T_{1,2}T_{2,1} - T_{1,1}T_{2,2}) = 0$. In each case $\operatorname{Ker}\hat{\pi}$ consists only of the binomial $T_{1,2}T_{2,1} - T_{1,1}T_{2,2}$.
The result follows since every binomial is a minimal Gröbner basis. □

2. Finding Quadratic Gröbner bases

Let n and d be two integer positive numbers.
Let $\mathcal{A} = \{\underline{a}_1, ..., \underline{a}_n\}$ a fixed subset of lattice vectors in Z^d.
We consider the homomorphism of semigroups:

$$\pi : N^n \longrightarrow Z^d$$

$$\underline{u} \longrightarrow u_1\underline{a}_1 + ... + u_n\underline{a}_n$$

where $\underline{u} = (u_1, ..., u_n)$,and the image of π is the semigroup generated by $\mathcal{A}$,
$N\mathcal{A} = \{\lambda_1\underline{a}_1 + ... + \lambda_n\underline{a}_n, \lambda_1, ..., \lambda_n \in N\}$.
The map π can be lifted to the algebras homomorphism:

$$\hat{\pi} : K[x_1, ..., x_n] \to K[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}] = K[\underline{t}^{\pm 1}]$$

$$X_i \to \underline{t}^{\underline{a}_i} = t_1^{a_{i1}} t_2^{a_{i2}} ... t_d^{a_{id}}$$

$$i = 1, ..., n \qquad \underline{a}_i = (a_{i1}, a_{i2}, ..., a_{id}).$$

$\operatorname{Ker}\hat{\pi}$ is denoted by $I_{\mathcal{A}}$ and it's called toric ideal of $\mathcal{A}$. Since $\operatorname{Im}\hat{\pi} \subset K[t^{\pm 1}]$ that is a domain,$I_{\mathcal{A}}$ is a prime ideal and
$K[X_1, ..., X_n]/I_{\mathcal{A}} \cong K[\underline{t}^{\underline{a}_1}, ..., \underline{t}^{\underline{a}_n}]$. Then the affine variety $V(I_{\mathcal{A}})$ in K^n is irreducible and it's called affine toric variety.

For the following, the semigroup N^n has to be considered as a semigroup of Z^n, otherwise we couldn't speak of $\operatorname{Ker}\pi$.

Let $\underline{u} \in Z^n$. It can be written uniquely $\underline{u} = \underline{u}^+ - \underline{u}^-$, where $\underline{u}^+$ and $\underline{u}^-$ are vectors with non-negative components and disjoint support,where $\operatorname{Supp}\underline{u} = \{i \in N, u_i \neq 0\}$.

We come back to the map $\pi : N^n \longrightarrow Z^d$ .We assume that $\mathcal{A} = \{\underline{a}_1, ..., \underline{a}_n\}$ lies in $N^d \backslash \{0\}$ that is each lattice vector $\underline{a}_i$ is non-zero and non-negative and let $\underline{b} \in N\mathcal{A} \subset N^d$.

Theorem 2.1. *Let $d = s + t$, $n = st$, $s, t > 1$.*
The map $\pi : N^{s\times t} \longrightarrow N^{s+t}$ computes the sum of the rows and of the columns of a given matrix and the map π acts in the following way:
if $\underline{u} = (u_{ij}) \in N^{s\times t}$ then

$$\pi((u_{ij})) = \left(\sum_{j=1}^{t} u_{1j}, ..., \sum_{j=1}^{t} u_{sj}, \sum_{i=1}^{s} u_{i1}, ..., \sum_{i=1}^{s} u_{it}\right) = (r_1, ..., r_s; c_1, ..., c_t) = (\underline{r}; \underline{c})$$

Proof. See Ref. 6, theorem 1.6. □

Theorem 2.2. *If we consider the linear orders L_i, $i = 1, ..., 4$ on the set of variables $T_{1,1}, T_{1,2}, ..., T_{s,t}$, then $\operatorname{Ker}\hat{\pi}$ has a lexicographical quadratic square-free Gröbner basis given by the set of binomials*

$$G_1 = \{T_{j,k}T_{i,l} - \underline{T_{j,l}T_{i,k}}, 1 \le i < j \le s, 1 \le k < l \le t\}$$

$$G_2 = \{\underline{T_{i,l}T_{j,k}} - T_{i,k}T_{j,l}, 1 \le i < j \le s, 1 \le k < l \le t\}$$

$$G_3 = \{\underline{T_{j,k}T_{i,l}} - T_{j,l}T_{i,k}, 1 \le i < j \le s, 1 \le k < l \le t\}$$

$$G_4 = \{T_{i,l}T_{j,k} - \underline{T_{i,k}T_{j,l}}, 1 \le i < j \le s, 1 \le k < l \le t\}$$

where in each binomial the initial term is underlined.

Proof. Consider the linear order of variables L_1. Then $T_{j,l} > T_{j,k} > T_{i,l} > T_{i,k}$. We choose the lex order on the monomials and we proceed by induction on $s+t$ to prove that G_1 is a reduced Gröbner basis. If $s+t=4$ it's true by Proposition (1.2). For $s+t \geq 5$, G_1 is a reduced Gröbner basis for all the matrices $\underline{u} \in N^{(s-1)\times t}$ or $\underline{u} \in N^{s\times(t-1)}$by inductive hypothesis.
Now suppose that G_1 is not a reduced Gröbner basis. Then there exists $\underline{u} \in \operatorname{Ker}\pi$, $\underline{u} = \underline{u}^+ - \underline{u}^-$, such that the correspondent binomial $f = \underline{T}^{u^+} - \underline{T}^{u^-}$ has the initial term that is not a multiple of the initials terms of the binomials of G_1.
Since $\underline{u} \in \operatorname{Ker}\pi$, then $\pi(\underline{u}) = 0$ and $\pi(\underline{u}^+) = \pi(\underline{u}^-)$. By definition $\pi(\underline{u}^+) = \pi(\underline{u}^-) := (\underline{r}, \underline{c}) = (r_1, ..., r_s; c_1, ..., c_t)$. It results that all the coordinates of $(\underline{r}, \underline{c})$ are not zero. In fact if we suppose that $r_1 = 0$, that is $\sum_{j=1}^{t} u^+_{1j} = 0$. Since every element in the matrix $(\underline{u}^+_{ij})$is positive, then $u^+_{1j} = 0$ for every j and the first row in $(\underline{u}^+_{ij})$ is zero, it follows that $(\underline{u}^+_{ij}) \in N^{(s-1)\times t}$. By inductive hypothesis $T^{\underline{u}^+}$ is divisible for $T_{j,l}T_{i,k}$ where $i < j \leq s-1$ and G_1 would be a Gröbner basis. In particular we suppose that $r_1 \geq 1$ and $c_1 \geq 1$. Since $T^{\underline{u}^+}$ is not divisible for $T_{j,l}T_{1,1}$, $j, l \geq 2$ then $T^{\underline{u}^+}$ is divisible for $T_{1,2}$. The same procedure is valid for $T^{\underline{u}^-}$. But this implies that $\operatorname{Supp}(\underline{u}^+) = \operatorname{Supp}(\underline{u}^-)$. Contradiction.
With the same procedure we obtain that G_i for $i = 2, 3, 4$ are Gröbner basis. Precisely:
-With respect the order of variables L_2, $T_{i,l} > T_{i,k} > T_{j,l} > T_{j,k}$ and the element that is not divisible for the initial term $\underline{T_{i,l}T_{j,k}}$ is $T_{1,1}$.
-With respect the order of variables L_3, $T_{j,k} > T_{j,l} > T_{i,k} > T_{i,l}$ and the element that is not divisible for the initial term $\underline{T_{j,k}T_{i,l}}$ is $T_{1,1}$.
-With respect the order of variables L_4, $T_{i,k} > T_{i,l} > T_{j,k} > T_{j,l}$ and the element that is not divisible for the initial term $\underline{T_{i,k}T_{j,l}}$ is $T_{1,2}$. □

The analogous result is true for the inverse linear orders L'_i, $i = 1, \ldots, 4$, and for the reverse lexicographical order on monomials. Precisely we have:

$$L'_1 : T_{1,1} > T_{2,1} > ... > T_{s,1} > ... > T_{1,t} > ... > T_{s,t}$$

$$L'_2 : T_{s,1} > T_{s-1,1} > ... > T_{1,1} > ... > T_{s,t} > ... > T_{1,t}$$

$$L'_3 : T_{1,t} > T_{2,t} > ... > T_{s,t} > ... > T_{1,1} > ... > T_{s,1}$$

$$L'_4 : T_{s,t} > T_{s-1,t} > ... > T_{1,t} > ... > T_{s,1} > ... > T_{1,1}$$

Theorem 2.3. *If we consider the linear orders L'_i, $i = 1, ..., 4$ on the set of variables $T_{1,1}, T_{1,2}, ..., T_{s,t}$, then $\operatorname{Ker}\hat{\pi}$ has a reverse lexicographical quadratic square-free Gröbner basis given by the set of binomials*

$$G'_1 = \{\underline{T_{j,k}T_{i,l}} - T_{i,k}T_{j,l}, 1 \leq i < j \leq s, 1 \leq k < l \leq t\}$$

$$G'_2 = \{T_{j,k}T_{i,l} - \underline{T_{i,k}T_{j,l}}, 1 \leq i < j \leq s, 1 \leq k < l \leq t\}$$

$$G'_3 = \{T_{i,l}T_{j,k} - \underline{T_{j,l}T_{i,k}}, 1 \leq i < j \leq s, 1 \leq k < l \leq t\}$$

$$G'_4 = \{\underline{T_{i,l}T_{j,k}} - T_{j,l}T_{i,k}, 1 \leq i < j \leq s, 1 \leq k < l \leq t\}$$

where in each binomial the initial term is underlined.

Proof. Consider the linear order of variables L'_1. Then $T_{i,k} > T_{j,k} > T_{i,l} > T_{j,l}$. We choose the revlex order on the monomials and we proceed by induction on $s+t$ to prove that G'_1 is a reduced Gröbner basis. If $s+t=4$ it's true by Proposition (1.2). For $s+t \geq 5$, G'_1 is a reduced Gröbner basis for all the matrices $\underline{u} \in N^{(s-1)\times t}$ or $\underline{u} \in N^{s\times(t-1)}$ by inductive hypothesis.
Now suppose that G'_1 is not a reduced Gröbner basis. Then there exists $\underline{u} \in \operatorname{Ker}\pi$, $\underline{u} = \underline{u}^+ - \underline{u}^-$, such that the correspondent binomial $f = \underline{T}^{u^+} - \underline{T}^{u^-}$ has the initial term that is not a multiple of the initials terms of the binomials of G_1. Since $\underline{u} \in \operatorname{Ker}\pi$, then $\pi(\underline{u}) = 0$ and $\pi(\underline{u}^+) = \pi(\underline{u}^-)$. By definition $\pi(\underline{u}^+) = \pi(\underline{u}^-) := (\underline{r}, \underline{c}) = (r_1, ..., r_s; c_1, ..., c_t)$. It results that all the coordinates of $(\underline{r}, \underline{c})$ are not zero. In fact if we suppose that $r_1 = 0$, that is $\sum_{j=1}^{t} u^+_{1j} = 0$. Since every element in the matrix $(\underline{u}^+_{ij})$is positive, then $u^+_{1j} = 0$ for every j and the first row in $(\underline{u}^+_{ij})$ is zero. It follows that $(\underline{u}^+_{ij}) \in N^{(s-1)\times t}$. By inductive hypothesis $T^{\underline{u}^+}$ is divisible for $T_{jk}T_{il}$ where $i < j \leq s-1$ and G'_1 would be a Gröbner basis.
In particular we suppose that $r_1 \geq 1$ and $c_1 \geq 1$. Since $T^{\underline{u}^+}$ is not divisible for $T_{j,l}T_{1,l}$, $j, l \geq 2$ then $T^{\underline{u}^+}$ is divisible for $T_{1,1}$. The same procedure is valid for $T^{\underline{u}^-}$. But this implies that $\operatorname{Supp}(\underline{u}^+) = \operatorname{Supp}(\underline{u}^-)$. Contradiction.
With the same procedure we obtain that G'_i for $i = 2, 3, 4$ are Gröbner basis. Precisely:
-With respect the order of variables L'_2, $T_{j,k} > T_{i,k} > T_{j,l} > T_{i,l}$ and the element that is not divisible for the initial term $\underline{T_{i,k}T_{j,l}}$ is $T_{1,2}$.
-With respect the order of variables L'_3, $T_{i,l} > T_{j,l} > T_{i,k} > T_{j,k}$ and the element that is not divisible for the initial term $\underline{T_{j,l}T_{i,k}}$ is $T_{1,2}$.
-With respect the order of variables L'_4, $T_{j,l} > T_{i,l} > T_{j,k} > T_{i,k}$ and the element that is not divisible for the initial term $\underline{T_{i,l}T_{j,k}}$ is $T_{1,1}$. □

Remark 2.1. We can have lexicographic quadratic Grobner basis induced by not linear orders. Let $s=t=3$, $k=1$, for the order on the variables $T_{1,1} > T_{1,2} > T_{2,2} > T_{2,1} > T_{1,3} > T_{2,3} > T_{3,2} > T_{3,3}$ we have a lexicographical quadratic Gröbner basis for the toric ideal $Ker\widehat{\pi}$. But L is not a linear order according to the definition.

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