Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 2 (2007)
DOI: 10.1685/CSC06160

BINOMIAL IDEALS AND APPLICATIONS

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Abstract.

We utilize methods from affine semigroups theory in computational commutative algebra to study semigroups map, whose fibers can be reviewed as possible transportation plans in the exchange of units of an indivisible good among factories and $2-$block of stores. The results come from the description of the toric ideal of the Segre product between a polynomial ring and a $2-$Veronese square-free ring.

Keywords: Polynomial ideals; Gröbner bases

Introduction

Many scientific problems can be modelled by polynomial equations.
In this field, the special class of toric ideals lay the foundation of many applications in integer programming and computational statistics.
The toric ideals are characterized as those prime ideals that are generated by monomial or as the defining ideals of algebraic toric varieties.
Toric ideals are related to recent progress in computational geometry.
In this paper we consider the problem associated with a linear map $\pi : \mathbb{N}^n \rightarrow \mathbb{N}^d$.
Each fiber $\pi^{-1}(\underline{b}), \underline{b} \in \mathbb{N}^d$, consists of lattice points in a polyhedron in $\mathbb{R}^n$.
The toric ideal $I_{\mathcal{A}}$ which is used to model these questions is the ideal of algebraic relations on the monomials with exponent vectors the columns of a matrix representing π.
In Ref. 4 the integer programming problem associated to the set $\mathcal{A} = \{\underline{e}_i \oplus \underline{e}'_j, i = 1, ..., s, j = 1, ..., t\} \subset \mathbb{N}^d$, $\underline{e}_1, ..., \underline{e}_s$ the unit vectors in $\mathbb{N}^s$, $\underline{e}'_1, ..., \underline{e}'_t$ the unit vectors in $\mathbb{N}^t$, $d = s + t$, is called the transportation problem and it is studied "via" the projective toric variety $\mathcal{V}(I_{\mathcal{A}})$, called the Segre embedding of $\mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$ into $\mathbb{P}^{st-1}$, where $\mathbb{P}^n$ is the projective space of dimension n over a field of characteristic zero.
In our paper we study the integer programming problem associated with the set $\mathcal{A}' = \{\underline{e}_i \oplus (\underline{e}'_j + \underline{e}'_l)\}$. We call it again the transportation problem and it is studied "via" the projective toric variety $\mathcal{V}(I_{\mathcal{A}'})$, called the Segre embedding of $\mathbb{P}^{s-1} \times W^{(2)}$ into $\mathbb{P}^{s\binom{t}{2}}$, where $W^{(2)}$ is the $2-$square free Veronese variety of $\mathbb{P}^{\binom{t}{2}}$.
We utilize methods from semigroups theory and results obtained in Ref. 3, where the Segre product of two $2-$Veronese square-free subrings are considered.
More precisely in N.1 we consider the map π of semigroup and we give the rule to obtain

special fibers that we will utilize for applications.
In N.2 we look at the equivalence between the affine semigroup theory contained in $\mathbb{N}^d$ and the semigroup ring theory, contained in $K[X_1, ..., X_n]$, the polynomial ring over a field K in the $X_1, ..., X_n$ variables. We obtain in this way the moves of π.
In N.3, we consider the possible transportation plans to ship a number of units of an indivisible good from s factories $F_1, ..., F_s$ to a $\binom{t}{2}$ blocks of a 2 stores $S_1, ..., S_t$ and we apply the model obtained in N.2 to study all possible transportation plans.

1. Main Results

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}$ the set of integer numbers. We consider $n, d \in \mathbb{Z}$ positive numbers. Let $\mathcal{A} = \{\underline{a}_1, ..., \underline{a}_n\}$ be a fixed subset of lattice vectors in $\mathbb{Z}^d$. We consider the homomorphism of semigroups:

$$\pi : \mathbb{N}^n \longrightarrow \mathbb{Z}^d$$

$$\underline{u} \longrightarrow u_1\underline{a}_1 + ... + u_n\underline{a}_n$$

where $\underline{u} = (u_1, ..., u_n)$, and the image of π is the semigroup generated by $\mathcal{A}$, $\mathbb{N}\mathcal{A} = \{\lambda_1\underline{a}_1 + ... + \lambda_n\underline{a}_n, \lambda_1, ..., \lambda_n \in \mathbb{N}\}$. The map π can be lifted to the algebras homomorphism:

$$\hat{\pi} : K[X_1, ..., X_n] \to K[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}] = K[\underline{t}^{\pm 1}]$$

$$X_i \to \underline{t}^{\underline{a}_i} = t_1^{a_{i1}} t_2^{a_{i2}} ... t_d^{a_{id}} \qquad i = 1, ..., n \qquad \underline{a}_i = (a_{i1}, a_{i2}, ..., a_{id}).$$

$Ker\hat{\pi}$ is denoted by $I_{\mathcal{A}}$ and it is called toric ideal of $\mathcal{A}$. Since $\mathrm{Im}\hat{\pi} \subset K[t^{\pm 1}]$, that is a domain, $I_{\mathcal{A}}$ is a prime ideal and $K[X_1, ..., X_n]/I_{\mathcal{A}} \cong K[\underline{t}^{\underline{a}_1}, ..., \underline{t}^{\underline{a}_n}]$. Then the affine variety $V(I_{\mathcal{A}})$ in K^n is irreducible and it is called affine toric variety.

Definition 1.1. Let t be a positive integer number, $t \geq 4$. We define Square-free $2-$Veronese matrix any $\binom{t}{2} \times t$ matrix

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\
2 & 0 & 2 & 0 & 0 & \dots & \dots & \dots & 0 \\
3 & 0 & 0 & 3 & 0 & \dots & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
t-1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & t-1 \\
0 & t & t & 0 & \dots & \dots & \dots & 0 & 0 \\
0 & t+1 & 0 & t+1 & 0 & \dots & \dots & \dots & 0 \\
0 & . & . & . & . & . & . & \dots & \dots \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 2t-3 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & \binom{t}{2} & \binom{t}{2}
\end{pmatrix}$$

with the following properties:

1. In every column, there are exactly $t - 1$ entries different from zero.
2. In every row, there are only two entries different from zero.
3. The rule of the first block of $t - 1$ rows and of the second block of $t - 2$ rows will continue until the matrix is completed.

In the same way we can define a $t \times \binom{t}{2}$ matrix transposed of the $\binom{t}{2} \times t$ matrix.

The matrix defined above, has been introduced in Ref. 3 by the author.

In the following we will always use the lexicographic order on the vectors of $\mathbb{N}^s \oplus \mathbb{N}^t$. Let $e_1, ..., e_s$ be the canonical vectors of $\mathbb{N}^s$ and $e'_1, ..., e'_t$ the canonical vectors of $\mathbb{N}^t$. By now we consider the map π with $n = s\binom{t}{2}$ and $d = s + t$, where $s > 1$.

Theorem 1.1. *Let*

$$\pi : \mathbb{N}^{s\times\binom{t}{2}} \to \mathbb{N}^s \oplus \mathbb{N}^t$$

be the map of semigroups correspondent to the configuration of $\mathbb{N}^{s+t}$

$$\mathcal{A} = \{e_i \oplus (e'_k + e'_l), 1 \leq i \leq s, 1 \leq k < l \leq t\}.$$

Then, by identifying $\mathbb{N}^{s\times\binom{t}{2}}$ *with the semigroup of* $s \times \binom{t}{2}$ *matrices* (u_{ij})*, we have:*

$$\pi(u_{ij}) = \left(r_1, r_2, ..., r_s; \sum_j C_j^1, ..., \sum_j C_j^t \right)$$

where $r_i = \sum_{j=1}^{\binom{t}{2}} u_{ij}$*,* $1 \leq i \leq s$*, and in every summation* $\sum_j C_j^l$ *there are only* $t-1$ *elements different from zero,* j *takes the value of the* l*-th column of the square-free Segre* $2-$*Veronese* $\binom{t}{2} \times t$ *matrix and, for every* l*,* C_j^l *is the sum of the elements of the* $j-$*th column of the matrix* (u_{ij})*.*

Proof.

Each lattice vector of $\mathbb{N}^{s+t}$ has length $s+t$. The lattice vectors in the lexicographic order are:

$\{v_{1,j}\}_{j=1,...,\binom{t}{2}}$ that have 1 as first coordinate in the first s components and two coordinates equal to 1 in the last t components:

$$v_{1,1} = (1, 0, 0, ..., 0; 1, 1, 0, ..., 0) = z_{1,12}$$

$$v_{1,2} = (1, 0, 0, ..., 0; 1, 0, 1, 0, ..., 0) = z_{1,13}$$

$$\cdots\cdots\cdots\cdots$$

$$v_{1,\binom{t}{2}} = (1, 0, 0, ..., 0; 0, ..., 0, 1, 1) = z_{1,(t-1)t}.$$

$\{v_{2,j}\}_{j=1,...,\binom{t}{2}}$ that have 1 as second coordinate in the first s components and two coordinates equal to 1 in the last t components:

$$v_{2,1} = (0, 1, 0, 0, ..., 0; 1, 1, 0, ..., 0) = z_{2,12}$$

$$v_{2,2} = (0, 1, 0, ..., 0; 1, 0, 1, ..., 0) = z_{2,13}$$

$$\cdots\cdots\cdots\cdots$$

$$v_{2,\binom{t}{2}} = (0, 1, 0, ..., 0; 0, ..., 0, 1, 1) = z_{2,(t-1)t}$$

and so on we obtain

$$v_{s,1} = (0, ..., 0, 1; 1, 1, 0, ..., 0) = z_{s,12}$$

$$v_{s,2} = (0, ..., 0, 1; 1, 0, 1, ..., 0) = z_{s,13}$$

$$\dots\dots\dots\dots$$

$$v_{s,\binom{t}{2}} = (0, ..., 0, 1; 0, ..., 0, 1, 1) = z_{s,(t-1)t}$$

We have that

$$\pi(u_{ij}) = \sum_{i=1}^{s} \sum_{j=1}^{\binom{t}{2}} u_{ij} v_{i,j} = (w_1, ..., w_s; w'_1, ..., w'_t), \qquad u_{i,j} \in \mathbb{N}$$

Computation of $w_1, ..., w_s$: w_1 is obtained by all vectors $v_{i,j}$ that have 1 as first component, hence by the first group $v_{1,1}, v_{1,2}, ..., v_{1,\binom{t}{2}}$. Then in correspondence we obtain the sum

$$u_{11} + u_{12} + ... + u_{1\binom{t}{2}} = \sum_{j=1}^{\binom{t}{2}} u_{1j} =$$

$$= r_1 = \text{sum of the elements of the first row of the matrix } (u_{ij}),$$

and so on, w_s is obtained by all vectors that have 1 as $s-$component,hence by the $s-$group $v_{s,1}, v_{s,2}, ..., v_{s,\binom{t}{2}}$. Computation of $w'_1, ..., w'_t$: w'_1 is obtained by all vectors that have 1 as $s+$ 1-component, hence by the vectors $v_{1,1}, v_{1,2}, ..., v_{1,t-1}; v_{2,1}, v_{2,2}, ..., v_{2,t-1}; ...; v_{s,1}, v_{s,2}, ..., v_{s,t-1}$. Then in correspondence we obtain the sum

$$u_{11} + u_{12} + ... + u_{1(t-1)} + u_{21} + u_{22} + ... + u_{2(t-1)} + ... + u_{s1} + u_{s2} + ... + u_{s(t-1)} =$$

$$= (u_{11} + u_{21} + ... + u_{s1}) + (u_{12} + u_{22} + ... + u_{s2}) + ... + (u_{1(t-1)} + u_{2(t-1)} + ... + u_{s(t-1)}) =$$

$$= C_1^1 + ... + C_{t-1}^1 \qquad \square$$

Example 1.1. $s = 4$ and $t = 4$. We have the square-free Segre $2-$Veronese matrix $\binom{t}{2} \times t = 6 \times 4$:

$$\begin{pmatrix} 1\,1\,0\,0 \\ 2\,0\,2\,0 \\ 3\,0\,0\,3 \\ 0\,4\,4\,0 \\ 0\,5\,0\,5 \\ 0\,0\,6\,6 \end{pmatrix}$$

and

$$\pi(u_{ij}) = (r_1, r_2, r_3, r_4; C_1^1 + C_2^1 + C_3^1, C_1^2 + C_4^2 + C_5^2, C_2^3 + C_4^3 + C_6^3, C_3^4 + C_5^4 + C_6^4)$$

where

$$(u_{ij}) \in N^{4\times 6}, \qquad r_i = \sum_{j=1}^{6} u_{ij} \qquad\qquad C_j^l = \sum_{i=1}^{4} u_{ij},$$

$$j = 1, ..., 6, \qquad l = 1, ..., 4.$$

2. Moves

Now we consider $R = K[X_1, ..., X_s]$ and the square-free 2−Veronese ring $S^{(2)} = K[Y_1Y_2, Y_1Y_3, ..., Y_{t-1}Y_t]$, subring of $S = K[Y_1, ..., Y_t]$, generated by all square-free monomials of degree two in the variables $Y_1, ..., Y_t$. Then we have the Segre product

$$R *_s S^{(2)} = K[X_1Y_1Y_2, X_1Y_1Y_3, ..., X_1Y_{t-1}Y_t, X_2Y_1Y_2, X_2Y_1Y_3, ...,$$
$$X_2Y_{t-1}Y_t, X_sY_1Y_2, X_sY_1Y_3, ..., X_sY_{t-1}Y_t].$$

Let J be the toric ideal of $S^{(2)}$ and let $W^{(2)} \subset \mathbb{P}^{\binom{t}{2}-1}$ be the toric algebraic variety defined by J, where $\mathbb{P}^{\binom{t}{2}-1}$ is the projective space on an algebraic closed field K.
Then the algebraic variety defined by the toric ideal of $R *_s S^{(2)}$ is contained in $\mathbb{P}^{N-1}$, where $N = s\binom{t}{2}$.

Definition 2.1. We call

$$\mathbb{P}^{s-1} *_s W^{(2)} \hookrightarrow \mathbb{P}^{N-1}$$

the Segre immersion in $\mathbb{P}^{N-1}$ of $\mathbb{P}^{s-1}$ and of the toric variety $W^{(2)}$.

Definition 2.2. We call fiber of the immersion

$$\mathbb{P}^{s-1} *_s W^{(2)} \hookrightarrow \mathbb{P}^{N-1}$$

a fiber of the homomorphism of rings

$$\hat{\pi} : K[U_1, ..., U_N] \rightarrow$$

$$\rightarrow K[X_1Y_1Y_2, X_1Y_1Y_3, ..., X_1Y_{t-1}Y_t, ..., X_sY_1Y_2, X_sY_1Y_3, ..., X_sY_{t-1}Y_t]$$

Theorem 2.1. *$\hat{\pi}$ is the lifting of the homomorphism of semigroups*

$$\pi : \mathbb{N}^{s\times\binom{t}{2}} \rightarrow \mathbb{N}^s \oplus \mathbb{N}^t$$

for the configuration $\mathcal{A} = \{e_j \oplus (e'_k + e'_l), 1 \leq j \leq s, 1 \leq k < l \leq t\}$ of $\mathbb{N}^s \oplus \mathbb{N}^t$.

In the following, *sort*$(\cdot)$ will be the operator that takes any string of non-negative integers to a weakly increasing string.

Theorem 2.2. *Let $A = K[X_1, ..., X_n]$ be a polynomial ring, let $A^{(2)}$ be the 2− Veronese square-free subring of A.*
Then $A^{(2)} = K[T_{ij}]/I$ $1 \leq i < j \leq n$, where I is generated by the set of binomials

$$T_{i_1i_2}T_{j_1j_2} - T_{k_1k_3}T_{k_2k_4}$$

with $sort(i_1j_1i_2j_2) = k_1k_2k_3k_4$,

Proof. See Ref. 4, Remark 14.1. □

With the notations of theorem2.2, if we order lexicographically the variables T_{ij}, we have:

$$K[T_{ij}] = K[T_1, ..., T_{\binom{n}{2}}]$$

under the ring homomorphism:

$$T_1 \to T_{11},\ T_2 \to T_{12},\ ...,\ T_{\binom{n}{2}} \to T_{(n-1)n}$$

and $T_1 < T_2 < ... < T_{\binom{n}{2}}$ in the lexicographic order.

Let $A = K[X_1, ..., X_s]$ and $B = K[Y_1, ..., Y_t]$ be polynomial rings and let $C = K[\{Z_{ij}, 1 \leq i \leq s, i \leq j \leq t\}]$ be the polynomial ring in st variables.
Given a homogeneous polynomial $f = \sum_{i_1 \leq i_2 \leq ... \leq i_k} a_{i_1...i_k} X_{i_1} ... X_{i_k}$ of A and given any sequence of numbers $1 \leq j_1 \leq j_2 \leq ... \leq j_k \leq t$, we can define the homogeneous polynomial

$$f_{j_1,...,j_k} = \sum a_{i_1...i_k} Z_{i_1 j_1} ... Z_{i_k j_k}$$

in C.
In the same way, for any homogeneous polynomial
$g = \sum_{j_1 \leq j_2 \leq ... \leq j_k} b_{j_1...j_k} Y_{j_1} ... Y_{j_k}$ and for any sequence of numbers $1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq s$, we define

$$g_{i_1,...,i_k} = \sum b_{j_1...j_k} Z_{i_1 j_1} ... Z_{i_k j_k}.$$

Moreover, we consider the lexicographic order.

Theorem 2.3. *Let $A = K[X_1, ..., X_s]$ and $B = K[Y_1, ..., Y_t]$ two polynomial rings and let $B^{(2)} = K\left[V_1, ..., V_{\binom{t}{2}}\right]/J$ be the $2-$ Veronese square-free subring of B.*
Then the Segre product $A * B^{(2)}$ is $C = K[\{Z_{ij}, 1 \leq i \leq s, i \leq j \leq t\}]/L$, $K[Z_{ij}]$ the polynomial ring in $s \times \binom{t}{2}$ variables, where L is generated by the following set of binomials:
1. $\{g^{(j)}_{i_1,i_2}\}, j = 1, ..., q$ *where* $J = (g^{(1)}, ..., g^{(q)}) \subset B^{(2)}$, $g^{(j)} = V_{j_1} V_{j_2} - V_{j_3} V_{j_4}$
2. $Z_{ij} Z_{kl} - Z_{il} Z_{kj}, 1 \leq i \leq s, i \leq j, l \leq t.$

Proof. See Ref. 1. □

The moves(generators of $\operatorname{Ker} \pi$) can be read on the generators of the toric ideal arising from the map $\hat{\pi}$. In theorem 1.7 we translate results of the theory of $K-$algebras to the theory of semigroup rings.

Theorem 2.4. *Let $\pi : \mathbb{N}^{s \times \binom{t}{2}} \to \mathbb{N}^s \oplus \mathbb{N}^t$. Then $\operatorname{Ker} \pi$ is generated by matrices $s \times \binom{t}{2}$ of the following type:*

1. *Only one row is non-zero and it contains two entries equal to 1 and two entries equal to -1, either only 2 rows are non-zero and each one contains 1 and -1.*
2. *Only a minor of order two is different from zero and it is of the form*

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Proof. With the notations as in Theorem 2.3, let

$$\pi_2 : \mathbb{N}^{s \times \binom{t}{2}} \to \mathbb{N}^s \oplus \mathbb{N}^t$$

be the map of semigroup rings. Then we have the following facts:

1. Consider a relation $g^{(j)}_{i_1,i_2} = Z_{i_1j_1}Z_{i_2j_2} - Z_{i_1j_3}Z_{i_2j_4}$ that comes from the relation $V_{j_1}V_{j_2} - V_{j_2}V_{j_3}$.

   The correspondent movie has the elements 1 and -1 on the same i_1-th-row (they come from $Z_{i_1j_1}$ and $Z_{i_1j_3}$) and the elements 1 and -1 on the same i_2-th-row(they come from $Z_{i_2j_2}$ and $Z_{i_2j_4}$).

2. Consider a relation $Z_{ij}Z_{kl} - Z_{il}Z_{kj}, 1 \leq i,k \leq s, 1 \leq j,l \leq t$.
   The correspondent movie contains a minor of order *two* of the form

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad or \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

If $i < k$, we find 1 in the $i-$th row and in the $j-$th column (by Z_{ij}) and -1 in the $l-$th row and in the $j-$h column (by Z_{il}). Moreover we find 1 in the $k-$th row and in the $l-$th column (by Z_{kl}) and -1 in the $k-$th row and in $j-$th column and so we have the first minor. By reasoning in the same way, we obtain the second minor. □

Example 2.1. $s = t = 4$, the monomials

$$Y_1Y_2 \qquad Y_1Y_3 \qquad Y_1Y_4 \qquad Y_2Y_3 \qquad Y_2Y_4 \qquad Y_3Y_4$$

are the generators of square-free 2− Veronese ring $K[Y_1Y_2, ..., Y_3Y_4]$.
We write the generators in the lexicographic order:

$$\begin{array}{l} X_1Y_1Y_2,\ X_1Y_1Y_3,\ X_1Y_1Y_4,\ X_1Y_2Y_3,\ X_1Y_2Y_4,\ X_1Y_3Y_4 \\ X_2Y_1Y_2,\ X_2Y_1Y_3,\ X_2Y_1Y_4,\ X_2Y_2Y_3,\ X_2Y_2Y_4,\ X_2Y_3Y_4 \\ X_3Y_1Y_2,\ X_3Y_1Y_3,\ X_3Y_1Y_4,\ X_3Y_2Y_3,\ X_3Y_2Y_4,\ X_3Y_3Y_4 \\ X_4Y_1Y_2,\ X_4Y_1Y_3,\ X_4Y_1Y_4,\ X_4Y_2Y_3,\ X_4Y_2Y_4,\ X_4Y_3Y_4 \end{array}$$

and consider

$$\hat{\pi}_2 : K[U_1, U_2, ..., U_{24}] \to K[X_1Y_1Y_2, ..., X_4Y_3Y_4]$$

$$U_1 \to X_1Y_1Y_2$$

$$........$$

$$U_{24} \to X_4Y_3Y_4.$$

Two generators of the toric ideal are $g_1 = U_1U_6 - U_2U_5$ and $g_2 = U_7U_{12} - U_8U_{11}$.
The binomial $U_1U_6 - U_2U_5$ corresponds to the element

$$(1, 0, 0, 0, 0, 1, 0, ..., 0_{24}) - (0, 1, 0, 0, 1, 0, 0, ..., 0_{24}) = (1, -1, 0, 0, -1, 1, 0, ..., 0_{24})$$

that, as a matrix $(u_{ij}) \in \mathbb{N}^4 \times \mathbb{N}^6 = \mathbb{N}^4 \times \mathbb{N}^{\binom{4}{2}}$, is the following:

$$u_{ij} = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} r_1 = 1 - 1 - 1 + 1 = 0, \\ r_2 = 0 \\ r_3 = 0 \\ r_4 = 0. \end{cases} \qquad \begin{cases} c_1 = C_1^1 + C_2^1 + C_3^1 = 1 - 1 + 0 = 0, \\ c_2 = C_1^2 + C_4^2 + C_5^2 = 1 + 0 - 1 = 0 \\ c_3 = C_2^3 + C_4^3 + C_6^3 = -1 + 0 + 1 = 0 \\ c_4 = C_3^4 + C_5^4 + C_6^4 = 0 - 1 + 1 = 0. \end{cases}$$

For the binomial $U_1U_8 - U_2U_7$ we have the matrix

$$u_{ij} = \begin{pmatrix} 1 & -1 & 0\,0\,0\,0 \\ -1 & 1 & 0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0 \end{pmatrix}$$

$$\begin{cases} r_1 = 1 - 1 = 0 \\ r_2 = -1 + 1 = 0, \\ r_3 = 0 \\ r_4 = 0. \end{cases} \qquad \begin{cases} c_1 = C_1^1 + C_2^1 + C_3^1 = 1 - 1 - 1 + 1 = 0, \\ c_2 = C_1^2 + C_4^2 + C_5^2 = 1 - 1 - 1 + 1 = 0 \\ c_3 = C_2^3 + C_4^3 + C_6^3 = -1 + 1 = 0 \\ c_4 = C_3^4 + C_5^4 + C_6^4 = 0 - 1 + 1 = 0. \end{cases}$$

The remaining generators of the toric ideal will give all moves.

3. Application to transportation problem

We want to consider an application of the results obtained in the previous section. We recall the classical problem of manufacturing:
consider s factories $F_1, ..., F_s$ which produce a supply respectively of $f_1, ..., f_s$ units of an indivisible good.
Consider also t stores $S_1, S_2, ..., S_t$ with demands respectively of $s_1, ..., s_t$ units and such that $\sum_{i=1}^{s} f_i = \sum_{i=1}^{t} s_i \qquad (1).$
All exchanges $f_1s_1, ..., f_1s_t, f_2s_1, ..., f_ss_t$ can produce a $s \times t-$ matrix (u_{ij}), where each entry represents the exchange f_is_j.
This matrix is called a transportation plan. It's very special, because all produced good must satisfy all demands, hence the entries must reflect the link (1).
This problem was studied in Ref. 4 by using algebraic models that come from theory of affine semigroups, semigroup rings and in particular it can be visualized by the Segre variety in the projective space $\mathbb{P}^{s+t-1}$ (Ref. 4).
By using Segre Product of a polynomial ring and of a 2−Veronese square-free variety, we can study the following problem:
we have s factories $F_1, ..., F_s$ and t stores $S_1, ..., S_t$ such that (1) is satisfied.
In the new direction the exchanges are done among the s factories $F_1, ..., F_s$ and the $\binom{t}{2}$ blocks S_iS_j, $1 \leq i < j \leq t$.
The model can answer to a business plane which provides for exchanges among single factories and groups of two stores in all possible ways.
The motivations can depend from requirement to balance a possible loss, or, on the contrary, to have better economic conditions. For our aim we utilize the Segre product between the polynomial ring with as many variables as many factories and the 2−square-free Veronese subring of the polynomial ring with as many variables as many stores. It is possible also that we have all groups of two factories and they exchange the good with

some groups of two stores (not all).
The last problem is not easy, since it involves Segre products of $K-$algebras that are not square-free Veronese subrings, but their subrings.
Let $(\underline{r};\underline{c}) \in \mathbb{N}\mathcal{A}$. The fiber $\pi^{-1}(\underline{r};\underline{c})$ is a set of $s \times \binom{t}{2}$ matrices such that r_i is the sum of the i-th row and $c_j = \sum_{j=1}^{t} C_l^j$.
The elements of this fiber are all possible transportation plans of a fixed exchange among factories and groups of two stores.

Example 3.1. We consider two factories F_1 and F_2 and three stores S_1, S_2, S_3. The block of 2−stores are S_1S_2, S_1S_3, S_2S_3. The units of indivisible goods that F_1 and F_2 appear, are respectively 62 and 10.
The demands of S_1, S_2, S_3 are respectively $21, 6, 9$. A possible transportation plan is

$$\begin{pmatrix} F_1S_1S_2 & F_1S_1S_3 & F_1S_2S_3 \\ F_2S_1S_2 & F_2S_1S_3 & F_2S_2S_3 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 26 & 11 \\ 2 & 4 & 4 \end{pmatrix}$$

where $F_1S_iS_j$ and $F_2S_iS_j$ are the goods that F_1 and F_2 gives to the block S_iS_j:

	S_1	S_2	S_3
F_1	20	5	6
F_2	1	1	3

The 2×3 matrix (u_{ij}) is an element of the fiber $\pi^{-1}(62, 10; 57, 42, 45)$. Infact

$$R_1 = 25 + 26 + 11 = 62$$

$$R_2 = 2 + 4 + 4 = 10$$

$$C_1 = 27, C_2 = 30, C_3 = 15$$

$$C_1 + C_2 = 27 + 30 = 57$$

$$C_1 + C_3 = 27 + 15 = 42$$

$$C_2 + C_3 = 30 + 15 = 45.$$

In this transportation plan F_1 gives $20, 5, 6$ to S_1, S_2, S_3 and F_2 gives $1, 1, 3$ to S_1, S_2, S_3. Then F_1 and F_2 together satisfy the demand $21 = 20 + 1$ of S_1, the demand $9 = 6 + 3$ of S_3,

$$F_1 + F_2 = 62 + 10 = 72$$

$$F_1S_1S_2 + F_1S_1S_3 + F_1S_2S_3+$$

$$+F_2S_1S_2 + F_2S_1S_3 + F_2S_2S_3 = 25 + 26 + 11 + 2 + 4 + 4 = 72.$$

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