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Hybridization of Galerkin and Petrov–Galerkin Mixed Finite Element Methods for 2nd Order Elliptic Problems

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Abstract.

Starting from the characterization of the static condensation procedure for mixed hybridized methods introduced in Refs. 6 and 7, we use Helmholtz decompositions to obtain simpler local mapping problems that can be inverted in a more accurate way. Moreover, we extend the variational characterization of static condensation to more general saddle–point formulations.

Keywords: Mixed finite elements; Hybrid finite elements; Petrov-Galerkin formulations.

1. Introduction

We consider the following elliptic model problem written in mixed form on the polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary Γ:
Given $f \in L^2(\Omega)$, find $(\boldsymbol{q}, u) \in (H(\mathrm{div}\,;\Omega) \times L^2(\Omega))$, such that

$$(1) \qquad \begin{cases} \boldsymbol{q} = -\kappa \nabla u, \quad \mathrm{div}\, \boldsymbol{q} + du = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where $\kappa(\mathbf{x})$ is a bounded diffusion coefficient, such that $\kappa(\mathbf{x}) \geq \kappa_0 > 0$ almost everywhere in Ω, and $d(\mathbf{x})$ is a nonnegative bounded reaction coefficient. Homogeneous Dirichlet boundary conditions are considered only for ease of presentation. In the numerical discretization of problem (1), we seek a simultaneous approximation of both primal and dual variables; this approach gives rise to a nondefinite linear algebraic system of large size the solution of which is computationally expensive. The so-called "hybridization" procedure introduces an additional field to relax the interelement continuity requirement for the normal component of the vector field (see Ref. 1), in such a way that static elimination of primal and dual variables can be performed, yielding a much smaller matrix equation in the sole hybrid field. The resulting method is called Dual–Mixed Hybridized (DMH), and hybridization is traditionally carried out via an algebraic manipulation of

the full linear system acting on the primal, dual and hybrid variables. However, such a manipulation may become rather involved and dissuade from its actual use in computer implementation. In Refs. 6 and 7, a novel characterization of the static condensation is proposed, showing that the reduced system may be derived from the solution of suitable local mapping problems. This interpretation provides a variational framework to the hybridized formulation and an explicit expression of the entries of the condensed local coefficient matrix and right-hand side. Based on this variational procedure, in the first part of the paper we propose to use a Helmholtz decomposition of the local finite element space for the flux variable, in order to decompose the elementwise mappings into a set of simpler problems, that yield local matrices easier to be inverted accurately.
In the second part of the article, we discuss an extension of the variational characterization of the static condensation to more general saddle–point formulations. As an example, we focus our attention on the Discontinuous Petrov–Galerkin method of lowest order (DPG_0) of Refs. 2 and 4. As already done for the DMH formulation, we use a Helmholtz decomposition technique to identify a sub-structure in the local mapping problems.
The paper is organized as follows. In Sec. 2 we introduce the notation; in Sec. 3 we recall the DMH formulation of (1); in Sec. 4 we review the steps of the abstract variational characterization given in Ref. 6; in Sec. 5 we introduce the orthogonal decomposition of the flux discrete space; in Sec. 6 we provide a variational characterization to the static condensation of the DPG_0 formulation; eventually, in Sec. 7 we draw the conclusions.

2. Mathematical Preliminaries

We let $\overline{\Omega} = \bigcup \overline{K}$ be a regular partition $\mathcal{T}_h$ of the domain Ω into triangular elements K of area $|K|$, diameter h_K and barycenter $\mathbf{x}_{CG} = (x_{CG}, y_{CG})^T$. For each $K \in \mathcal{T}_h$, we denote by ∂K and $\boldsymbol{n}_{\partial K}$ the boundary of the element and its outward unit normal vector (according to a counterclockwise orientation along ∂K), respectively. If v is any function defined in Ω, we denote by v^K its restriction to K and by $v_{\partial K}$ its restriction on ∂K. We denote by $\mathcal{E}_h$ the set of all the edges of $\mathcal{T}_h$. The set of the internal edges of $\mathcal{E}_h$ is denoted by $\mathcal{E}_{h,i}$. For each edge $e \in \partial K$, we indicate by $|e|$ the length of e and by $\boldsymbol{n}_{\partial K}|_e$ the restriction of $\boldsymbol{n}_{\partial K}$ on $e \in \mathcal{E}_{h,i}$ and by $\boldsymbol{n}_{\Gamma}|_e$ the restriction of $\boldsymbol{n}_{\partial K}$ on $e \in \Gamma$.

For $k \geq 0$, we let $\mathbb{P}_k(K)$ to be the space of polynomials in two variables of total degree at most k on K and by $R_k(\partial K)$ the space of polynomials in two variables of total degree at most k on each edge of ∂K. Furthermore, we denote by $\mathbb{RT}_k(K)$ the k-order Raviart–Thomas finite element space of Ref. 12 defined as

$$\mathbb{RT}_k(K) = (\mathbb{P}_k(K))^2 \oplus \mathcal{P}_k(K)\,\mathbf{x}, \tag{1}$$

where $\mathbf{x} = (x,\, y)^T$ and where $\mathcal{P}_k(K) = \mathrm{span}\{x^\alpha y^\beta, \alpha + \beta = k\}$. We also introduce the nonconforming Crouzeix–Raviart finite element space (see Ref. 8)

$$\mathbb{P}_1^{nc}(K) = \mathrm{span}\,\{\widetilde{\varphi}_i\}_{i=1}^3\,, \qquad \widetilde{\varphi}_i(\mathbf{x}_{m,j}) = \delta_{ij}, \quad i,j = 1,2,3 \quad \forall K \in \mathcal{T}_h \tag{2}$$

where each basis function $\widetilde{\varphi}_i \in \mathbb{P}_1(K)$ and $\mathbf{x}_{m,j}$ is the midpoint of each edge $e_j \in \partial K$. We denote by $\mathbb{P}_1^{nc}(\mathcal{T}_h)$ the space of functions whose restriction to each triangle K of $\mathcal{T}_h$ belongs to $\mathbb{P}_1^{nc}(K)$. Finally, for any set $S \subseteq \Omega$, we denote by $\mathcal{P}_S^k\,\eta$ the L^2 projection of a function $\eta \in L^2(S)$ onto $\mathbb{P}_k(S)$.

3. The Dual–Mixed Hybridized Method

For each $K \in \mathcal{T}_h$, we define the local finite element spaces

(1) $$V_h(K) = \mathbb{RT}_k(K), \quad W_h(K) = \mathbb{P}_k(K), \quad L_h(\partial K) = R_k(\partial K).$$

The corresponding global finite element spaces are

(2) $$V_h = \prod_{K \in \mathcal{T}_h} V_h(K), \quad W_h = \prod_{K \in \mathcal{T}_h} W_h(K), \quad L_h = \prod_{K \in \mathcal{T}_h} L_h(\partial K),$$

with the constraint $\lambda_{\partial K_1}|_e = \lambda_{\partial K_2}|_e \, \forall e \in \mathcal{E}_{h,i}$ and $\lambda_{\partial K_1}|_e = 0 \, \forall e \in \Gamma$, $\lambda \in L_h$. Then, the Dual–Mixed Hybridized formulation of (1), (see Ref. 1), reads: Find $(\boldsymbol{q}_h, u_h, \lambda_h) \in (V_h \times W_h \times L_h)$ such that for all $(\boldsymbol{v}_h, w_h, \eta_h) \in (V_h \times W_h \times L_h)$ we have

(3) $$\begin{aligned} &\int_\Omega \mathcal{K}\, \boldsymbol{q}_h \cdot \boldsymbol{v}_h \, dx - \sum_{K \in \mathcal{T}_h} \int_K u_h \operatorname{div} \boldsymbol{v}_h \, dx + \sum_{e \in \mathcal{E}_{h,i}} \int_e \lambda_h \, [\![\boldsymbol{v}_h]\!] \, ds = 0, \\ &- \sum_{K \in \mathcal{T}_h} \int_K w_h \operatorname{div} \boldsymbol{q}_h \, dx - \int_\Omega d\, w_h\, u_h \, dx = - \int_\Omega f\, w_h \, dx, \\ &\sum_{e \in \mathcal{E}_{h,i}} \int_e \eta_h \, [\![\boldsymbol{q}_h]\!] \, ds = 0, \end{aligned}$$

where we have set $\mathcal{K} := \kappa^{-1}$. Problem (3) admits a unique solution (see Ref. 3 for a proof).

4. Variational Characterization of the DMH Method

In this section, we recall the steps of the characterization of the mixed–hybridized method (3) given in Ref. 6.

4.1. *Generalized Displacement DMH Problem from Superposition of Effects*

The triple $(\boldsymbol{q}_h, u_h, \lambda_h)$, solution of (3), can be characterized as follows.
The pair $(\boldsymbol{q}_h, u_h) \in (V_h \times W_h)$ is given by

(1) $$(\boldsymbol{q}_h, u_h) = (\boldsymbol{q}_{\lambda_h}, u_{\lambda_h}) + (\boldsymbol{q}_f, u_f),$$

where the two members at the right–hand side are suitable lifting operators associated with λ_h and f, respectively.
The Lagrange multiplier $\lambda_h \in L_h$ is the unique solution of

(2) $$a_h(\lambda_h, z_h) = b_h(z_h) \qquad \forall z_h \in L_h,$$

where

(3) $$a_h(\lambda_h, z_h) = \int_\Omega \mathcal{K}\, \boldsymbol{q}_{\lambda_h} \cdot \boldsymbol{q}_{z_h} \, dx + \int_\Omega d\, u_{\lambda_h}\, u_{z_h} \, dx, \quad b_h(z_h) = \int_\Omega f\, u_{z_h} \, dx.$$

The variational problem (2) is a generalized displacement approximation of the model problem (1) and can be written in matrix form as

(4) $$\mathbb{E}\, \Lambda = \mathbb{H},$$

where $\mathbb{E}$ is the global, symmetric and positive definite stiffness matrix, Λ is the vector of the degrees of freedom of λ_h over $\mathcal{E}_{h,i}$ and $\mathbb{H}$ is the load vector.

4.2. *Local Mappings*

The pair $(\boldsymbol{q}_{\mathtt{m}}, u_{\mathtt{m}})$ is the local lifting of a given hybrid variable function $\mathtt{m} \in L_h(\partial K)$, such that $\mathtt{m} = 0$ for each edge $e = \partial K \cap \Gamma$, while $(\boldsymbol{q}_f, u_f)$ is the local mapping of a given source term $f \in L^2(K)$. Let

$$\begin{aligned} a^K(\boldsymbol{q},\, \boldsymbol{v}) &= \int_K \mathcal{K}\, \boldsymbol{q} \cdot \boldsymbol{v}\, dx \quad : (V_h(K) \times V_h(K)) \to \mathbb{R}, \\ b^K(u,\, \boldsymbol{v}) &= -\int_K u \operatorname{div} \boldsymbol{v}\, dx : (U_h(K) \times V_h(K)) \to \mathbb{R}, \\ c^K(u,\, w) &= \int_K d\, u\, w\, dx \quad : (U_h(K) \times U_h(K)) \to \mathbb{R}. \end{aligned}$$

The first local mapping reads: Given $\mathtt{m} \in L_h(\partial K)$, find $(\boldsymbol{q}_{\mathtt{m}},\, u_{\mathtt{m}}) \in (V_h(K) \times W_h(K))$ such that for all $K \in \mathcal{T}_h$

$$\begin{aligned} a^K(\boldsymbol{q}_{\mathtt{m}},\, \boldsymbol{v}_h) + b^K(u_{\mathtt{m}},\, \boldsymbol{v}_h) &= G^K_{\mathtt{m}}(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in V_h(K), \\ b^K(w_h,\, \boldsymbol{q}_{\mathtt{m}}) - c^K(w_h,\, u_{\mathtt{m}}) &= 0 && \forall w_h \in W_h(K), \end{aligned} \tag{5}$$

where we have introduced the local linear form

$$G^K_\xi(\boldsymbol{v}_h) = -\int_{\partial K} \xi\, \boldsymbol{v}_h \cdot \boldsymbol{n}_{\partial K}\, ds : V_h(K) \to \mathbb{R},$$

which is parametrically depending on the given function $\xi \in L_h(\partial K)$.

The second local mapping reads: Given $f \in L^2(K)$, find $(\boldsymbol{q}_f,\, u_f) \in (V_h(K) \times W_h(K))$ such that for all $K \in \mathcal{T}_h$

$$\begin{aligned} a^K(\boldsymbol{q}_f,\, \boldsymbol{v}_h) + b^K(u_f,\, \boldsymbol{v}_h) &= 0 && \forall \boldsymbol{v}_h \in V_h(K), \\ b^K(w_h,\, \boldsymbol{q}_f) - c^K(w_h,\, u_f) &= F^K_f(w_h) && \forall w_h \in W_h(K), \end{aligned} \tag{6}$$

where we have introduced the local linear form

$$F^K_\phi(w_h) = -\int_K \phi\, w_h\, dx : W_h(K) \to \mathbb{R},$$

which is parametrically depending on the given function $\phi \in L^2(K)$.

4.3. *Generalized Displacement Formulation*

Using superposition of effects in $(3)_3$, yields

$$\sum_{e \in \mathcal{E}_{h,i}} \int_e \eta_h\ [\![\boldsymbol{q}_{\lambda_h}]\!]\, ds = -\sum_{e \in \mathcal{E}_{h,i}} \int_e \eta_h\ [\![\boldsymbol{q}_f]\!]\, ds \qquad \forall \eta_h \in L_h. \tag{7}$$

Substituting into (7) the characterizations of $\boldsymbol{q}_{\lambda_h}$ and $\boldsymbol{q}_f$, and using Lemma 2.2 of Ref. 6, leads to the generalized displacement formulation (2), or, equivalently, to the algebraic form (4).

5. Orthogonal Decomposition as a Constructive Tool for Local Mappings

As the finite element degree k increases, the solution of the local problems (5) and (6) becomes involved. Helmholtz decomposition may be used as a way to obtain in the mappings easy and well conditioned local matrices. This allows for an easier and more stable implementation (see, *e.g.*, Ref. 13, Sec. 8.18), especially in view of p-type refinement or variable degree formulations (see *e.g.* Ref. 9).

5.1. *The Algebraic Subsystems*

For all $K \in \mathcal{T}_h$, and for a given $\sigma \in L^2(K)$, we introduce the affine manifold (see Ref. 11, Chpt. 7)

$$(\mathcal{Z}_h^K)^\sigma := \{\boldsymbol{v} \in V_h(K) \,|\, b^K(w,\, \boldsymbol{v}) = (\sigma,\, w)_K,\, \forall w \in W_h(K)\} = V_h^0(K),$$

where $(\cdot,\, \cdot)_K$ is the inner product in $L^2(K)$. The space $V_h^0(K)$ induces the following Helmholtz decomposition of $V_h(K)$

$$V_h(K) = V_h^0(K) \oplus (V_h^0(K))^\perp \qquad \forall K \in \mathcal{T}_h, \tag{1}$$

where

$$(V_h^0(K))^\perp = \left\{\boldsymbol{v} \in V_h(K) \,|\, a^K(\boldsymbol{v}, \boldsymbol{v}^0) = 0,\, \forall \boldsymbol{v}^0 \in V_h^0(K)\right\}. \tag{2}$$

The representation (1) of the space $V_h(K)$ into a solenoidal and a weakly irrotational part is unique. We have the following result (for the proof see Ref. 5).

Proposition 5.1. *For $d \geq 0$, we have*

$$\boldsymbol{q}_f = \boldsymbol{q}_f^\perp \in (V_h^0(K))^\perp. \tag{3}$$

If $d = 0$, then

$$\boldsymbol{q}_{\mathtt{m}} = \boldsymbol{q}_{\mathtt{m}}^0 \in V_h^0(K). \tag{4}$$

The above result suggests that the local problems (5) and (6) may be furtherly split into smaller subproblems. Let

$$\underbrace{\dim(V_h(K))}_{M_k} = \underbrace{\dim(V_h^0(K)}_{M_k^0} + \underbrace{\dim(V_h^0(K))^\perp}_{M_k^\perp},$$

and consider the local problem (5). In the case $d = 0$, using (1), relation $(5)_1$ can be written as

$$\begin{aligned} a^K(\boldsymbol{q}_{\mathtt{m}},\, \boldsymbol{v}_h^0) \qquad\qquad &= G_{\mathtt{m}}^K(\boldsymbol{v}_h^0) \;\; \forall \boldsymbol{v}_h^0 \in V_h^0(K), \\ b^K(u_{\mathtt{m}},\, \boldsymbol{v}_h^\perp) &= G_{\mathtt{m}}^K(\boldsymbol{v}_h^\perp) \;\; \forall \boldsymbol{v}_h^\perp \in (V_h^0(K))^\perp, \end{aligned} \tag{5}$$

which yields the block diagonal system

$$\begin{bmatrix} A^0 & 0_{(M_k^0, M_k^\perp)} \\ 0^T_{(M_k^0, M_k^\perp)} & A^\perp \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathtt{m}} \\ u_{\mathtt{m}} \end{bmatrix} = \begin{bmatrix} \mathtt{m}^0 \\ \mathtt{m}^\perp \end{bmatrix}, \tag{6}$$

where A^0 and $A^\perp$ are square matrices of size M_k^0 and $M_k^\perp$, respectively.
In the case $d > 0$, the unknown $\boldsymbol{q}_{\mathtt{m}}^0$ can be obtained from the square invertible system of size M_k^0

$$A^0 \boldsymbol{q}_{\mathtt{m}}^0 = \mathtt{m}^0,$$

whilst the (coupled) variables $\boldsymbol{q}_{\mathtt{m}}^\perp$ and $u_{\mathtt{m}}$ are the solution of system

$$\begin{bmatrix} A^\perp & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathtt{m}}^\perp \\ u_{\mathtt{m}} \end{bmatrix} = \begin{bmatrix} \mathtt{m}^\perp \\ 0_{(M_k^\perp, 1)} \end{bmatrix}.$$

Each matrix in the above system is square of size equal to $M_k^\perp$.
Let us now consider the local problem (6). Since $\boldsymbol{q}_f = \boldsymbol{q}_f^\perp$, problem (6) can be rewritten as

$$\begin{aligned} b^K(w_h, \boldsymbol{q}_f) - c^K(w_h, u_f) &= F_f^K(w_h) && \forall w_h \in W_h(K), \\ a^K(\boldsymbol{q}_f, \boldsymbol{v}_h^\perp) + b^K(u_f, \boldsymbol{v}_h^\perp) &= 0 && \forall \boldsymbol{v}_h^\perp \in (V_h^0(K))^\perp, \end{aligned}$$

or, equivalently, as the system

$$\begin{bmatrix} B^\perp & C \\ A^\perp & (B^\perp)^T \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_f \\ u_f \end{bmatrix} = \begin{bmatrix} F \\ 0_{(M_k^\perp,1)} \end{bmatrix}. \tag{7}$$

Each matrix in (7) is square of size equal to $M_k^\perp$. If $d = 0$, then $C = 0_{(M_k^\perp, M_k^\perp)}$ and system (7) is block lower triangular.

5.2. *Construction of the Basis for V_h^0 and $(V_h^0)^\perp$ based on Helmholtz decomposition*

In order to compute the solution of the systems in Sect. 5.1, we need a basis for functions belonging to $V_h^0(K)$ and $(V_h^0(K))^\perp$, respectively. The tool we use is the classical Helmholtz decomposition principle of the space $\mathbb{RT}_k(K)$ (see Ref. 5 for an alternative approach based on hierarchical splitting). We carry out the computations on the reference triangle $\widehat{K}$ with coordinate vector $\widehat{\mathbf{x}} = (\widehat{x}, \widehat{y})^T$, and then we use Piola's transformation to map $\widehat{K}$ into the generic element K of the triangulation (see Ref. 3, Sect. III.1.3). Moreover, throughout this section, we assume that the diffusion coefficient κ and the reaction term d are piecewise constant.
Let $\widehat{\phi} \in H^1(\widehat{K})$ and let $\mathbf{curl} : \widehat{\phi} \to \mathbf{curl}\,\widehat{\phi} = \left(\partial\widehat{\phi}/\partial\widehat{y},\, -\partial\widehat{\phi}/\partial\widehat{x}\right)^T$. Then, we have (see Ref. 3, Corollary 3.2)

$$V_h^0(\widehat{K}) = \mathbf{curl}\,\mathbb{P}_{k+1}(\widehat{K}). \tag{8}$$

The computation of the basis for $V_h^0(\widehat{K})$ can be done by using directly (8). The computation of the basis for the orthogonal complement $(V_h^0(\widehat{K}))^\perp$ requires solving the following linear algebraic system associated with (2)

$$\mathcal{V}^\perp \mathbf{x}^\perp = \mathbf{0}_{(M_k^0,1)}, \tag{9}$$

where $\mathcal{V}^\perp \in \mathbb{R}^{M_k^0 \times M_k^\perp}$ and $\mathbf{x}^\perp \in \mathbb{R}^{M_k^\perp}$. For given $k \geq 0$, the solution of (9) can be found once for all, for example, by computing an orthonormal basis for the null space of $\mathcal{V}^\perp$ via a singular value decomposition. We refer to Ref. 5 for examples of practical computations in the cases $k = 0, 1$.

6. The Discontinuous Petrov–Galerkin Method

For each $K \in \mathcal{T}_h$, we introduce the local trial finite element spaces

$$Q_h(K) = (\mathbb{P}_0(K))^2, \quad U_h(K) = \mathbb{P}_0(K), \quad L_h(\partial K) = M_h(\partial K) = R_0(\partial K),$$

and the local test spaces

$$V_h(K) = \mathbb{RT}_0(K), \quad W_h(K) = \mathbb{P}_1(K).$$

The global trial and test spaces $Q_h, U_h, L_h, M_h, V_h, W_h$ are obtained as product over the triangulation of the respective local spaces. Additionally, the space L_h satisfies the same continuity constraint as in the DMH formulation, while the space M_h satisfies the constraint of traction reciprocity $\mu_{\partial K_1}|_e + \mu_{\partial K_2}|_e = 0$, $\forall e \in \mathcal{E}_{h,i}$, $\mu \in M_h$. The DPG$_0$ finite element approximation of problem (1) reads:
Find $(\boldsymbol{q}_h, u_h, \lambda_h, \mu_h) \in (Q_h \times U_h \times L_h \times M_h)$ such that for all $(\boldsymbol{v}_h, w_h) \in (V_h \times W_h)$ we have

$$
\begin{aligned}
&\int_\Omega \mathcal{K}\, \boldsymbol{q}_h \cdot \boldsymbol{v}_h \, dx - \sum_{K\in\mathcal{T}_h} \int_K u_h \operatorname{div} \boldsymbol{v}_h \, dx + \sum_{e\in\mathcal{E}_{h,i}} \int_e \lambda_h \; [\![\boldsymbol{v}_h]\!] \, ds = 0, \\
&- \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{q}_h \cdot \nabla w_h \, dx + \int_\Omega d\, u_h \, w_h \, dx \\
&+ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h \, w_h \, ds = \int_\Omega f \, w_h \, dx.
\end{aligned} \tag{1}
$$

Problem (1) admits a unique solution, the proof being an extension of the one given in Ref. 4 in the case $d = 0$ (see Ref. 5 for details).

6.1. *Local Mappings*

We define the two pairs $(\boldsymbol{q}_{\mathtt{m}}, u_{\mathtt{m}}, \mu_{\mathtt{m}})$ and $(\boldsymbol{q}_f, u_f, \mu_f)$ as the solutions of local dual–primal mixed problems, which provide the lifting of the given data $\mathtt{m}$ on ∂K and f in K, respectively. Let

$$
\begin{aligned}
A^K(\boldsymbol{q}, \boldsymbol{v}) &= \int_K \mathcal{K}^K \boldsymbol{q} \cdot \boldsymbol{v} \, dx && : (Q_h(K) \times V_h(K)) \to \mathbb{R}, \\
C^K(u, w) &= \int_K d^K \, u \, w \, dx && : (U_h(K) \times W_h(K)) \to \mathbb{R}, \\
\widehat{B}_1^K((u,\lambda), \boldsymbol{v}) &= -\int_K u \operatorname{div} \boldsymbol{v} \, dx \\
&\quad + \int_{\partial K} \lambda \boldsymbol{v} \cdot \boldsymbol{n}_{\partial K} \, ds && : ((U_h(K) \times L_h(\partial K)) \times V_h(K)) \to \mathbb{R}, \\
\widehat{B}_2^K((\boldsymbol{q},\mu), w) &= -\int_K \boldsymbol{q} \cdot \nabla w \, dx \\
&\quad + \int_{\partial K} \mu \, w \, ds && : ((Q_h(K) \times M_h(\partial K)) \times W_h(K)) \to \mathbb{R}.
\end{aligned}
$$

Then, the first local mapping reads: Given $\mathtt{m} \in L_h(\partial K)$, find $(\boldsymbol{q}_{\mathtt{m}}, u_{\mathtt{m}}, \mu_{\mathtt{m}}) \in (Q_h(K) \times U_h(K) \times M_h(\partial K))$ such that for all $K \in \mathcal{T}_h$

$$
\begin{aligned}
&A^K(\boldsymbol{q}_{\mathtt{m}}, \boldsymbol{v}_h) + \widehat{B}_1^K((u_{\mathtt{m}}, 0), \boldsymbol{v}_h) = G_{\mathtt{m}}^K(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in V_h(K), \\
&\widehat{B}_2^K((\boldsymbol{q}_{\mathtt{m}}, \mu_{\mathtt{m}}), w_h) + C^K(u_{\mathtt{m}}, w_h) = 0 && \forall w_h \in W_h(K),
\end{aligned} \tag{2}
$$

where we have introduced the local linear form

$$
G_\xi^K(\boldsymbol{v}_h) = -\int_{\partial K} \xi \, \boldsymbol{v}_h \cdot \boldsymbol{n}_{\partial K} \, ds : V_h(K) \to \mathbb{R},
$$

which is parametrically depending on the given function $\xi \in L_h(\partial K)$. This lifting can be thought as the discretization of the local problem

$$
\begin{aligned}
&\boldsymbol{q}_{\mathtt{m}} = -\kappa \nabla u_{\mathtt{m}}, \quad \operatorname{div} \boldsymbol{q}_{\mathtt{m}} + d u_{\mathtt{m}} = 0 && \text{in } K, \\
&\mu_{\mathtt{m}} = \boldsymbol{q}_{\mathtt{m}} \cdot \boldsymbol{n}_{\partial K}, \quad u_{\mathtt{m}} = \mathtt{m} && \text{on } \partial K.
\end{aligned} \tag{3}
$$

The second local mapping reads: Given $f \in L^2(K)$, find $(\boldsymbol{q}_f,\, u_f,\, \mu_f) \in (Q_h(K) \times U_h(K) \times M_h(\partial K))$ such that for all $K \in \mathcal{T}_h$

$$
\begin{aligned}
&A^K(\boldsymbol{q}_f, \boldsymbol{v}_h) + \widehat{B}_1^K((u_f, 0),\, \boldsymbol{v}_h) = 0 && \forall \boldsymbol{v}_h \in V_h(K),\\
&\widehat{B}_2^K((\boldsymbol{q}_f, \mu_f),\, w_h) + C^K(u_f,\, w_h) = -F_f^K(w_h) && \forall w_h \in W_h(K),
\end{aligned}
\tag{4}
$$

where we have introduced the local linear form

$$F_\phi^K(w_h) = -\int_K \phi\, w_h\, dx : W_h(K) \to \mathbb{R},$$

which is parametrically depending on the given function $\phi \in L^2(K)$. This lifting can be thought as the discretization of the local problem

$$
\begin{aligned}
&\boldsymbol{q}_f = -\kappa \nabla u_f, \quad \operatorname{div} \boldsymbol{q}_f + d u_f = f && \text{in } K,\\
&\mu_f = \boldsymbol{q}_f \cdot \boldsymbol{n}_{\partial K}, \quad u_f = 0 && \text{on } \partial K.
\end{aligned}
\tag{5}
$$

6.2. *Generalized Displacement Formulation*

The following result holds (see Ref. 5 for the proof).

Lemma 6.1. *Let $(\boldsymbol{q}_h, u_h, \lambda_h, \mu_h)$ be the unique solution of problem (1). Then*

$$\boldsymbol{q}_h = \boldsymbol{q}_{\lambda_h} + \boldsymbol{q}_f, \qquad u_h = u_{\lambda_h} + u_f, \qquad \mu_h = \mu_{\lambda_h} + \mu_f, \tag{6}$$

and the Lagrange multiplier $\lambda_h \in L_h$ is the unique solution of

$$A_h(\lambda_h, \zeta_h) = F_h(\zeta_h) \qquad \forall \zeta_h \in L_h, \tag{7}$$

with

$$
\begin{aligned}
A_h(\lambda_h, \zeta_h) &= \sum_{K \in \mathcal{T}_h} (A^K(\boldsymbol{q}_{\lambda_h},\, \boldsymbol{q}_{\zeta_h}) + C^K(u_{\lambda_h}, w_{\zeta_h})),\\
F_h(\zeta_h) &= -\sum_{K \in \mathcal{T}_h} F_f^K(w_{\zeta_h}),
\end{aligned}
\tag{8}
$$

$w_{\zeta_h} \in \mathcal{V}_{h,0}$ being the nonconforming piecewise linear function such that $\mathcal{P}_e^0 w_{\zeta_h} = \zeta_h$ for all $e \in \mathcal{E}_h$, $\zeta_h \in L_h$.

6.3. *Orthogonal Decomposition as a Constructive Tool for Local Mappings*

Let us introduce the null space (see Ref. 10)

$$\mathcal{Z}_{1,h}^K = \ker \widehat{B}_1^K((u,\, 0),\, \boldsymbol{v}) = \{\boldsymbol{v} \in V_h(K) \,|\, \operatorname{div} \boldsymbol{v} = 0\} = (\mathcal{Z}_h^K)^0,$$

which induces the decomposition $V_h(K) = \mathcal{Z}_{1,h}^K \oplus (\mathcal{Z}_{1,h}^K)^\perp$, orthogonal with respect to the L^2 innner product induced by the bilinear form $A^K(\cdot,\cdot)$. Moreover, let us introduce the affine manifold

$$(\mathcal{Z}_{2,h}^K)^\sigma := \{\boldsymbol{q} \in Q_h(K),\, \mu \in M_h(\partial K) \,|\, \widehat{B}_2^K((\boldsymbol{q},\, \mu),\, w) = (\sigma,\, w)_K\, \forall w \in W_h(K)\}.$$

We have the following result (see Ref. 5 for the proof).

Lemma 6.2. *Let $\sigma_d := -d^K u_{\mathtt{m}}$. The local liftings $(\boldsymbol{q}_{\mathtt{m}}, \mu_{\mathtt{m}})$ and $(\boldsymbol{q}_f, \mu_f)$ are such that*

$$(\boldsymbol{q}_{\mathtt{m}}, \mu_{\mathtt{m}}) \in (\mathcal{Z}_{2,h}^K)^{\sigma_d}, \quad (\boldsymbol{q}_f, \mu_f) \in (\mathcal{Z}_{2,h}^K)^f.$$

The above results suggests again that smaller subproblems may be obtained from the local mappings (2) and (4). Eq. $(2)_1$ is rewritten as

$$(9) \qquad \begin{aligned} A^K(\boldsymbol{q}_{\mathtt{m}}, \boldsymbol{v}_h^0) &= G_{\mathtt{m}}^K(\boldsymbol{v}_h^0) \ \ \forall \boldsymbol{v}_h^0 \in V_h^0(K), \\ \widehat{B}_1^K(u_{\mathtt{m}}, \boldsymbol{v}_h^\perp) &= G_{\mathtt{m}}^K(\boldsymbol{v}_h^\perp) \ \ \forall \boldsymbol{v}_h^\perp \in (V_h^0(K))^\perp, \end{aligned}$$

which yields the block diagonal system

$$(10) \qquad \begin{bmatrix} A^0 & 0_{(2,1)} \\ 0_{(2,1)}^T & A^\perp \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathtt{m}} \\ u_{\mathtt{m}} \end{bmatrix} = \begin{bmatrix} \mathtt{m}^0 \\ \mathtt{m}^\perp \end{bmatrix},$$

where A^0 and $A^\perp$ are square matrices of size 2×2 and 1×1. Eventually, the approximate normal flux $\mu_{\mathtt{m}}$ can be recovered, via a post-processing like procedure, by solving a local 3×3 system, $\boldsymbol{q}_{\mathtt{m}}$ and $u_{\mathtt{m}}$ being already known.

7. Conclusions

Stemming from the characterization of the static condensation procedure of Refs. 6 and 7, we have used the Helmholtz decomposition to derive a simpler local mapping problems, in view of a possible p-type refinement or variable degree strategy. Moreover, we have extended the variational characterization to the DPG_0 scheme, which represents a more general saddle point formulation. Also in this latter case, Helmholtz decomposition has been used to yield simpler local subproblems.

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