

# SHOCK PROPAGATION IN A FLOW THROUGH DEFORMABLE POROUS MEDIA

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## Abstract.

We consider a one dimensional incompressible flow through a porous medium undergoing deformations such that the porosity and the hydraulic conductivity can be considered to be functions of the flux intensity. The medium is initially dry and we neglect capillarity, so that a sharp wetting front proceeds into the medium under the influence of a given pressure on the surface. We study the problem of the continuation in time of the solution in presence of singularities, under the assumption that the porosity is a non increasing function of the volumetric velocity. This assumption implies that the hyperbolic equation expressing the conservation law can degenerate.

*Keywords:* free boundary; porous media; shocks.

## 1. Introduction

In the recent papers Refs. [2](#), [3](#), [8](#) a generalization of the classical Green-Ampt model<sup>10</sup> for the penetration of a wetting front in a dry porous medium has been considered.

The intense research performed in the past decade about some non-standard phenomena occurring in the process of water injection for the oil extraction, in the coffee brewing process, and on the filter clogging has evidenced the presence of a mutual action between the flow and the porous medium (see e.g. Refs. [9](#), [11](#), [12](#); for survey papers also Refs. [6](#), [7](#)).

Such an action can be both of mechanical and of chemical origin. Leaving aside any chemical interaction (a series of experiments was performed at the laboratories of Illycaffè, Trieste, Italy, in the nineties, using compact beds and low temperature water, in order to isolate mechanical effects), the model takes into account the possibility of flow induced deformations on the microscopic scale, however negligible on the macroscopic scale. This is done assuming that the physical parameters  $k$  (hydraulic conductivity) and  $\epsilon$  (porosity) depend on the volumetric velocity  $q$ , i.e. they are instantly affected, with no relaxation, by the flow.

We remark that, according to the behaviour of the pressure injection rate of the fluid at the external surface, the problem may exhibit some peculiar feature, which can be

interpreted as a local collapse of the medium, observable in the experiments with the presence of straight lines crossing the wet region (see Ref. 1).

In summary the model, neglecting capillarity effects and gravity, gives rise to the following free boundary problem

$$\begin{aligned}
(1) \quad & p_x = -\frac{q}{k(q)}, \quad 0 < x < s(t), \quad t > 0, \\
(2) \quad & q_x + \epsilon'(q)q_t = 0, \quad 0 < x < s(t), \quad t > 0, \\
(3) \quad & p(0, t) = p_0(t), \quad t > 0, \\
(4) \quad & p(s(t), t) = 0, \quad t > 0, \\
(5) \quad & \dot{s} = \frac{q(s(t), t)}{\epsilon(q(s(t), t))}, \quad t > 0, \quad s(0) = 0,
\end{aligned}$$

where  $p$  is the pressure,  $p_0(t)$  is a given function, such that

$$p_0 \in C^2, \quad p_0(t) > 0 \quad \forall t > 0, \quad p_0(0) = 0, \quad \dot{p}_0(t) > 0, \quad \forall t \geq 0;$$

$s(t)$  is the thickness of the wet region,  $\epsilon$  and  $k$  satisfy

$$\begin{aligned}
(6) \quad & \epsilon \in C^3, \quad \epsilon' < 0, \quad \epsilon'' \geq 0, \quad 0 < \epsilon_\infty < \epsilon(z) < \epsilon_0, \quad \forall z > 0, \\
(7) \quad & k \in C^3, \quad k' \leq 0, \quad k'' \geq 0, \quad 0 < k_\infty < k(z) < k_0, \quad \forall z > 0.
\end{aligned}$$

In the Green-Ampt model  $\epsilon$  and  $k$  are constant, then the incompressibility condition  $\frac{\partial q}{\partial x} = 0$  yields that  $q$  depends only on  $t$ , and the condition that the wetting front moves with the speed of the penetrating liquid (5) leads immediately to the equation of motion of the wetting front  $p_0(t) = \frac{qs}{k} = \frac{\epsilon}{k}ss$ , then the problem has an explicit solution.

When we assume that  $\epsilon$  and  $k$  vary with  $q$ , according with (6), (7), the problem (1)-(5) becomes a non-standard hyperbolic problem, since the datum  $p_0(t)$  is given on the plane  $x = 0$ , while the characteristics travel from the free boundary  $x = s(t)$  towards  $x = 0$ .

A local in time existence result is to be found in Ref. 8.

In Refs. 2, 3 it has been proved that when  $\ddot{p}_0 \leq 0$  (non-increasing injection rate) the local solution can be continued in time and remains regular. When  $\ddot{p}_0 > 0$  (increasing injection rate), shocks can appear and we have a ‘‘classical entropy solution’’ for  $q$ , that is a piecewise continuous and smooth solution satisfying the Rankine-Hugoniot jump condition (see Ref. 3, Sect. 3). This happens because for  $\ddot{p}_0 > 0$  the characteristics ‘‘thicken’’ inside the wet region while for  $\ddot{p}_0 < 0$  they ‘‘enlarge’’.

Moreover if a first shock appears then an infinite sequence of shocks is generated. This behaviour is well described by an example considered in Ref. 2, in which the first shock is ‘‘forced’’ on the free boundary by a discontinuity of  $\dot{p}_0$  (see Remark 2.1)

All the previous results are strongly based on the assumption of  $\epsilon'$  strictly negative, which means that the problem (1)-(5) is strictly hyperbolic: that allowed us to exploit the information provided by the pressure boundary data in order to obtain a relationship between  $q(s(t), t)$  and  $q(0, t)$  (Ref. 8) and to solve the problem using the methods of the characteristics.

Then the natural question arises of what happens of the solutions of the hyperbolic problem, especially with shocks, if  $\epsilon \leq 0$ , recalling that the Green-Ampt solution has no shocks for  $p_0(t)$  continuous.

We consider first the case in which  $\epsilon$  approximates a constant and we give a convergence result of the solution with  $\epsilon$  and  $k$  varying with  $q$  to the Green-Ampt solution (see Theorem (2.1) of Sect.2). Moreover we show that there exists the possibility of the “permanence” of shocks in any approximating solution.

The convergence result is proved by means of the comparison with the solution of the same problem with  $\epsilon$  constant and  $k$  varying with  $q$  (for the proofs see Ref. 4).

In Section 3 we have instead considered a physically reasonable case in which there is a threshold value of the volumetric velocity  $\hat{q}$  after which the porosity remains constant. In this assumption the hyperbolic problem becomes degenerate, however, when no shock is present, existence and uniqueness of the classical solution are preserved. The situation is quite different if shocks can appear: the analysis evidentiates that the behaviour of the sequence of shocks strongly depends on how  $\epsilon'$  vanishes. In particular either the solution has a finite number of shocks or it has an infinite number of shocks accumulating in a finite time. After the time in which the shocks extinguish, the solution becomes classical again. From a physical point of view this situation corresponds to the presence of a non uniform zone observed between two uniform zones of the wet region (for detailed proofs see Ref. 5).

## 2. Convergence results

We start this section recalling some known properties of the solution of the Green-Ampt model which correctly predicts various physical properties.

For a given injection pressure  $p_0$ , let  $(q_1, s_1)$ ,  $(q_2, s_2)$  be the solutions of the Green-Ampt model corresponding to different values of the porosity and conductivity, say  $\epsilon_1, k_1$  and  $\epsilon_2, k_2$ . For any  $t > 0$  we have the following results, that can be easily deduced by the explicit solution:

- (i) Suppose  $\epsilon_1 < \epsilon_2$ ,  $k_1 = k_2$ , then  $s_1(t) > s_2(t)$  and  $q_1(t) < q_2(t)$ .
- (ii) Suppose  $\epsilon_1 = \epsilon_2$ ,  $k_1 < k_2$ , then  $s_1(t) < s_2(t)$  and  $q_1(t) < q_2(t)$ .
- (iii) If  $\dot{p}_0$  is constant then  $q$  and  $\dot{s}$  are constant.
- (iv) If  $\ddot{p}_0 > 0$  then  $q' > 0$  and  $\ddot{s} > 0$ . Moreover  $q^2(t) < \dot{p}_o(t)\epsilon k$ .
- (v) If  $\ddot{p}_0 < 0$  then  $q' < 0$  and  $\ddot{s} < 0$ . Moreover  $q^2(t) > \dot{p}_o(t)\epsilon k$ .

Let us notice that, in the case of strictly decreasing  $\epsilon$ , some of these properties have been proved also for the general problem (1)-(5). We recall that in Refs. 8, 2, 3 the existence and uniqueness of a bounded solution of problem (1)-(5) was proved, and in particular

- (1) if  $\ddot{p}_0(t) = 0$ , then  $q \equiv q^*$ ,
- (2) if  $\ddot{p}_0(t) > 0$ , then  $q^* < q < Q(t)$ ,  $q'_0 > 0$ ,  $q'_s > 0$ ,
- (3) if  $\ddot{p}_0(t) < 0$ , then  $Q(t) < q < q^*$ ,  $q'_0 < 0$ ,  $q'_s < 0$ ,

where, denoted by  $F(z) = \frac{z^2}{\epsilon k}$ ,

- (4)  $q_0(t) = q(0, t)$ ,  $q_s(t) = q(s(t), t)$ ,  
 $q^* : F(q^*) = \dot{p}_0(0)$ ,  $Q(t) : F(Q(t)) = \dot{p}_0(t)$ ,

Those results hold both for classical solutions, and for entropy solutions, that is in the presence of shocks.

Actually the same properties hold also if we assume that the porosity  $\epsilon$  is constant and the hydraulic conductivity  $k$  is a function of  $q$  satisfying hypothesis (7), (see Ref. 4, Sect. 3).

Let us define a sequence of functions  $\epsilon_n(z)$ , satisfying, for any  $n$ , hypotheses (6) and

$$(5) \quad |\epsilon'_n| < \gamma_n, \quad \lim_{n \rightarrow \infty} \gamma_n = 0,$$

and a sequence  $k_n(z)$ , with  $k_n$  satisfying the hypotheses (7).

For any  $n$ , let us denote by  $s_n(t)$ ,  $q_n(x, t)$  the solution of the problem with data  $\epsilon_n(t)$ ,  $k_n(t)$ , and by  $\hat{S}(t)$ ,  $\hat{Q}(t)$  the solution of the Green-Ampt problem with constant data  $\epsilon = \hat{\epsilon}$ ,  $k = \hat{k}$ .

**Theorem 2.1.** *Let  $\epsilon_n(z) \rightarrow \hat{\epsilon}$ ,  $k_n(z) \rightarrow \hat{k}$  uniformly, according to (5). Then*

$$(6) \quad s_n(t) \rightarrow \hat{S}(t), \quad q_n(x, t) \rightarrow \hat{Q}(t),$$

*uniformly in  $[0, T]$ .*

The proof is based on the following three comparison Lemmata.<sup>4</sup>

Let  $(S_0, Q_0)$  be the solution of problem (1)-(5) with data  $k(z)$  and  $\epsilon = \epsilon_0$ ,  $(S_\infty, Q_\infty)$  be the solution of problem (1)-(5) with data  $k(z)$  and  $\epsilon = \epsilon_\infty$ .

The first Lemma gives a comparison between the free boundaries.

**Lemma 2.1.** *Suppose (6), (7) hold, then we have that*

$$(7) \quad \text{if } \ddot{p}_0 \geq 0 \text{ then } S_0(t) \leq s(t) \leq \frac{C}{\gamma} S_\infty(\bar{\gamma}t), \quad 0 \leq t < \frac{T}{\bar{\gamma}},$$

$$(8) \quad \text{if } \ddot{p}_0 \leq 0 \text{ then } \frac{1}{C\hat{\gamma}} S_0(\hat{\gamma}t) \leq s(t) \leq S_\infty(t), \quad t \geq 0,$$

where

$$(9) \quad C = \left( \frac{k_0}{k_\infty} \right)^{\frac{3}{2}} \left( \frac{\epsilon_0}{\epsilon_\infty} \right)^{\frac{1}{2}}, \quad \gamma = \sup_{z \geq 0} (-\epsilon'(z)),$$

$$(10) \quad \bar{\gamma} = 1 + \gamma \frac{Q(T)}{\epsilon(Q(T))}, \quad Q(T) = F^{-1}(\dot{p}_0(T)),$$

$$(11) \quad \hat{\gamma} = 1 + \gamma \frac{q^*}{\epsilon(q^*)}, \quad q^* = F^{-1}(\dot{p}_0(0)).$$

Let us remark that the main tool to prove Lemma 2.1 is an estimate of  $q(\xi, t)$  for some  $\xi \in (0, s(t))$ , and the characteristic line of (2) for  $(\xi, t)$  which starts at  $(s(\tau), \tau)$  is defined by

$$(12) \quad t = \tau + \epsilon'(q_s(\tau))(\xi - s(\tau)).$$

The proof of the estimates follow from a uniform estimate on  $\tau$ , once  $t$  is fixed, obtained recalling that, for any arbitrary  $T$ , we have

$$(13) \quad s(t) < \frac{Q(T)}{\epsilon(Q(T))}t = \alpha t, \quad 0 < t < T.$$

Let us fix  $t$ , then

$$(14) \quad \tau(t) > \bar{\tau}(t), \quad \bar{\tau}(t) : t = \bar{\tau}(t) + \gamma\alpha\bar{\tau}(t) = \bar{\gamma}\bar{\tau}(t),$$

where we denoted by  $\gamma = \sup_{z \geq 0} -e'(z)$ , so that  $\bar{\tau}$  is the intersection between the straight line for  $(0, t)$  with slope  $-\gamma$  and the straight line  $x = \alpha t$ .

The second Lemma gives a comparison for the mean value of the gradient of the pressure  $R(q)$ .

**Lemma 2.2.** *Suppose that the hypotheses of Lemma 2.1 hold, then*

$$(15) \quad \text{if } \dot{p}_0 \geq 0, \quad \frac{\bar{\gamma}}{C_1} \left( \frac{P(t)}{P(\bar{\gamma}(t))} \right)^{\frac{1}{2}} R(Q_\infty(t)) \leq \bar{R}(t) \leq R(Q_0(t)), \quad 0 \leq t \leq \frac{T}{\bar{\gamma}},$$

$$(16) \quad \text{if } \dot{p}_0 \leq 0, \quad R(Q_\infty(t)) \leq \bar{R}(t) \leq \hat{\gamma}C_1 \left( \frac{P(t)}{P(\hat{\gamma}(t))} \right)^{\frac{1}{2}} R(Q_0(t)), \quad t \geq 0,$$

where  $C_1 = C \left( \frac{k_0}{k_\infty} \right)^{\frac{1}{2}}$ ,  $C, \bar{\gamma}, \hat{\gamma}$  are defined in (2.9)-(2.11) and

$$P(t) = \int_0^t p_0(\tau) d\tau.$$

**Remark 2.1.**

The comparison between mean values makes sense because in the presence of shocks a pointwise comparison does not hold.

In order to make that clearer, let us consider the example studied in Ref. 2 under the assumption that  $p_0$  has a piecewise constant derivative: with such assumptions we obtained explicit formula for the solution of the problem.

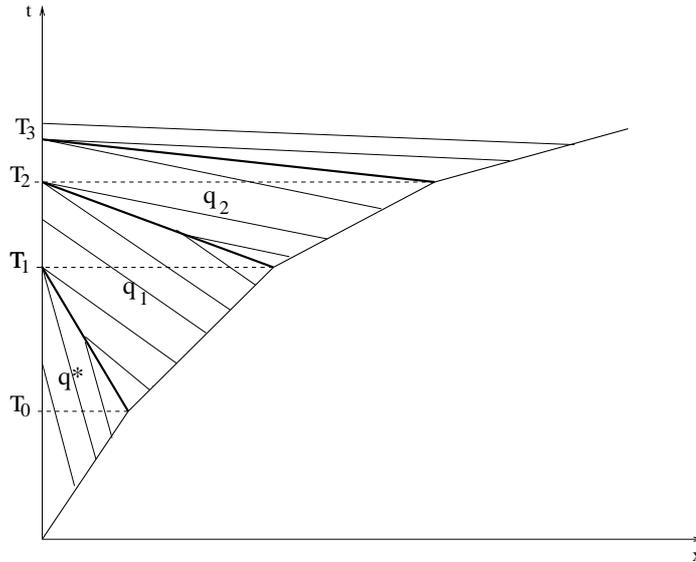


Fig. 1. Domain with shocks

Assume

$$p_0(0) = 0, \quad \dot{p}_0(t) = \begin{cases} \dot{p}_0^-, & 0 < t < T_0, \\ \dot{p}_0^+, & t > T_0, \end{cases}$$

with  $T_0, \dot{p}_0^-, \dot{p}_0^+$  positive constants.

In the case  $\dot{p}_0$  increasing, that is  $\dot{p}_0^- < \dot{p}_0^+$ , if  $\epsilon'$  is strictly increasing, there exists a unique piecewise constant solution for any  $t > 0$ , see Fig.1, given by:

$$\begin{aligned} q &= q_0, & s &= \frac{q_0}{\epsilon(q_0)}t, & 0 < x < s(t), & 0 < t < T_0, \\ q(x, t) &= \begin{cases} q_i, & 0 < x < \Sigma_{i+1}(t), \\ q_{i+1}, & \Sigma_{i+1}(t) < x < s(t), \end{cases} & T_i < t < T_{i+1}, & i = 0, 1, \dots \\ s(t) &= s(T_i) + \frac{q_{i+1}}{\epsilon(q_{i+1})}(t - T_i), \\ \Sigma_{i+1}(t) &= s(T_i) + (t - T_i)\dot{\Sigma}_{i+1}, \end{aligned}$$

where  $q_i$  is the following increasing sequence

$$\begin{cases} q_0 & \text{for } t \leq T_0, \\ q_{i+1} : \dot{p}_0^+ = A_i(q_{i+1}) = F(q_{i+1}) + f_i(q_{i+1}), & i = 0, 1, \dots \end{cases}$$

with

$$f_i(z) = \frac{q_i - z}{\epsilon(q_i) - \epsilon(z)} \left( \frac{q_i}{k(q_i)} - \frac{z}{k(z)} \right),$$

and

$$\begin{aligned} \dot{\Sigma}_i &= \frac{q_{i-1} - q_i}{\epsilon(q_{i-1}) - \epsilon(q_i)} < 0, & i = 1, 2, \dots \\ s(T_0) &= \frac{q_0}{\epsilon(q_0)}T_0, \\ s(T_{i+1}) &= s(T_i) + \frac{q_{i+1}}{\epsilon(q_{i+1})}(T_{i+1} - T_i), \end{aligned}$$

$$T_{i+1} = T_i - \frac{s(T_i)}{\dot{\Sigma}_{i+1}}, \quad i = 0, 1, \dots$$

If we take  $\epsilon_0 = \epsilon(q_0)$ , then  $Q_0(t)$  is a continuous function for any  $t$  and it is equal to  $q_0$  for  $t \leq T_0$ .

On the contrary the solution of the problem with data  $\epsilon(q)$ , in the interval of time  $(T_0, T_0 + \delta)$  sufficiently small, is given by

$$(17) \quad q(x, t) = \begin{cases} q_0, & 0 < x < \Sigma_1(t), \\ q_1, & \Sigma_1(t) < x < s(t), \end{cases}$$

with  $q_0 < Q_0(t)$  according to Lemma 2.1 and  $q_1 > Q_0(t)$ .

**Remark 2.2.**

For any constant  $a > 1$ , let us define the function  $g(t; a) = \frac{P(t)}{P(at)}$ .

Then  $g(t; a) \leq 1$  and

$$\text{if } \ddot{p}_0 \geq 0, \quad g(t; a) \geq \frac{1}{1 + \frac{\dot{p}_0(aT)}{\dot{p}_0(0)}(a^2 - 1)} = C_2(T), \quad 0 \leq t \leq T.$$

Using the results of previous Lemmata, the next step consists in comparing  $\sup q(x, t)$  and  $\inf q(x, t)$  for any  $t$ .

**Lemma 2.3.** *Under the hypotheses of Lemma 2.1 we have*

if  $\ddot{p}_0 \geq 0$ ,

$$\begin{aligned} \bar{\gamma} \frac{C_2}{C_1} k_\infty R(Q_\infty(t)) &\leq q_s^-(t) \leq q_s^+(t) \leq k_0 R(Q_0(\bar{\gamma}t)), \quad 0 \leq t \leq \frac{T}{\bar{\gamma}}, \\ \bar{\gamma} \frac{C_2}{C_1} k_\infty R\left(Q_\infty\left(\frac{t}{\bar{\gamma}}\right)\right) &\leq q^-(0, t) \leq q^+(0, t) \leq k_0 R(Q_0(t)), \quad 0 \leq t \leq T, \end{aligned}$$

if  $\ddot{p}_0 \leq 0$ ,

$$\begin{aligned} k_\infty R(Q_\infty(\hat{\gamma}t)) &\leq q_s(t) \leq k_0 \hat{\gamma} C_1 R(Q_0(t)), \\ k_\infty R(Q_\infty(t)) &\leq q(0, t) \leq k_0 \hat{\gamma} C_1 R\left(Q_0\left(\frac{t}{\hat{\gamma}}\right)\right), \quad t \geq 0. \end{aligned}$$

The proof of Theorem 2.1 is an immediate consequence of the three Lemmata above, remarking that when  $n \rightarrow \infty$  the constants  $C, C_1, C_2, \hat{\gamma}, \bar{\gamma}$  tend to 1,  $k_n$  tends to  $\hat{k}$ ,  $S_0$  and  $S_\infty$  tend to  $\hat{S}$ ,  $Q_0$  and  $Q_\infty$  tends to  $\hat{Q}$ .

**Remark 2.3.**

We emphasize that when  $n \rightarrow \infty$  if there are shocks, they do not vanish, but they thicken while the jump of  $q$  tends to 0.

To be more precise let us consider the example of Ref. 2, where the first shock is caused by a discontinuity of  $\dot{p}_0$  (see Remark 2.1).

Under the hypotheses of  $\dot{p}_0$  increasing, let us fix a value  $\tilde{q}$  such that

$$\hat{Q}(0) < \tilde{q} < \hat{Q}(\infty), \quad \hat{Q}(0) = F^{-1}(\dot{p}_0^-), \quad \hat{Q}(\infty) = F^{-1}(\dot{p}_0^+).$$

Recalling that  $\hat{Q}(t)$  is increasing, then there exists a unique  $\tilde{T} > T_0$  such that  $\hat{Q}(\tilde{T}) = \tilde{q}$ .

For any  $n > 0$  the jump of  $\dot{p}_0$  in  $T_0$  yields the first shock that starts in  $(s(T_0), T_0)$  and reflects infinitely many times between  $(0, T_i)$ , and  $(s(T_i), T_i)$ ,  $i = 1, \dots$ , (see Fig.1), where  $T_i \rightarrow \infty$  as  $i \rightarrow \infty$  is an increasing sequence, and the sequence of the values  $q_i$  tends to  $Q_n = F_n^{-1}(\dot{p}_0^+)$ . Then for any  $n$  let us define  $\tilde{j} = j(n)$  such that  $T_{\tilde{j}} < \tilde{T}$ ,  $T_{\tilde{j}+1} > \tilde{T}$ . Then  $\tilde{j}$  represents the number of the shocks of the solution  $q_n(x, t)$  before  $\tilde{T}$  and we can prove that  $\tilde{j} \rightarrow +\infty$  as  $n \rightarrow \infty$ .

### 3. Threshold for $\epsilon$

In the present section we will consider the case in which there is a threshold value of the volumetric velocity  $\hat{q}$  after which the porosity remains constant. To be precise instead of (6) we will have the following

$$(1) \quad 0 < \hat{\epsilon} < \epsilon < \epsilon_0, \quad \epsilon' < 0, \quad \epsilon'' \geq 0, \quad \text{for } q < \hat{q}, \quad \epsilon \equiv \hat{\epsilon} = \epsilon(\hat{q}), \quad \text{for } q \geq \hat{q}.$$

This assumption implies that the hyperbolic problem becomes degenerate.

The interesting case is when  $\ddot{p}_0 > 0$ , and we consider the example of Ref. 2 (see Remark 2.1), where the first shock is caused by a discontinuity of  $\dot{p}_0$ , under the hypothesis that we will assume in the following of the section:

$$(2) \quad F^{-1}(\dot{p}_0(0)) = q^* < \hat{q} < Q(T) = F^{-1}(\dot{p}_0(T)).$$

In assumption (2), if a term of the sequence reaches or goes beyond the value  $\hat{q}$ , the mechanism of reflection of the shock lines cannot be carried on anymore by the characteristic lines since they become “horizontal”; then two situations are possible:

- (a) -  $\forall i \quad q_i < \hat{q} < Q$ ,
- (b) -  $\exists j : q_j < \hat{q} < Q$  and  $q_{j+1} \geq \hat{q} < Q$ .

Both cases (a) and (b) are possible, in particular, the following theorem proves, in case (a), the most interesting one, the existence of a solution with infinitely many shocks in finite time.

**Theorem 3.1.** *Suppose that hypothesis (1) holds, and that case (a) is verified. Then there exists a finite time  $T_\infty$  such that*

- (i) - for  $0 \leq t \leq T_\infty$ ,  $q$  is given as in Ref. 2 by (2.4), with  $\lim_{i \rightarrow \infty} T_i = T_\infty$ ,
- (ii) - for  $t \geq T_\infty$ ,  $q$  is the Green-Ampt solution with initial datum  $\hat{q}$ .

**Remark 3.1.**

Recalling that  $q_i \geq q^* = q_0$ , we can prove an inferior estimate for  $T_\infty$ :

$$(3) \quad T_0 < \frac{(\dot{p}_0^+ - \dot{p}_0^-)T_0}{\dot{p}_0^+ - \frac{q^*}{\epsilon(q^*)}R(\hat{q})} < T_\infty.$$

Let us now consider case (b): then we have the following

**Theorem 3.2.** *Suppose that hypothesis (1) holds, and that case (b) is verified. Then*

- (i) - for  $0 \leq t \leq T_{j+1}$ ,  $q$  is given as in Remark 2.1,
- (ii) - for  $t \geq T_{j+1}$ ,  $q$  is the Green-Ampt solution with initial datum  $q_{j+1}$ , and  $\lim_{t \rightarrow \infty} q = Q$ .

Let us characterize the occurrence of cases (a) and (b).

Let us define the function

$$\hat{A}(y) = F(\hat{q}) + \frac{y - \hat{q}}{\epsilon(y) - \hat{\epsilon}}(R(y) - R(\hat{q})), \text{ for } y < \hat{q}.$$

For any  $q_i < \hat{q}$  we have

$$\hat{A}(q_i) = A_i(\hat{q}).$$

It follows immediately

**Proposition 3.1.**

$$(4) \quad \begin{aligned} (a) &\iff \hat{A}(q_i) > \dot{p}_0^+ \quad \forall i, \\ (b) &\iff \exists j : \hat{A}(q_j) \leq \dot{p}_0^+, \quad i = 0, 1, \dots \end{aligned}$$

We remark that the function  $\hat{A}(y)$  is known once  $\epsilon(z)$  and  $k(z)$  have been given, then (4) yields a characterization of case (a) and (b) in dependence on the jump of  $p_0$ . In particular we have the following:

**Proposition 3.2.** *If  $\lim_{q \rightarrow \hat{q}^-} \epsilon'(q) = -\gamma < 0$ , then only case (b) is possible, that is, case (a) verifies only if  $\epsilon'$  is continuous in  $\hat{q}$ .*

Let us conclude remarking that an analogous of Theorem 2.1 holds, but Remark 2.3 fails; infact the behaviour of the approximating solutions strongly depends on the behaviour of the sequence  $\epsilon_n$ , as it will be made clear from the following two Lemmata.

**Lemma 3.1.** *Let us consider a sequence of functions  $\{\epsilon_i(z)\}$  satisfying hypotheses (1) for any  $i$ , with  $\lim_{i \rightarrow \infty} \hat{q}_i = 0$ .*

*Then there exists a certain  $\hat{j}$  such that the solutions coincide with the Green-Ampt solution for  $i > \hat{j}$ .*

**Lemma 3.2.** *Let us consider a sequence of functions  $\{\epsilon_i(z)\}$  satisfying hypotheses (1) for any  $i$ , with  $\hat{q}$  constant for any  $i$ , and a sequence  $k_i(z)$  satisfying (7) such that*

$$\lim_{i \rightarrow \infty} \epsilon_i = \hat{\epsilon}, \quad \lim_{i \rightarrow \infty} k_i = \hat{k}.$$

*Let us suppose moreover that for any  $i$ ,  $\epsilon_i$  is such that condition (a) holds.*

*Then, defining  $T_\infty^i$  according to Theorem 3.1, we have that*

$$\lim_{i \rightarrow \infty} T_\infty^i = \hat{T} > T_0.$$

We can remark that the sequence of shocks does not shrinks at time  $T_0$  as one could conjecture, but it goes beyond  $T_0$ .

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