Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 2 (2007)

DOI: 10.1685/CSC06175

EINSTEIN RELATION ON FRACTAL OBJECTS

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Abstract.

Many physical phenomena proceed in or on irregular objects which are often modeled by fractal sets. Using the model case of the Sierpinski gasket, the notions of Hausdorff, spectral and walk dimension are introduced in a survey style. These "characteristic" numbers of the fractal are essential for the Einstein relation, expressing the interaction of geometric, analytic and stochastic aspects of a set.

Keywords: fractals, self-similarity, Hausdorff dimension, Dirichlet form, Laplacian, spectral dimension, (strong) Markovian process, walk dimension, Einstein relation.

1. Introduction

Classical second order partial differential equations as for example heat, wave, or Schrödinger equation rely on the assumption that the underlying space is smooth. However, many things in the world around us are "wild", "irregular", and "rough". They can be modeled by non–smooth "fractal" sets. Due to this fact there were many efforts to develop some tools of analysis on fractals. The main problem is obvious: Fractals are non–differentiable objects because the notion of a "tangent space" is not available.

But there are ways out. One of the main approaches uses probability theory and founds on the following observation: In $\mathbb{R}^n$, the Laplacian is the infinitesimal generator of the standard Brownian motion, which can be obtained as the limit of renormalized random walks. On the other hand, the construction of a random walk does not require a differentiable structure. Hence, one defines the Brownian motion on suitable fractals as the limit of a sequence of "suitable" normalized random walks and calls the infinitesimal generator of the limit process "Laplacian" on this fractal. This construction was done for the class of so–called *nested fractals* (see Kusuoka[12] and Lindstrøm[13]); as an introduction we recommend the very nice survey by Barlow.[1] Another approach – somehow the analytic counterpart of the probabilistic one – goes back to Kigami, see as a standard reference his monograph[10] and the references listed therein (for a short and very simple introduction into the subject see the recent paper of the author[4]). An energy form (and hence – via the Gauß–Green–formula – a Laplacian) is constructed on so–called *post critically finite self–similar fractals* as a limit of approximating energies which are defined by suitable

difference schemes on a sequence of "pre–fractals". Note that sets from both these families "nested" and "post critically finite" are in particular self-similar and finitely ramified. The construction of the energy form as well as of the random walk deeply relies on this property.
"Self–similarity" means that the fractals consist of smaller similar copies of themselves, that is they are allowed to be "irregular", but at the same time they carry a very rigid recursive structure. For self–similar fractals, satisfying in addition the open set condition, the calculation of the Hausdorff dimension is very simple (see Sec. 2). Of course, a lot of applications require dealing with "wilder" than self-similar objects. First results in this direction have been obtained using a Lagrangian approach, which allows to define a Laplacian for example on junctions[5] and deformations[6] of nested fractals.
"Finitely ramification" means that the fractal may become a disconnected set by removing a finite number of points. A standard example of a finitely ramified self–similar fractal is the well–known Sierpinski gasket (see Fig. 1 on page 4). Finitely ramified fractals can be approximated by an increasing sequence of points (which form the "nodes" of the underlying "resistance networks", see Fig. 3 on page 5) - this is fundamental in the analytic as well as in the stochastic approach. Note that every nested fractals is post–critically finite. On the intersection of both classes both approaches – probabilistic and analytic – determine the same object.

As mentioned above, there is a connection between the analytic and the stochastic behaviour of a set, which can brought into connection with its dimension and expressed in terms of Einstein relation. Einstein relation in its "dimension form" reads

$$2d_H = d_S d_W, \tag{1}$$

where d_H, d_S and d_W denote Hausdorff, spectral and walk dimension respectively. The Hausdorff dimension is a geometrical object, it describes the relation between volume and length scaling of a set and will be introduced in Sec. 2. The spectral dimension is the double of the exponent appearing in the eigenvalue asymptotics of the Laplacian. For a better understanding, we shortly recall the classical result in the Euclidean case. Let be Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We regard the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_n u = \lambda u \text{ on } \Omega \\ u_{|\partial\Omega} \equiv 0, \end{cases}$$

where $\Delta_n = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the classical Laplacian in $\mathbb{R}^n$. It is well–known (see Weyl[17]) that the eigenvalue counting function

$$N_n(x) := \#\{\lambda_k \le x \quad : \quad -\Delta_n u = \lambda_k u \text{ for some } u \neq 0\},$$

(counting the eigenvalues according multiplicities) is well–defined. Moreover, for any $n \in \mathbb{N}$ it holds that

$$N_n(x) = (2\pi)^{-n} c_n \mathrm{vol}^n(\Omega) x^{n/2} + o(x^{n/2}), \qquad \text{as} \quad x \to \infty, \tag{2}$$

where $\mathrm{vol}^n(\Omega)$ denotes the n–dimensional volume of Ω and c_n is the n–dimensional volume of the unit ball in $\mathbb{R}^n$. In Sec. 3, we will sketch the definition of the "natural" Laplacian on the Sierpinski gasket and give the fractal analogue of Eq. (2). Finally, in Sec. 4, we introduce the walk dimension which expresses the space-time-scaling of a Markov process. It is well known that the walk dimension of Brownian motion in Euclidean domains equals 2. (Readers not familiar with this fact will certainly become convinced by taking a careful look at the formula of the Gaussian bell shaped curve, which is in fact the normal distribution density, – and hence,– the heat kernel of the Brownian motion.)

Note that Einstein relations appear in mathematical as well as physical literature in many different versions. For example, in Ref. 8, time–space–scaling exponents of random walks on finitely ramified fractal lattices have been obtained by Monte–Carlo simulation. Then, using group theory, exact relations have been proved which can be expressed in the form of Eq. (1). Nyberg[14] described the Einstein relation on certain classes of graphs in the language of electric networks relating masses, resistances, and mean commute times between vertices. In the setting of a quite concrete physical application, namely an interface model in a finite tube, the Einstein relation is expressed in terms of mobility, inverse temperature and the diffusion coefficient of long–time–fluctuations (see Ref. 15). Finally, in Telcs,[16] Eq. (1) reads as a relation between volume growth of balls, expected exit times from balls, and resistances of annuli (see also Zhou[18]).

As outlined above, for domains Ω in $\mathbb{R}^n$, the Einstein relation is trivially satisfied by $d_H = n$, $d_S = n$ and $d_W = 2$. Studying these quantities on a fractal is much more interesting and will even lead to a better understanding of the Euclidean case.

2. Fractals as geometric objects – the Hausdorff dimension

We will call "fractal" a set for which the Hausdorff dimension strictly exceeds the topological dimension. In most cases, the Hausdorff dimension of such sets is a non–integer number. For the definition of Hausdorff measure as well as Hausdorff dimension we recommend the book of Falconer.[3] In order to understand the meaning of "dimension", let us recall the connection between measures and dimension: somehow, we call a (rectifiable) curve *one–dimensional,* if the notion of its "length" makes sense. Analogously, we call a plane or a more general surface in a higher dimensional space *two–dimensional* if it is measurable by a two–dimensional (i.e. area) measure, or a solid body *three–dimensional,* if we may measure its volume. Interpolating these observations, the definition of the Hausdorff dimension is done by introducing Hausdorff measures. Roughly spoken, the Hausdorff dimension a fractal F is d_H, if F has a non-trivial d_H-dimensional Hausdorff measure. Or, equivalently, the measure of a small ball $B(x,r)\cap F$ around a point $x \in F$ scales like r^{d_H}.

The best studied and somehow simplest fractals are so-called self-similar fractals. These – maybe most popular – examples of fractal sets have the property that they consist of a finite number of smaller copies of themselves. Due to a result of Hutchinson,[9] the Hausdorff dimension of such sets calculates very easily, if the copies do not overlap "too much". We will recall the construction of self-similar sets and self-similar measures as well as the dimension result here.
Let $S = \{S_1, ..., S_M\}$, $M \geq 2$, be a finite family of contractive similitudes acting on $\mathbb{R}^n$,

$n \geq 1$, that is

$$|S_i(x) - S_i(y)| = r_i|x - y|, \qquad x, y \in \mathbb{R}^n,$$

for some numbers $r_i \in (0, 1)$, $i = 1, \ldots, M$. Further, we are given a M–dimensional vector of weights $\rho = (\rho_1, \ldots, \rho_M)$, that is $\rho_1, \ldots, \rho_M$ are real numbers from the interval $(0, 1)$ and $\sum_{i=1}^M \rho_i = 1$.
We call a subset F of $\mathbb{R}^n$ *self–similar with respect to* S if $F = \bigcup_{i=1}^M S_i(F)$, and a Borel probability measure μ *self–similar with respect to* S *and* ρ if $\mu(A) = \sum_{i=1}^M \rho_i \mu(S_i^{-1}(A))$ for any Borel set A in $\mathbb{R}^n$.
Results of Hutchinson[9] imply the existence and uniqueness of such self-similar sets F and measures μ for any finite family $S = \{S_1, ..., S_M\}$ and any vector $\rho = (\rho_1, ..., \rho_M)$. Moreover, he proved that the Hausdorff dimension of F is given by the unique positive solution d of $\sum_{i=1}^M r_i^d = 1$, where the numbers r_i are the contraction ratios of the mappings S_i, $i = 1, \ldots, M$, if the family S satisfies the so–called *open set condition,* that is we assume that there exists a non–empty bounded open set O such that $S_i(O) \subseteq O$, $i = 1, \ldots, M$, and $S_i(O) \cap S_j(O) = \emptyset$, $i \neq j$. Finally, if $\rho_i = s_i^{d_H}$ (which is the "natural choice" of the weights), then the unique self–similar measure is just given by the normalized d_H–dimensional Hausdorff measure on F. Note that in the special case that all mappings of the family S have the same contraction ratio r, the Hausdorff dimension is given by $d_H = \frac{\ln M}{-\ln r}$.
Let us consider the famous **Sierpinski gasket**, which in the Sections 3 and 4 will serve

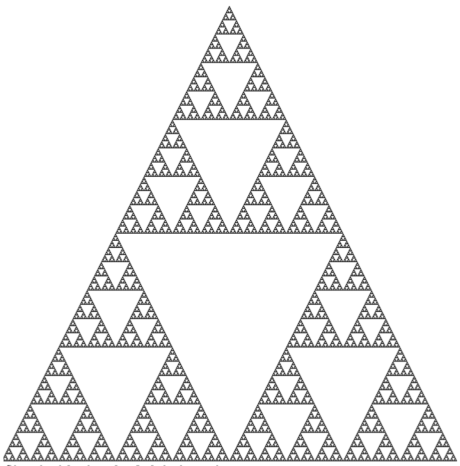

Fig. 1. The Sierpinski gasket

as our standard example in explaining the notions of spectral and walk dimension on a nested fractal. Pose $A := (0, 0)$, $B := (1, 0)$ and $C := (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The Sierpinski gasket K is defined to be the unique nonempty compact set which is self–similar with respect to the family of affine contractions $\Psi := \{\psi_1, \psi_2, \psi_3\}$ (i.e. $K = \bigcup_{i=1}^3 \psi_i(K)$) where the mappings $\psi_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are just given by the unique contractive similitude with contraction ratio $\frac{1}{2}$ and fixed point A, B and C respectively. It is easy to check that the open set condition is satisfied by choosing the open set O to be the interior of the triangle ABC. Hence, the Hausdorff dimension of the Sierpinski gasket equals $d_H = \frac{\ln 3}{\ln 2}$. Obviously, the Sierpinski gasket is finitely ramified: Removing the middle points a, b and c from the line segments BC, AC and AB respectively, makes it a disconnected set (see also Fig. 1 on page 8).
In this paper, we will not consider self-similar sets like the well-known **Cantor set** or

Sierpinski carpet. The first one is totally disconnected while the latter one is obviously infinitely ramified. Nevertheless, both sets in Fig. 2 satisfy the open set condition, hence their Hausdorff dimensions are given by $\frac{\ln 2}{\ln 3}$ and $\frac{\ln 8}{\ln 3}$ respectively.

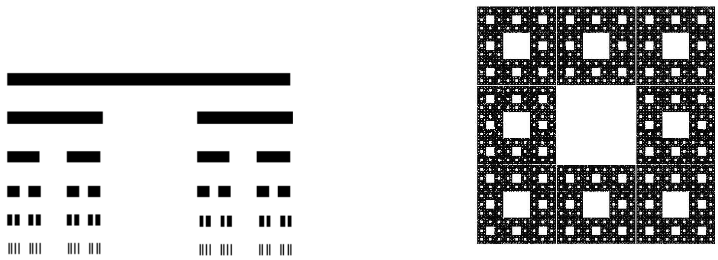

Fig. 2. a: Cantor set b: Sierpinski carpet

In classical analysis and stochastics, continuous quantities are often approximated by discrete structures. For example, the derivative of a function is given a s the limit of difference quotients, or the Brownian motion is obtained as the limit of a sequence of random walk. The approximating objects are defined on geometrical objects approximating the limit set. The same procedure serves for defining the energy of a function on the Sierpinski gasket where the "fractal analogue" of the Euclidean standard energy form $\mathcal{E}[u] = \int |\nabla u|^2 dx$ is obtained as the limit of certain discrete "pre–energies" defined on finite sets (V_n) approximating the fractal (see Sec. 3). Similarly, Brownian motion will be obtained as the limit of a sequence of random walks on (V_n) (see Sec. 4). Both approaches deeply rely on the self–similarity and the finite ramification of the underlying set.
In preparation of the following two sections we explain here how approximation of Sierpinski gasket K by an increasing sequence of finite sets $(V_n)_{n\geq 0}$ is done: Setting $V_0 := \{A, B, C\}$, we define for arbitrary n–tuples of indices $(j_1, ..., j_n) \in \{1, 2, 3\}^n$: $\psi_{j_1 \ldots j_n} := \psi_{j_1} \circ \ldots \circ \psi_{j_n}$, $V_{j_1 \ldots j_n} := \psi_{j_1 \ldots j_n}(V_0)$ and $V_n := \bigcup_{j_1 \ldots j_n = 1}^{3} V_{j_1 \ldots j_n}$. We say that

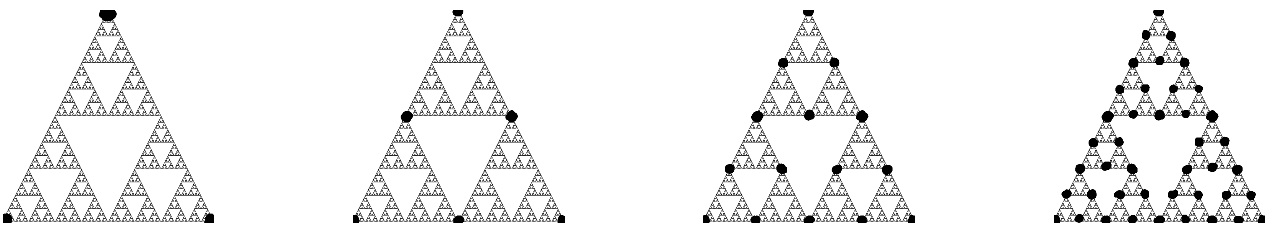

Fig. 3. The approximating sets V_0, V_1, V_2 and V_3

$p, q \in V_n$ are n*–neighbors* if there exists a n–tuple of indices $(j_1, ..., j_n) \in \{1, 2, 3\}^n$ such that $p, q \in V_{j_1 \ldots j_n}$. Every point p in $V_n \backslash V_0$ has four n–neighbors $q \in V_n$ denoted in the following by $q \sim_n p$. Every n–neighbor q of p has distance 2^{-n} from p. We say that any two points from V_0 form a pair of zero–neighbors. Further, we set $V_* := \bigcup_{n\geq 0} V_n = \lim_{n\to\infty} V_n$. It holds that $K = \overline{V_*}$ (the bar denotes the closure).
The corresponding approximation of the normalized $\frac{\ln 3}{\ln 2}$–dimensional Hausdorff measure, restricted to K by discrete measures supported on the sets $(V_n)_{n\geq 0}$ is done as follows. For any $n \geq 0$, we define a discrete measure on V_n by $\mu_n := \frac{2}{3^{n+1}} \sum_{p \in V_n} \delta_{\{p\}}$, where $\delta_{\{p\}}$ denotes the Dirac measure at the point p. Note that $\hat{\mu}_n(V_n) = 1 + \frac{1}{3^n}$, because the cardinality of

V_n is given by $\sharp V_n = \frac{3}{2}(3^n+1)$. The sequence $(\mu_n)_{n\geq 1}$ is weakly convergent to the measure μ.

3. Fractals as analytic objects – the spectral dimension

In this section, we present the construction of the energy form on the Sierpinski gasket K which is based on finite difference schemes. For a general outline of the theory see Ref. 10. Here we follow the general lines described in Ref. 12 for nested fractals.
For any function $u : V_* \longrightarrow \mathbb{R}$, we define

$$\mathcal{E}_n[u] := \frac{1}{2}\left(\frac{5}{3}\right)^n \sum_{p\in V_n}\sum_{q\sim_n p}(u(p)-u(q))^2, \qquad n\geq 0. \tag{1}$$

The number $\frac{5}{3}$ is the energy scaling factor determined by the Gaussian principle as follows: Suppose we are given the values of a function u on the set V_0, say $u(A)=u_A$, $u(B)=u_B$ and $u(C)=u_C$. According to (1), we have that

$$\mathcal{E}_0[u] = (u_A-u_B)^2+(u_A-u_C)^2+(u_B-u_C)^2 \tag{2}$$

and

$$\begin{aligned}\mathcal{E}_1[u] = \frac{5}{3}\big[&(u(a)-u_B)^2+(u(a)-u_C)^2+(u_B-u(c))^2\\ &+(u_A-u(b))^2+(u_A-u(c))^2+(u(b)-u_C)^2\\ &+(u(a)-u(c))^2+(u(a)-u(b))^2+(u(b)-u(c))^2\big]\end{aligned} \tag{3}$$

(see also Fig. 1 on page 8 for a better illustration). We minimize $\mathcal{E}_1[u]$ with respect to the values $u(a), u(b), u(c)$, i.e. we are seeking for the *harmonic extension* of the function u from V_0 to V_1 (see Ref. 2 for a recently developed algorithm calculating such harmonic extensions). Here, a simple calculation leads to the minimizers $u(a) = (u_A+2u_B+2u_C)/5$, $u(b) = (2u_A+u_B+2u_C)/5$ and $u(c) = (2u_A+2u_B+u_C)/5$, and in view of (2) and (3) we obtain that $\mathcal{E}_1[u] = \mathcal{E}_0[u]$ for this harmonic extension. In other words, $5/3$ is the unique number ϱ, satisfying $\min\{\mathcal{E}_1[v] \ : \ v_{|V_0} = u\} = \mathcal{E}_0[u]$, if we would use the general ansatz

$$\mathcal{E}_n[u] := \frac{1}{2}\,\varrho^n \sum_{p\in V_n}\sum_{q\sim_n p}(u(p)-u(q))^2, \qquad n\geq 0.$$

From the self–similarity and the finite ramification we obtain this minimizing property on each magnification level (with the same scaling factor $5/3$), and so one easily verifies that the sequence $(\mathcal{E}_n[u])_{n\geq 0}$ is non–decreasing. We define the limit form $\mathcal{E}_K[u] := \lim_{n\to\infty}\mathcal{E}_n[u]$ on the domain $\mathcal{D}_*(\mathcal{E}_K) := \{u : V_* \longrightarrow \mathbb{R} \ : \ \mathcal{E}_K[u] < \infty\}$. Every function $u \in \mathcal{D}_*(\mathcal{E}_K)$ has an unique extension to an element of $\mathcal{C}(K)$ which is actually Hölder continuous with Hölder exponent $\beta = \frac{\ln(5/3)}{2\ln 2}$. We denote this extension still by u and we set $\mathcal{D} := \{u \in \mathcal{C}(K) \ : \ \mathcal{E}_K[u] < \infty\}$, where $\mathcal{E}_K[u] := \mathcal{E}_K[u_{|V_*}]$. Hence $\mathcal{D} \subseteq \mathcal{C}(K) \subseteq L^2(K,\mu)$ where $L^2(K,\mu)$ is the Hilbert space of square summable functions on K with respect to the self–similar measure μ. We now define the space $\mathcal{D}(\mathcal{E}_K)$ to be the completion of $\mathcal{D}$ with respect to the norm

$$||u||_{\mathcal{E}_K} := \left(||u||^2_{L^2(K,\mu)} + \mathcal{E}_K[u]\right)^{1/2}. \tag{4}$$

$\mathcal{D}(\mathcal{E}_K)$ is injected in $L^2(K,\mu)$ and is a Hilbert space with the scalar product associated to the norm (4). Then we extend $\mathcal{E}_K$ as usual on the completed space $\mathcal{D}(\mathcal{E}_K)$. By $\mathcal{E}_K(\cdot,\cdot)$ we denote the bilinear form defined on $\mathcal{D}(\mathcal{E}_K)\times\mathcal{D}(\mathcal{E}_K)$ by polarization, that is

$$\mathcal{E}_K(u,v) := \frac{1}{2}\left(\mathcal{E}_K[u+v]-\mathcal{E}_K[u]-\mathcal{E}_K[v]\right), \quad u,v\in\mathcal{D}(\mathcal{E}_K)\,.$$

$\mathcal{E}_K(\cdot,\cdot)$ with domain $\mathcal{D}(\mathcal{E}_K)$ is a Dirichlet form on the Hilbert space $L^2(K,\mu)$. Moreover, the form $\mathcal{E}_K$ is regular and local. Regularity means that $\mathcal{D}(\mathcal{E}_K)\cap\mathcal{C}(K)$ is dense both in $\mathcal{C}(K)$ with respect to the uniform norm and in $\mathcal{D}(\mathcal{E}_K)$ with respect to the intrinsic norm (4). Locality means that $\mathcal{E}_K(u,v)=0$ whenever the functions u and v are supported on disjoint sets. These properties imply that there is an associated stochastic process $\mathbf{X}$ with continuous paths on K (see Sec. 4).

As $(\mathcal{E}_K,\mathcal{D}(\mathcal{E}_K))$ is a closed form on $L_2(K,\mu)$ there exists a unique self–adjoint, non–positive operator Δ_K on $L_2(K,\mu)$ – with domain $\mathcal{D}(\Delta_K)\subseteq\mathcal{D}(\mathcal{E}_K)$, dense in $L_2(K,\mu)$ – such that

$$\mathcal{E}_K(u,v) = -\int_K (\Delta_K u)\, v d\mu, \qquad u\in\mathcal{D}(\Delta_K)\,, v\in\mathcal{D}(\mathcal{E}_K), \tag{5}$$

for a general outline of the theory see Fukushima et. al.[7]The operator Δ_K is called the *Laplacian* on K. Let us give a look to its spectral asymptotics. The natural analog of Weyl's result (2) would be

$$N_d(x) = c_d\mathcal{H}^d(\Omega)x^{d/2}+o(x^{d/2}), \qquad \text{as} \quad x\to\infty, \tag{6}$$

where Ω is a fractal set of Hausdorff dimension $d=\dim_{\mathrm{H}}(\Omega)$, $\mathcal{H}^d$ denotes the d–dimensional Hausdorff measure, and c_d is a constant independent of the set Ω. In fact, (6) has been conjectured in the early 80's. Later it turned out that (6) fails for most self–similar finitely ramified fractals (see Ref. 11). In particular, the eigenvalue counting function of the Laplacian Δ_K on the Sierpinski gasket K behaves asymptotically like $x^{d_S/2}$, where $d_S=\frac{\log 9}{\log 5}$. Obviously, d_S differs from the Hausdorff dimension. This is a "fractal feature" which we want to emphasize: Spectral asymptotics on fractals do not only depend on the Hausdorff dimension, but also on the ramification properties. In fact, one could construct two fractals of the same Hausdorff dimension with "eigenfrequencies" of different order.

In general, it is rather difficult to determine the spectral dimension on a fractal. However, for fractals being "such symmetric" as the Sierpinski gasket, it is determined by $d_S/2=\ln M/\ln(M\varrho)$, where M denotes the number of similitudes and ϱ the energy scaling factor introduced above.

4. Fractals as state spaces of stochastic processes – the walk dimension

As the Dirichlet form $(\mathcal{E}_K,\mathcal{D}(\mathcal{E}_K))$ introduced in Sec. 3 is regular and local, it follows that there exists a strong Markovian process $\mathbf{X}=(X_t)_{t\geq 0}$ with continuous paths in K. The connection between $\mathbf{X}$ and $(\mathcal{E}_K,\mathcal{D}(\mathcal{E}_K))$ is expressed by the fact that the Laplacian Δ_K (which is related to $(\mathcal{E}_K,\mathcal{D}(\mathcal{E}_K))$ according to formula (5)) is the so–called infinitesimal generator of the process $\mathbf{X}$. For a general outline of the theory see Fukushima et. al.[7]
Moreover, $\mathbf{X}$ can be obtained as the limit of a sequence of suitable renormalized random

walks $\mathbf{X}^{(n)}$ moving on the prefractal graphs, i.e. $\mathbf{X}^{(n)}$ is a symmetric random walk on the graph with vertex set V_n and edges connecting n–neighbors (see Lindstrøm[13]).
Now we are going to introduce the notion of walk dimension in a "heuristic" way. While the Hausdorff dimension somehow relates the volume of small balls with their radii, the walk dimension relates mean exit times from balls with their radii. Hence, the walk dimension describes the time-space-scaling of a random walk or a stochastic process.
Let $\tau(B(x,R))$ denote the exit time of a stochastic process $\mathbf{X}$ starting at time 0 in x from a ball with radius R. Then the walk dimension is defined by

$$d_W := \frac{\ln \mathbb{E}^x \tau(B(x,R))}{\ln R}.$$

In graph theory, the limit for $R \to \infty$ of the latter term is taken, but we will see, that in the case of fractals (due to the self–similarity) we will have a reasonable value for d_W independent of R.
The local symmetry and the self–similarity of the sets V_n in connection with the arising (natural) strong reflection principle of the corresponding random walks lead to the helpful observation that exit times from balls equal crossing times through subgraphs of V_n. Thus, we can reformulate our "leaving–a–ball"–problem as follows: Supposing that the random walk starts in a vertex of V_0, we ask for the expected time of the first hitting of *another* vertex of V_0 provided that we move along the edges of V_1. To be more concrete, we assume that we start in A and we want to pass through the graph of V_1 until we reach B or C. In the following considerations, $\mathbb{E}\tau^P$ denotes the expected time moving from a point P to the set $\{B,C\}$.
Starting in point A and making one step, we can reach either point b or point c, both of

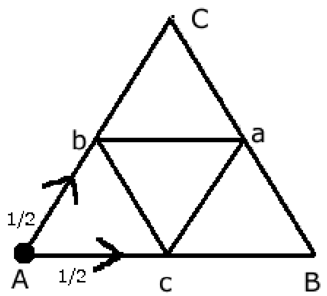

Fig. 1. Random walk on the graph with vertex set V_1

them with probability 1/2 (all notations of this paragraph refer to Figure 4). Hence, we have

$$\mathbb{E}\tau^A = \frac{1}{2}\left(\mathbb{E}\tau^b + \mathbb{E}\tau^c\right) + 1 = \mathbb{E}\tau^c + 1.$$

Note that our hitting problem is symmetric with respect to the symmetry axis of the triangle mapping b to c. This implies $\mathbb{E}\tau^b = \mathbb{E}\tau^c$. Similar observations lead to

$$\mathbb{E}\tau^c = \frac{1}{4}\left(\mathbb{E}\tau^A + \mathbb{E}\tau^b + \mathbb{E}\tau^a + \mathbb{E}\tau^B\right) + 1 = \frac{1}{4}\left(\mathbb{E}\tau^A + \mathbb{E}\tau^c + \mathbb{E}\tau^a\right) + 1$$

and

$$\mathbb{E}\tau^a = \frac{1}{4}\left(\mathbb{E}\tau^C + \mathbb{E}\tau^b + \mathbb{E}\tau^c + \mathbb{E}\tau^B\right) + 1 = \frac{1}{2}\mathbb{E}\tau^c + 1,$$

taking into account that $\mathbb{E}\tau^B = \mathbb{E}\tau^C = 0$. From the last three equations one easily calculates that $\mathbb{E}\tau^A = 5$. Hence, it takes in expectation five steps (of length one) to leave a ball of radius two. By the self similarity of the Sierpinski gasket and the Markov property of the random walks it readily verifies that this time–space–scaling will occur on all magnification scales, and also in the limit approaching continuous time (see Lindstrøm[13]). Thus, the walk dimension of the Sierpinski gasket equals $d_W = \frac{\ln 5}{\ln 2}$.

Concluding remarks

For the model case of the Sierpinski gasket it now easily verifies that Einstein's relation (1) holds, because we have $d_H = \ln 3/\ln 2$, $d_S = \ln 9/\ln 5$ and $d_W = \ln 5/\ln 2$. We hope that the reader got some better imagination how the geometric, analytic and stochastic properties of a physical object are interwoven. From the viewpoint of applications, the following interpretation of the Einstein relation might be useful: If one can investigate a porous set visually (leading to d_H), acoustically (leading to d_S), or with a "blind–and–deaf–ant"–sense (leading to d_W), then (1) tells us, that it is sufficient to do two of these "experiments". The third quantity can be calculated using the Einstein relation.

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