Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 2 (2007)

DOI: 10.1685/CSC06057

On Software Development for Financial Evaluation of Participating Life Insurance Policies

S. CORSARO*, P.L. DE ANGELIS*, Z. MARINO*, F. PERLA*
University of Naples Parthenope
Naples, Italy
E-mail: {corsaro,deangelis,marino,perla}@uniparthenope.it

Abstract.

In this work we focus on the numerical issues in the evaluation of an important class of financial derivatives: participating life insurance contracts. We investigate the impact of different numerical methods on accuracy and efficiency in the solution of main computational kernels generally arising from mathematical models describing the financial problem. The main kernels involved in the evaluation of these financial derivatives are multidimensional integrals and stochastic differential equations. For this reason we consider different Monte Carlo simulations and various stochastic differential equations discretization schemes. We have established that a combination of the Monte Carlo method with the Antithetic Variates variance reduction technique and the fully implicit Euler scheme developed by Brigo and Alfonsi allows to obtain high efficiency and good accuracy.

Keywords: Life insurance policies, multidimensional integrals, stochastic differential equations.

1. Introduction

This work has been carried out within a project focused on the development of efficient and accurate software for financial evaluation of participating life insurance policies. Some preliminary results concerning previous work by authors on this subject are shown in.[1] Our aim is the analysis of numerical methods for the solution of the main computational kernels in the mathematical models describing the problem, that is, the evaluation of multidimensional integrals and the solution of stochastic differential equations (SDE). The former can represent expected values, the latter often model diffusion processes describing the time evolution of interest rate risk á la Cox, Ingersoll and Ross[2] as well as stock index á la Black and Scholes.[3] We discuss the development of algorithms and software based on different numerical schemes, and investigate their impact on accuracy and efficiency in the solution.
High-dimensional integrals are usually solved via Monte Carlo (MC) method, since its convergence rate does not decrease dramatically as dimension increases. Evaluating financial derivatives in many cases reduces to computing expectations that can be written as integrals of large dimension. In this setting MC proves very promising, but its rate of convergence is quite low. The expected error of the classical MC method depends on the variance of the integrand, therefore, it decreases if the variance is reduced. Different variance reduction techniques are well-known in literature.[4,5] One of the simplest and most widely used one is the *Antithetic Variates*, which we consider here. Many deter-

ministic methods have been proposed as well. One class of such deterministic algorithms, the Quasi-Monte Carlo methods (QMC), is based on *low discrepancy sequences*, that is, deterministic sequences chosen to be more evenly dispersed through the domain of integration than random ones. In this work we test the sequences proposed by Faure,[6] using the implementation reported in.[5] Concerning the solution of SDE, the first method we consider is the well known explicit stochastic Euler scheme. We take into account the Implicit Euler (IE) scheme as well. Further, in order to improve the accuracy of the discrete solution, we consider an higher order method that is a slightly different form of the method proposed by Milstein.[7] Finally, we test a fully implicit positivity-preserving Euler scheme proposed recently by Brigo and Alfonsi,[8] that preserves the monotonicity of the continuous CIR process.
The paper is organized as follows: in section 2 we introduce the financial problem with the aim of describing the main computational kernels involved in the solution. In section 3 we recall the numerical methods, to solve the kernels, we considered. In section 4 we present the mathematical model that we use as benchmark model. In section 5 we report the numerical experiments we performed using the considered numerical methods. Finally, in section 6, we give some conclusions.

2. Computational kernels in participating life insurance policies

In this paper we consider participating life insurance contracts. We analyze portfolios of level premium mixed life participating policies with benefits indexed to the annual return of a specified investment portfolio. This is a typical example of profit-sharing policy that has been widely sold in past years by Italian companies, and it is still widespread nowadays. The insurance company invests the mathematical reserve of the policy in a fund, whose yearly return is shared between the company and the insured. A readjustment rate is contractually defined. Due to the participating rule, the benefits of the policy are random variables with regard to both actuarial and financial uncertainties. The policy is a derivative contract, with underlying the return of the fund. Therefore, its numerical simulation is highly complex and computationally intensive, mainly due to the huge number of involved variables and conditions to take into account for accurate forecasts; the literature on this topic is rich, we recall[9,10,11] among the others. Much effort has been spent on the development of mathematical models to perform mark-to-market evaluation of this kind of contract, that allow to consider the financial uncertainty affecting the benefits.[12,9] We consider a market model with two sources of uncertainty: interest rate risk and stock market risk. We model the former through the one-factor CIR model;[2] stock market uncertainty is considered modelling the stock index $S(t)$ as a Black and Scholes log-normal process. The two fundamental SDE arising in our problem are therefore

$$dr(t) = \alpha[\gamma - r(t)]\, dt + \rho\sqrt{r(t)}\, dZ_r(t) \tag{1}$$

$$dS(t) = r(t)S(t)dt + \sigma S(t)\, dZ_S(t) \tag{2}$$

where Z_r, Z_S are standard Brownian motions correlated by a constant correlation factor. $\mu > 0$ and $\sigma > 0$ are the drift and the volatility parameters of the Black and Scholes process respectively. The parameters in (1) are set in such a way that, on one hand, the equations describe the risk-neutral dynamics of the state variables, on the other, the positivity of the process r is ensured. In this framework, a standard no-arbitrage argument shows that the market price at time t of a random payment $X(r, S; v)$ at time v with $t < v$, subject only to financial uncertainty, is given by:

$$V(t, X) = \mathrm{E}_t \left[X(r, S; v)\ \mathrm{e}^{-\int_t^v r(u)\,du} \right] \qquad (3)$$

where E_t is the risk-neutral expectation implied by the risk-neutral model, conditional to the market information at time t.

3. Numerical methods for the computational kernels

In this section we focus on numerical issues in the solution of the financial problem. The main computational kernels are the evaluation of multidimensional integrals representing expectation values and the solution of SDE.

High-dimensional integrals are usually solved via MC method. The MC algorithm relies on the replacement of a continuous average with a discrete one over randomly selected points. The expected error in MC method is proportional to $\sigma_f/\sqrt{N}$, where σ_f^2 is the variance of the integrand function and N is the number of computed trajectories. The value σ_f depends on the integrand function, but the factor $1/\sqrt{N}$ does not. This shows why MC becomes more and more attractive as the dimension of integral increases, in comparison with deterministic methods that are conversely characterized by a rate of convergence strongly decreasing with respect to the dimension. On the other hand, the rate of convergence is only proportional to $N^{-1/2}$ and special care has to be taken in generating independent pseudo-random points. Convergence can be speeded-up by decreasing the variance. One of the simplest and most widely used variance reduction techniques is the *Antithetic Variates* (AV), which we use in our experiments in conjunction with MC method. This method attempts to reduce variance by introducing negative dependence between pair of replications; we address to[5] for details. For brevity, in the following, we refer to the AV reduction technique combined with MC method as the *AV method*. Deterministic sequences can be used as well to improve the convergence rate of the MC method. These sequences are called *low-discrepancy sequences*; they are chosen to be more evenly dispersed through the region of integration than random sequences. Low discrepancy sequences are sometimes referred to as *quasi-random* sequences. Numerical integration methods based on them are named *low discrepancy methods* or Quasi-Monte Carlo methods. For a complete description of these methods we refer the reader to.[13] Different low discrepancy sequences are well-known in literature; here we confine ourselves to Faure ones. Studies concerning applications of low discrepancy sequences in finance have shown that the errors produced are substantially smaller than the corresponding ones generated by MC.[14,15,16,17] However, the existing theory of the worst case error bounds of QMC algorithms does not explain this phenomenon.[18] Therefore, we can not assume that QMC methods will always outperform MC ones.

Let us now focus on the solution of the SDE involved in the model. In particular, we just focus on equation (1), since all our considerations extend to (2) in a natural way. We consider four numerical schemes. Let $[0, T]$ be a time interval; we consider, for simplicity, a time grid $0 = t_0 < t_1 < \cdots < t_N = T$ with fixed time step $h > 0$, that is, $t_i = ih$, $i = 0, ..., N$. Let us furthermore denote by $\overline{r}$ a time-discrete approximation of function r in (1) on the mentioned time grid. In the following, we will use the notation $\overline{r}(t_i) = \overline{r}(ih) = \overline{r}_i$.

The first method we consider is the well-known explicit Euler stochastic scheme. The explicit Euler approximation of equation (1) is defined by

$$\overline{r}_{i+1} = \overline{r}_i + \alpha[\gamma - \overline{r}_i]\, h + \rho\sqrt{h\overline{r}_i}\; Zr_{i+1}. \tag{1}$$

with $\overline{r}_0 = r(0)$, $Zr_1, Zr_2, \dots$ independent, standard normal random variables. This scheme achieves order-1 weak convergence if appropriate hypotheses on the coefficients of the equation are satisfied.[19,5] Since implicit schemes can reveal better stability properties, we have taken into account the IE scheme reported in[19] too. This method is obtained by making implicit only the pure deterministic term of the equation, while at each time step, the coefficients of the random part of the equation are retained from the previous step. We have

$$\overline{r}_{i+1} = \overline{r}_i + \alpha[\gamma - \overline{r}_{i+1}]\, h + \rho\sqrt{h\overline{r}_i}\; Zr_{i+1}. \tag{2}$$

IE has the same weak order of convergence of corresponding explicit scheme. From the computational point of view, it is not more expensive than (1), but our numerical experiments revealed that it can provide better accuracy. Moreover, we consider a simplified version of the method proposed by Milstein,[7] having order-2 weak convergence.[5,19] It approximates the diffusion process (1) with the following expansion

$$\begin{aligned} \overline{r}_{i+1} = \overline{r}_i + ah + b\sqrt{h}\tilde{Z}_{i+1} + \frac{1}{2}\left(a'b + ab' + \frac{1}{2}b^2b''\right) h\sqrt{h}Zr_{i+1} + \\ + \frac{1}{2}bb'h\left[Zr_{i+1}^2 - 1\right] + \left(aa' + \frac{1}{2}b^2a''\right)\frac{1}{2}h^2 \end{aligned} \tag{3}$$

where $a = \alpha\,[\gamma - r(t)]$ and $b = \rho\sqrt{r(t)}$, and a, b and their derivatives are all evaluated at time t_i. This scheme is more accurate than the Euler method, but it is computationally more expensive.

It is well known that Euler scheme for CIR process can lead to negative values since the Gaussian increment is not bounded from below, thus violating the positivity of the continuous process. Then, finally, we test a fully implicit, positivity-preserving Euler scheme introduced by Brigo and Alfonsi.[8] According to it, if $\delta = \alpha\gamma - \rho^2/2$, the discrete values of r are obtained by means of the following recursion

$$\overline{r}_{i+1} = \Big(\frac{\rho\sqrt{h}Zr_{i+1} + \sqrt{\rho^2 hZr_{i+1}^2 + 4(\overline{r}_i + \delta h)(1 + \alpha h)}}{2(1 + \alpha h)}\Big)^2 \tag{4}$$

4. A benchmark mathematical model

In this section we briefly describe the mathematical model we used as benchmark one to evaluate the impact of the considered numerical methods on efficiency and accuracy in the solution of the problem. We address to[20] for details. Let us consider, at time t, a level premium mixed life participating policy, with term n years, for an insured of age x at the inception of the contract. Let P be the net constant annual premium paid by the policyholder at the beginning of the year, C_0 the initially insured sum, $i \geq 0$ the technical interest rate. We denote with a the time to expiry at time t. We assume that a is integer, therefore the policy starts exactly a years before the valuation date t. Finally, we set $m = n - a$. We suppose that benefit payments occur at integer payments dates

$a+1, a+2, ..., a+m$. Let C_a be the sum insured at the inception date $t-a$; the insured capital C_{a+k} at time $a+k$, $k=1,\ldots,m$, is given by

$$C_{a+k} = C_a \Phi(t,k) - \frac{C_0}{n}\Psi(t,m,k) \tag{1}$$

where, for $k>1$, $\Phi(t,k)$ and $\Psi(t,m,k)$ are random at time t, since they depend on the future values of the readjustment rate, whose expression, at the end of the just terminated year, is given by:

$$\rho_t = \max\left(\frac{\beta R_t - i}{1+i},\, s_{\min}\right). \tag{2}$$

R_t is the return of the segregated fund in the same year, $\beta \in (0,1]$ is the *participation coefficient*, and $s_{\min} \geq 0$ is the yearly minimum guaranteed. The technical rate i, $s_{\min}$ and β are contractually specified, thus their values are fixed at time zero. The quantity βR_t in (2) represents the portion of the fund return which is credited to the policyholder (by increasing the insured sum); the remaining portion $(1-\beta)R_t$ is retained by the insurer and determines his investment gain. The main problem in this framework is the evaluation of the statutory technical reserve of the policy, that is, the level of funding that the company has to maintain by law. In contrast to the traditional framework, the mark-to-market approach is able to consider the financial uncertainty affecting the benefits, coming from the market where the fund's manager invests the policy reserve. Then, we consider the *mark-to-market reserve* of the policy, also called *stochastic reserve*, hence considering a stochastic evolution of interest rates. The stochastic reserve $R(t)$ is defined as the difference between the future obligations of the company $R_{\mathrm{Y}}(t)$ and the future obligations of the insured $R_{\mathrm{X}}(t)$. For the considered policies, we obtain:

$$R_{\mathrm{X}}(t) = P\sum_{k=1}^{m-1} {}_{k}p_{x+a}\, v(t,t+k) \tag{3}$$

where ${}_{k}p_{x+a}$ is the expectation of life of an insured of age $x+a$ years after k years. On the other hand, we obtain:

$$R_{\mathrm{Y}}(t) = \sum_{k=1}^{m} V(t, C_{a+k})\, {}_{k-1|1}q_{x+a} + V(t, C_n)\, {}_{m}p_{x+a} \tag{4}$$

where ${}_{k-1|1}q_{x+a}$ is the probability that the insured of age $x+a$ dies between the year $k-1$ and the year k and V denotes the market price (3). Defining

$$\phi(t,k) = V\big(t, \Phi(t,k)\big) \tag{5a}$$

$$\psi(t,m,k) = V\big(t, \Psi(t,m,k)\big) \tag{5b}$$

it can be proved that

$$R_{\mathrm{Y}}(t) = \sum_{k=1}^{m} {}_{k-1|1}q_{x+a}\left[C_a\phi(t,k) - \frac{C_0}{n}\psi(t,m,k)\right] + {}_{m}p_{x+a}\left[C_a\phi(t,m) - \frac{C_0}{n}\psi(t,m,m)\right]. \tag{6}$$

From the computational point of view, the most intensive kernel is the evaluation of the factors (5a) and (5b), since they are affected by the market uncertainty.

5. Numerical experiments

In this section we show some of the numerical experiments we performed. The test case we use refers to the date of evaluation of bond market 1/4/1999. Bond market data have been estimated following Pacati;[21] in table 1 the parameters used for the CIR model are reported. All the experiments refer to a $n = 20$ years term policy for a 30 years old insured. The residual maturity is $a = 10$, the technical interest rate is set to 4% while the yearly minimum guaranteed is supposed to be $s_{min} = 0\%$ and $\beta = 0.8$. The initial capital is set to $C_0 = 100$. The values of the expectation of life have been computed by the life tables SIM81. Finally, the correlation factor between $dZ_S(t), dZ_r(t)$ in (1), (2) is set to -0.1. We focus on thw two fundamental computational kernels separately.

Table 1. Parameters for the CIR model

	$t = 04/01/1999$
$r(t)$	0.0261356909
α	0.0488239077
γ	0.1204548842
ρ	0.1056548588

In the discussion about techniques for the numerical computation of multidimensional integrals, we fix the Euler scheme for the solution of the SDE. We compared performances of MC, AV and QMC methods. In our simulations the routine snorm of the package ranlib, written by Brown, Lovato and Russell, available through Netlib repository, has been used to generate standard normally distributed values. In QMC method the values of the Faure sequences have been mapped to values from standard normal random variables via the routine dinvnr of the package dcdflib,[22] available through Netlib repository as well. The routine approximates the inverse normal cumulative function via Newton's method, as described in.[5] First we show our results in the estimation of the obligations of the insurance company R_Y, of functions ϕ, ψ (5), and of the spot interest rates r (1). A monthly discretisation step has been considered in the numerical solution of the SDE. Negative values resulting in the application of non-positivity preserving schemes have been set to zero. Finally, the integral of function r in (3) has been evaluated by means of the trapezoidal rule.

In table 2 we report the values of the obligations of the company estimated via the three integration methods, for different values of the number N of simulated trajectories. In order to estimate the error, an "almost true" values is needed: we assumed as true expected value the sample mean computed via AV method with $N = 20 \times 10^6$. We observe that with the classical MC method we obtain three significant digits for $N \geq 10^4$; applying AV method the same accuracy is reached just for $N = 10^3$. Moreover, to obtain four significant digits with AV we need $N = 2 \times 10^4$ simulations while with MC we need $N = 5 \times 10^4$ simulations. Since the application of AV technique at most doubles the computational cost, we deduce that efficiency is strongly improved in these cases. All the experiments we performed confirmed this. On the other hand, QMC method, with the Faure sequences, does not perform well in this framework; QMC needs, to deliver two significant digits, at least $N = 5 \times 10^4$. In table 3 we show the CPU time spent by MC and AV methods. In order to evaluate the overhead of AV with respect to MC method we also reported the ratio beetween the two values. We observe that the execution time of AV is

Table 2. Obligations of the company R_Y (6). N is the number of simulated trajectories; in the second column the expected value, that is, the sample mean computed via AV with $N = 20 \times 10^6$ is reported.

N	Expected Value	MC	AV	QMC
1000	85.530725	85.330736	85.538849	103.593865
2500	85.530725	85.446784	85.513984	94.824499
5000	85.530725	85.490321	85.526349	94.824499
10000	85.530725	85.515832	85.529525	89.144140
20000	85.530725	85.556925	85.532012	87.049162
50000	85.530725	85.531398	85.532053	85.537668
100000	85.530725	85.535615	85.532456	85.317408

never the double of execution time of MC method, even though it requires the generation of a number of simulations that is double with respect to MC. The ratio between the two execution times is always about 1.2. All the experiments done until now show that the

Table 3. MC and AV CPU times in seconds for different values of the number N of trajectories.

N	1000	2500	5000	10000	20000	50000	100000
MC	5.84	17.75	32.90	66.24	130.75	334.19	668.16
AV	7.87	19.73	39.53	79.01	157.13	393.59	786.06
$\frac{AV}{MC}$	1.34	1.11	1.2	1.19	1.2	1.18	1.18

use of AV method allows to obtain the same accuracy as MC method with a number of replications that is reduced by a factor near to four. Then AV method results in a large gain in terms of execution time.

We now turn to the numerical solution of SDE (1). We tested the four SDE discretization schemes Euler, IE, Milstein and Brigo-Alfonsi described above. In order to estimate the error we refer to the *deterministic solution* obtained neglecting the stochastic term in equation (1). Note that the Euler and IE schemes can lead to negative values for CIR process; when this happens we set to zero the computed negative value. In figure 1 we represent the values of the absolute errors of the interest rates computed at each time with the four different SDE methods and with both MC method and AV one, for $N = 5 \times 10^3$. A monthly discretisation step size is used. We observe that, as we expected, AV outperforms from the accuracy point of view MC method in the estimation of the spot interest rates too. For example, the obtained errors using AV are never greater than 10^{-3}. In figure 2, we plot the same variables obtained with $N = 5 \times 10^3$ replications of MC and with $N = 10^3$ replications of AV method; we observe that errors estimated via AV with $N = 10^3$ are almost always lower than those estimated via MC with $N = 5 \times 10^3$. We note that the absolute error lies for all the methods in the interval $]10^{-6}, 10^{-3}[$. In particular, for values of $N = 10^3$ and 5×10^3 the accuracy given by the four methods are almost comparable. The Euler method exhibits the worst behavior; IE is comparable with Milstein scheme, but the computational complexity of the latter is higher, while the Brigo-Alfonsi method reaches the highest level of accuracy. In table 4 we report the execution times for AV method using the four different schemes for the SDE. The results show that the Milstein scheme is more time-consuming than the other ones; the Brigo-Alfonsi method is slightly more expensive than the IE one. Finally, in figure 3 are reported, for MC and AV methods,

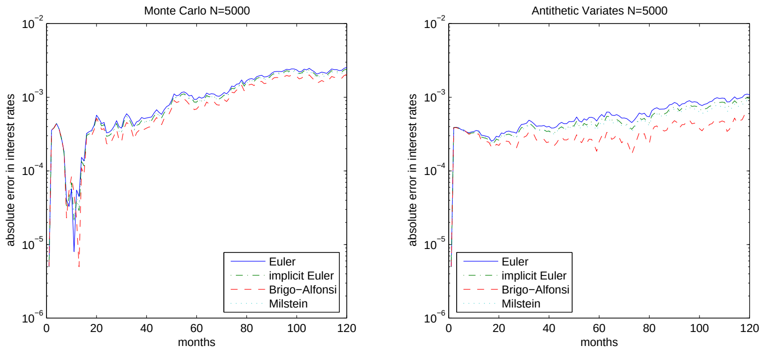

Fig. 1. Absolute errors in the estimation of interest rates computed with MC and AV vs time. $N = 5 \times 10^3$ trajectories have been simulated; the discretisation step size is monthly. At each time t the average over trajectories is represented.

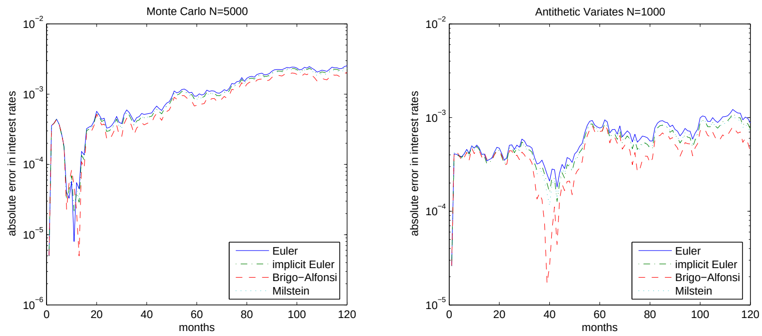

Fig. 2. Absolute errors in the estimation of interest rates computed with MC for $N = 5 \times 10^3$ and with AV for $N = 10^3$ vs time; the discretisation step size is monthly. At each time t the average over trajectories is represented.

Table 4. Execution times in seconds for MC and AV methods.

Antithetic Variates							
N	1000	2500	5000	10000	20000	50000	100000
Euler	7.87	19.73	39.53	79.01	157.13	393.59	786.06
Implicit Euler	8.30	20.80	41.63	83.69	166.34	414.45	829.81
Brigo-Alfonsi	8.65	21.67	43.30	86.69	173.52	433.25	868.01
Milstein	9.98	25.04	49.89	99.77	199.39	498.56	1000.37

the values of the root-mean-square absolute error (RMSE) defined by

$$RMSE = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\overline{r}_i - r_{det}(t_i))^2}$$

where M is the number of time steps from t_0 to t_N, $\overline{r}_i$ is the sample mean at time t_i and $r_{det}(t_i)$ is the value at time t_i of the deterministic solution. Figure 3 shows that the RMSE values of all the considered methods significantly reduce using AV. IE and Milstein have comparable behavior, while the Brigo-Alfonsi method outperforms all the other ones, especially for high values of N.

6. Conclusions

In this paper we discussed the development of an efficient and accurate software for the evaluation of participating life insurance policies. Our activity was motivated by

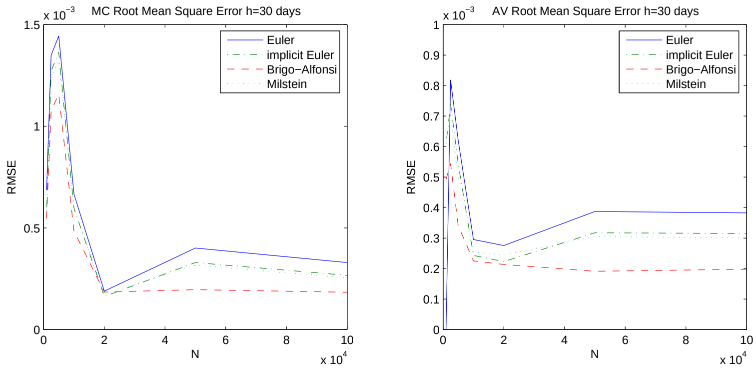

Fig. 3. On the left: RMSE in the SDE solution with MC method versus N; on the right: RMSE in the SDE solution with AV versus N. The discretisation step size is monthly.

actual needs of italian insurance companies: indeed, these contracts are widespread in italian insurance market, and their evaluation is computationally intensive. We focused on the fundamental computational kernels arising in most recent mathematical models; we tested different numerical schemes for their solution and we developed a software using the combination of numerical methods that exhibited the best behavior in terms of accuracy and efficiency.

REFERENCES

1. S. Corsaro, P. De Angelis, Z. Marino and F. Perla, Numerical aspects in financial evaluation of partecipating life insurance policies, in *VIII Italian-Spanish Meeting on Financial mathematics*, (Verbania, Italy, 2005).
2. J. Cox, J. Ingersoll and S. Ross, *Econometrica* **53**, 385 (1985).
3. F. Black and M. Scholes, *J. of Political Economy* **81** (1973).
4. P. Boyle, M. Broadie and P. Glasserman, *Monte Carlo Methods for Security pricing*, in *Monte Carlo Methodologies and Application for Pricing and Risk Management*, ed. B. Depire (Risk Books, 1998), pp. 15–44.
5. P. Glasserman, *Monte Carlo Methods in Financial Engineering* (Springer-Verlag, New York, 2004).
6. H. Faure, *J. of Number Theory* **41**, 47 (1992).
7. G. Milstein, *Theory of Probability and its Appl.* **23**, 396 (1978).
8. D. Brigo and A. Alfonsi, *Finance & Stochastics* **IX** (2005).
9. M. De Felice and F. Moriconi, *Astin Bulletin* **35**, 79 (2005).
10. A. Bacinello, *The J. of Risk and Insurance* **70**, 461 (2003).
11. L. Ballotta and S. Haberman, *Insurance: Math. and Economics* **38**, 195 (2006).
12. M. De Felice and F. Moriconi, *Embedded Value in Life Insurance*, working paper (Italy, 2005).
13. H. Niederreiter, *SIAM CBMS-NSF Regional Conf. Series in Appl. Math* **63** (1992).
14. C. Joy, P. Boyle, and K. Tan, *Quasi-Monte Carlo Methods in Numerical Finance*, in *Monte Carlo Methodologies and Application for Pricing and Risk Management*, ed. B. Depire (Risk Books, 1998), pp. 269–280.
15. A. Papageorgiou and J. Traub, *Computers in phisycs* **11**, 574 (1997).
16. S. Paskov and J. F. Traub, *J. of Portfolio Management* **Fall**, 113 (1995).
17. F. Perla, *Int. J. of Pure and Appl. Math.* **5**, 451 (2003).

18. I. Sloan and H. Wozniakowsky, *J. of Complexity* **14**, 1 (1998).
19. P. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer-Verlag, Berlin Hiedelberg, 1992).
20. C. Pacati, *Valutazione di Portafogli di Polizze Vita con Rivalutazione agli Ennesimi*, Modelli per la Finanza Matematica, Working paper 38 (Italy, 2000).
21. C. Pacati, *Estimating the Euro Term Structure of Interest Rates*, Models for Mathematical Finance, Working paper 32 (Italy, 1999).
22. B. Brown, J. Lovato and K. Russell, *DCDFLIB: A Library of C Routines for Cumulative Distribution Functions, Inverses, and Other Parameters*, Tech. Rep. 77030, Department of Biomathematics (The University of Texas, Houston, 1994).