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Crack initiation in elastic bodies

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In this talk, we illustrate the results obtained in [1] in collaboration with Antonin Chambolle (Ecole Polytechnique, Paris, France) and Marcello Ponsiglione (Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany) and concerning crack initiation in elastic bodies.

Griffith's criterion for crack propagation in hyper-elastic bodies [5] asserts that, during a load process, a crack Γ can grow only if the energy dissipated to enlarge the crack, which is basically assumed to be proportional to the area of the cracked surface, is balanced by the corresponding release of bulk energy. According to *Griffith's theory*, if Ω represents a two dimensional hyper-elastic body, ψ is a boundary datum and Γ is a curve in Ω parametrized by arc length, then the crack $\Gamma(l_0)$ is in equilibrium if

$$k(l_0) := \limsup_{l\to 0^+} \frac{W(u(l_0)) - W(u(l_0+l))}{l} \leq k, \tag{0.1}$$

where $u(l_0)$ and $u(l_0+l)$ are the displacements associated to ψ and to the cracks $\Gamma(l_0)$ and $\Gamma(l_0+l)$ respectively, W is the *bulk energy functional* and k is the *toughness* of the material. A *quasistatic crack evolution* is determined by an increasing function $t \to l(t)$ satisfying the *Griffith's criterion* for crack propagation, which asserts that for every t we have $k(l(t)) \leq k$ and

$$(k - k(l(t)))\dot{l}(t) = 0,$$

i.e., $\Gamma(l(t))$ propagates if and only if (0.1) holds with equality.

The case of *crack initiation*, i.e., when there is not a pre-existing crack in the body, corresponds to the situation $l_0 = 0$. A fundamental role in the problem is played by the singularities of the body, namely the behavior of the elastic energy concentration of the deformation. Experiments show that small cracks usually appear near sufficiently strong singular points of the body, whose position are essentially determined by its inhomogeneities. If the singularities of the body are sufficiently weak (for instance this is the case of homogeneous isotropic materials), a lot of results in the literature of material science show that the derivative in (0.1) for $l_0 = 0$ is equal to zero. The conclusion is that Griffith's criterion is not adequate to predict crack initiation (and, as a consequence, a crack evolution originating from an uncracked configuration). These results require that the path of the crack is sufficiently regular (a line or a smooth curve). In [1] we prove that the same conclusion holds in the class of all one dimensional closed sets with a finite number of connected components. More precisely we prove that the limit in (0.1) is zero if $\Gamma(l)$ is any family of closed sets with length less than l and with at most m connected components, with m independent of l. In particular we do not prescribe the path nor the shape of the cracks.

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The main tool we employ to address the problem of crack initiation is a local minimality result for the functional

$$\int_\Omega f(x,\nabla v)\,dx + k\mathcal{H}^1(\Gamma), \tag{0.2}$$

where Ω is a bounded Lipschitz open set in $\mathbb{R}^2$, $k>0$, and $f:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is a Carathéodory function strictly convex and C^1 in the second variable, satisfying standard p-growth estimates with $p>1$, and such that $f(x,0)=0$. $\mathcal{H}^1$ stands for the one-dimensional Hausdorff measure, which coincides with usual notion of length on the class of regular curves. The functional (0.2) is a variant of a functional which has first appeared in the theory of image segmentation, in a celebrated paper by Mumford and Shah [6].

In our setting, the crack Γ belongs to the class $\mathcal{K}_m(\bar\Omega)$, that is the family of closed sets in $\bar\Omega$, with at most m connected components, and with finite length, i.e., $\mathcal{H}^1(\Gamma)<+\infty$. The displacement v belongs to the Sobolev space $W^{1,p}(\Omega\setminus\Gamma)$ and satisfies the boundary condition

$$v=\psi \qquad \text{on } \partial_D\Omega\setminus\Gamma,$$

where $\partial_D\Omega\subseteq\partial\Omega$ is open in the relative topology, and ψ is (the trace of a function) in $W^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Let u_Γ be a minimum energy displacement relative to ψ and Γ, i.e., let u_Γ be a minimizer for

$$\min\left\{\int_\Omega f(x,\nabla v)\,dx \,:\, u\in W^{1,p}(\Omega\setminus\Gamma), v=\psi \text{ on } \partial_D\Omega\right\}. \tag{0.3}$$

We denote by u the *elastic configuration* of Ω relative to the boundary datum ψ, i.e., a solution of (0.3) with $\Gamma=\emptyset$, and we assume that u admits *uniformly weak singularities* in $\bar\Omega$, i.e.,

$$\|\nabla u\|^p_{L^p(B_r\cap\Omega)}\le Cr^\alpha \tag{0.4}$$

for some constants $\alpha>1$ and $C>0$ and for every ball B_r with radius r. Condition (0.4) means that the bulk energy of the elastic configuration u in a ball $B_r(x)$ is negligible with respect to the length of $\partial B_r(x)$ as r goes to zero, uniformly in $x\in\bar\Omega$.

Our main result is the following Theorem, which establishes that under the previous assumptions small cracks are not energetically convenient for the functional (0.2).

Theorem. Assume that u admits only uniformly weak singularities in $\bar\Omega$. Then there exists a critical length l^* depending only on Ω, f, k, ψ and m such that for all $\Gamma\in\mathcal{K}_m(\bar\Omega)$ with $\mathcal{H}^1(\Gamma)<l^*$ we have

$$\int_\Omega f(x,\nabla u)\,dx < \int_\Omega f(x,\nabla u_\Gamma)\,dx + k\mathcal{H}^1(\Gamma). \tag{0.5}$$

Let us briefly comment the assumption about the singularities of u. The minimality result is false if the elastic solution u has *strong singularities*, namely if there exists $x\in\bar\Omega$ such that

$$\limsup_{r\to0}\frac{1}{r}\int_{B_r(x)\cap\bar\Omega}|\nabla u|^p\,dx=+\infty. \tag{0.6}$$

In fact condition (0.6) ensures that it is energetically convenient to create a small crack $\Gamma := \partial B_r(x)$ around x: the surface energy needed to create such a crack is proportional to r, while the corresponding release of bulk energy is by (0.6) bigger than r if r is small enough.

The critical case when the right hand-side of (0.6) is a constant $0 < C < \infty$ corresponds to the singularity appearing around the tip of the crack. In this case the celebrated Irwin's formula states that the release of bulk energy per unit length along rectilinear increments of the crack is equal to the so called *mode III stress intensity factor* K_{III}, which is proportional to C. In our class of cracks $\mathcal{K}_m(\bar{\Omega})$ we have that if C is small enough, then the release of bulk energy per unit length is less than k, and therefore our minimality result still holds, while it is false if C is too large. We can not fill the gap, and therefore we do not achieve a sharp Irwin type formula in our class of cracks.

Our study is in part motivated by the variational model for quasistatic crack propagation proposed by Francfort and Marigo in [4], where the issue of *brutal crack initiation* is discussed. A crack evolution $\Gamma(t)$ is brutal when it is of the type

$$\Gamma(t) = \emptyset \quad \text{for every } t \leq t_i$$

and

$$\inf_{t>t_i} \mathcal{H}^1(\Gamma(t)) > 0,$$

where t_i is referred to as *time initiation* of the crack. As a consequence of our main result, we prove that, in the framework of Francfort-Marigo theory and within our class of admissible cracks $\mathcal{K}_m(\bar{\Omega})$, crack initiation is always brutal whenever the elastic displacement presents sufficiently weak singularities. On the contrary, in presence of a point x of strong singularity the crack initiation is progressive: a crack departs from x at the initial time of loading and with zero velocity.

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