NUMERICAL INTEGRATION SCHEMES
FOR HYPERSINGULAR INTEGRALS
ON THE REAL LINE

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Abstract.

In this paper we consider the numerical approximation of hypersingular integral equations coming from Neumann two-dimensional elliptic problems defined on a half-plane by using a Petrov-Galerkin infinite BEM approach as discretization technique. An analysis of the singularities, arising during the double integration process needed for the generation of the stiffness matrix elements related to the infinite mesh elements, is carried out and consequently efficient quadrature schemes are proposed. Several numerical results are presented.

Keywords: Petrov-Galerkin BEM; hypersingular integrals; infinite elements.

1. Introduction

Modelling unbounded domains is an important issue in engineering: electromagnetism, fluid dynamics, elastodynamics, soil and soil-structure mechanics are research areas where unbounded domains are of usual concern. During these last years, potential, linear elasticity and Helmoltz problems on a half-plane, with Dirichlet or pure Neumann boundary conditions, reformulated in terms of boundary integral equations on the real line, have been investigated (see e.g. Ref. [1,2,3,4]) and some numerical treatments in the form of boundary element methods have been suggested. Fundamental solutions for a half-space can be employed to solve the problem along the boundary.[5] Such an approach is not much used, due to the complexity of the involved kernels. An interesting alternative is represented by full-space solution coupled with infinite boundary elements, i.e. special elements extending to infinity, proposed in Ref. [6] to model the unknown displacement field along the boundary. These elements extend to infinity the discretization domain: a survey on this relatively recent topic can be found in Bettess' book.[7] In Ref. [8] efficient numerical integration schemes have been applied to the infinite element defined in Ref. [6]. In all cited works, numerical analysis and numerical tests are based on the boundary element collocation method. In the present work, using the fundamental solutions for a full-space, we consider hypersingular integral equations coming from Neumann two-dimensional elliptic problems defined over a half-plane with unbounded boundary coincident with the real line and we use a suitable Petrov-Galerkin infinite BEM approach as discretization technique. An analysis of the singularities, arising during the double integration process

needed for the generation of the stiffness matrix elements related to the infinite mesh elements is carried out. Then, a generalization of numerical quadrature schemes, appeared in Ref. 9, is proposed to compute the above-mentioned integrals. In the last Section, numerical results are presented.

2. Model problem and its discretization

Let $\Omega \subset \mathbb{R}^2$ be an half-plane with unbounded linear boundary $\Gamma = \mathbb{R}$. We consider the Neumann boundary value problem:

$$(1) \qquad \begin{cases} Lu(x) = 0 & \text{for } x \in \Omega \\ p(x) := (T_x u)(x) = \overline{p}(x) & \text{for } x \in \Gamma \end{cases},$$

where L is linear elliptic partial differential operator of second order acting on u, describing the field equation inside the domain Ω, $(T_x u)(x)$ the co-normal derivative of u for $x \in \Gamma$ and $\overline{p}$ is a given function. The behavior of solutions at infinity is prescribed too, and it can be, for instance, one of the following:

$$(2) \qquad u(x) = C \ln(|x|) + O(1) \quad \text{for } |x| \to \infty; \quad u(x) = C_\theta + O(|x|^{-1}) \quad \text{for } |x| \to \infty$$

where C, C_θ are suitable constants, the latter dependent on the inclination of the radius stemming from the origin of the reference system with an axis lying on Γ. Problem (1) may be rewritten in the integral form:

$$(3) \qquad \frac{1}{2} p(x) = (K'p)(x) - (Du)(x), \qquad x \in \Gamma,$$

using the adjoint double-layer potential and the hypersingular integral operator:

$$(K'p)(x) = \fint_\Gamma T_x U(x,y) p(y) \, \mathrm{d}\gamma_y, \qquad (Du)(x) = \fint_\Gamma T_x T_y U(x,y) u(y) \, \mathrm{d}\gamma_y.$$

The operator K' is defined by Cauchy singular integral and D is defined by a hypersingular finite part integral in the sense of Hadamard[10] due to the respective integral kernels singularity. In fact, the definition of these boundary potentials is based on a fundamental solution $U(x,y)$ of the operator L, which presents a log-singularity for $x = y$; as a consequence, for instance, integral kernel in D behaves like $O(|y-x|^{-2})$ as $y \to x$. By using Neumann boundary datum, we obtain from (3) a boundary integral equation of the first kind for the unknown function u on Γ, of the form:

$$(4) \qquad (Du)(x) = \overline{f}(x), \qquad x \in \Gamma,$$

where $\overline{f} = -\frac{1}{2}\overline{p} + K'\overline{p}$. Note that in scalar problems the opearator K', defined on the real line, vanishes.

Now, denoting with $\mathcal{D}'(\Gamma)$ the space of distributions on Γ and introducing the weight functions $(1+\rho^2)^\alpha$, $\alpha \in \mathbb{R}$, where, for $\xi \in \mathbb{R}$, $\rho(\xi) = |\xi|$, let us consider suitable weighted Hilbert spaces on the real line, as defined in Ref. 11:

$$\mathcal{H}_0^{\frac{1}{2}}(\Gamma) := \left\{ u \in \mathcal{D}'(\Gamma) : (1+\rho^2)^{-1/4} u \in L^2(\Gamma), \right.$$

$$\left. \int_{\mathbb{R}^+} \int_{\mathbb{R}} \frac{|u(\xi+t) - u(\xi)|^2}{t^2} \, \mathrm{d}\xi \, \mathrm{d}t < \infty \right\},$$

$$\mathcal{H}_\beta^{\frac{1}{2}}(\Gamma) := \left\{ u \in \mathcal{D}'(\Gamma) : (1+\rho^2)^{\beta/2} u \in \mathcal{H}_0^{\frac{1}{2}}(\Gamma) \right\}, \qquad \beta \in \mathbb{R}.$$

Then, having set:

$$W =: \mathcal{H}^{\frac{1}{2}}_{\beta}(\Gamma), \quad \beta < 0, \qquad V := \mathcal{H}^{\frac{1}{2}}_{0}(\Gamma)$$

and denoted with $V' := (\mathcal{H}^{\frac{1}{2}}_{0}(\Gamma))'$ the dual space of V with respect to the classical duality product in $L^2(\Gamma)$, it holds:

$$(5) \qquad\qquad\qquad\qquad D : W \to V'$$

and the weak form of (4) reads:

given $\overline{f} \in V'$, find $u \in W$ such that

$$(6) \qquad\qquad (Du, v)_{L^2(\Gamma)} = (\overline{f}, v)_{L^2(\Gamma)}, \qquad \forall\, v \in V.$$

Problem (6) admits a unique solution up to an additive constant.[12] For a Petrov-Galerkin BEM (PGBEM) discretization of problem (6), let us consider a bounded subset $\Gamma_B \subset \Gamma$, where we introduce a not necessarily uniform mesh $\mathcal{T}_h = \{e_1, \cdots, e_{M_h}\}$, such that $length\,(e_i) \leq h$ and $\overline{\Gamma}_B = \cup_{i=1}^{M_h} \overline{e}_i$. Let us denote by $\mathbb{P}_r$, $r \geq 1$, the space of polynomials of degree less than or equal r, and consider the space:

$$X^r_h = \{v_h \in C^0(\overline{\Gamma}_B) : v_{h|e_i} \in \mathbb{P}_r, \forall\, e_i \in \mathcal{T}_h\}$$

of finite elements of degree $\leq r$ related to $\mathcal{T}_h$. Further, we define two unbounded elements, let's say $e_{-\infty}, e_{+\infty}$, such that: $e_{-\infty} \cup e_{+\infty} = \Gamma \setminus \Gamma_B$. Denoted by x_i, $i = 1, \cdots, N_h$, the nodes on Γ_B, we introduce the finite-dimensional space of test functions $V_h = \text{span}\{\varphi_i, i = 1, \cdots, N_h\} \subset V$, where $\varphi_i, i = 2, \cdots, N_h - 1$, are the usual finite element basis functions, obtained with the standard assembling of the local lagrangian polynomials of degree $\leq r$ introduced on each element of the mesh $\mathcal{T}_h$. Functions φ_1, φ_{N_h} have unbounded supports, given by $e_{-\infty} \cup e_1$, $e_{M_h} \cup e_{+\infty}$ respectively, and must satisfy the decaying property:

$$\varphi_i(x) = O(|x|^{\alpha}), \quad \alpha < 0, \quad |x| \to \infty, \quad i = 1, N_h.$$

In any case, $\varphi_i(x_j) = \delta_{ij}$, $i, j = 1, \cdots, N_h$. Similarly, we consider the finite-dimensional space of trial functions $W_h = \text{span}\{\psi_i, i = 1, \cdots, N_h\} \subset W$, where $\psi_i = \varphi_i$, $i = 2, \cdots, N_h - 1$. Functions ψ_1, ψ_{N_h} have unbounded supports, given by $e_{-\infty} \cup e_1$, $e_{M_h} \cup e_{+\infty}$ respectively, and assume the same known behavior of the solution of (1) at infinity. Further, $\psi_i(x_j) = \delta_{ij}$, $i, j = 1, \cdots, N_h$.

Remark 2.1. Local basis functions on each mesh element can be obtained as usual, suitably "lifting" functions defined on the reference element $e = (0, 1)$. In particular, on the infinite element $e_{+\infty} = (a, +\infty)$, $a > 0$, one can define respectively trial and test functions starting from functions, for instance, of the type:

$$(7) \qquad\qquad \tilde{\psi}(s) = 1 - \log(1 - s) \quad \text{or} \quad \tilde{\psi}(s) = 1, \qquad s \in (0, 1),$$

$$(8) \qquad\qquad\qquad \tilde{\varphi}(s) = 1 - s, \qquad s \in (0, 1),$$

and then using the mapping $e_{+\infty} \to e$ defined by $s = 1 - \frac{a}{x}$. Similarly, we can define trial and test functions on $e_{-\infty}$.

At this stage, we can write the Petrov-Galerkin finite-dimensional discretization of the weak problem (6):

given $\overline{f} \in V'$, find $u_h \in W_h$ such that

$$(9) \qquad (Du_h, v_h)_{L^2(\Gamma)} = (\overline{f}, v_h)_{L^2(\Gamma)}, \qquad \forall\, v_h \in V_h.$$

Note that the algebraic restatement of (9) leads to a system of N_h linear equations of the form:

$$(10) \qquad A\,X = F$$

in the N_h unknowns $X_i = u_h(x_i)$, $i = 1, \cdots, N_h$, where the elements of A are double integrals with hypersingular kernel. Since block $A(2 : N_h-1, 2 : N_h-1)$ coincides with the standard symmetric Galerkin BEM stiffness matrix defined in correspondence to a bounded boundary, for the generation of its elements we can use the efficient quadrature schemes proposed in Ref. 9. For the remaining matrix elements A_{ij}, $i = 1, N_h$, $j = 1, \cdots, N_h$ and $i = 1, \cdots, N_h$, $j = 1, N_h$, we can use the classical FEM technique, working element by element on the mesh introduced on Γ and then suitably assembling partial results. In this way, we reduce their evaluation to the computation of double integrals of the form:

$$(11) \qquad \int_{e_m} \varphi_i(x) \int_{e_\ell} T_x T_y U(x, y) \psi_j(y) \, d\gamma_y \, d\gamma_x.$$

We observe that this decomposition is possible if, when $e_m = e_\ell$, we define both the inner and outer integrals in (11) as finite parts, while when e_m, e_ℓ are consecutive, only the outer integral in (11) is defined in the finite part sense. When $\overline{e}_m \cap \overline{e}_\ell = \emptyset$, the kernel singularity does not effectively arise, and the double integral in (11) can be efficiently treated by the product of two standard Gaussian quadrature rules. Further, when both e_m, e_ℓ are bounded, we can apply numerical schemes presented in Ref. 9. Hence, in the sequel, we will illustrate quadrature schemes for the approximation of double integrals over consecutive or coincident mesh elements, when one or both are unbounded.

Finally, note that, since system (10) is singular, we can proceed as described in Ref. 13 for its regularization, obtaining the approximate solution u_h such that $\int_{\Gamma_B} u_h(x)\, d\gamma_x = 0$.

3. Basic definitions and quadrature rules

Here, we will introduce some typical finite part integrals over unbounded intervals of the real line, which we will encounter in the inner and outer integrations. Firstly, we define for $a \in \mathbb{R}^+$ and $x \in (a, +\infty)$:

$$\fint_a^{+\infty} \frac{1}{(y-x)^2} \, dy = \lim_{\epsilon \to 0^+} \left[\int_a^{x-\epsilon} \frac{1}{(y-x)^2} \, dy + \int_{x+\epsilon}^{+\infty} \frac{1}{(y-x)^2} \, dy - \frac{2}{\epsilon} \right]$$

$$(1) \qquad = \lim_{\epsilon \to 0^+} \left[\int_a^{x-\epsilon} \frac{1}{(y-x)^2} \, dy + \lim_{M \to +\infty} \int_{x+\epsilon}^{M} \frac{1}{(y-x)^2} \, dy - \frac{2}{\epsilon} \right] = \frac{1}{a-x}.$$

The standard change of variable $t = \frac{a}{y}$ leads to:

$$(2) \qquad \fint_a^{+\infty} \frac{1}{(y-x)^2} \, dy = \frac{a}{x^2} \fint_0^1 \frac{1}{(t - \frac{a}{x})^2} \, dt$$

where the right-end side can be evaluated using the classical definition of Hadamard finite part integral over a bounded interval[14] .

If $f \in \mathcal{H}_{\beta}^{\frac{1}{2}}(\Gamma)$, $\beta < 0$, then we can generalize the definition (1) in the following way: let $x \in (a, +\infty)$, $a \in \mathbb{R}^+$, then

$$
(3) \qquad =\!\!\!\!\!\!\!\int_a^{+\infty} \frac{f(y)}{(y-x)^2}\, \mathrm{d}y = =\!\!\!\!\!\!\!\int_a^{+\infty} \frac{f(y) - f(x)}{(y-x)^2}\, \mathrm{d}y + =\!\!\!\!\!\!\!\int_a^{+\infty} \frac{f(x)}{(y-x)^2}\, \mathrm{d}y
$$

$$
= \lim_{\epsilon \to 0^+} \Big[\int_a^{x-\epsilon} \frac{f(y) - f(x)}{(y-x)^2}\, \mathrm{d}y + \lim_{M \to +\infty} \int_{x+\epsilon}^M \frac{f(y) - f(x)}{(y-x)^2}\, \mathrm{d}y \Big] + \frac{f(x)}{a-x}.
$$

For the outer integration in (11), we will have to deal with finite part integrals of the form $=\!\!\!\!\!\!\!\int_a^{+\infty} \frac{f(x)}{x-a}\, \mathrm{d}x$, where $a \in \mathbb{R}^+$ and $f \in \mathcal{H}_0^{\frac{1}{2}}(\Gamma)$. We will define them as:

$$
=\!\!\!\!\!\!\!\int_a^{+\infty} \frac{f(x)}{x-a}\, \mathrm{d}x = \lim_{\epsilon \to 0^+} \Big[\int_{a+\epsilon}^{+\infty} \frac{f(x)}{x-a}\, \mathrm{d}x + f(a) \ln(\epsilon) \Big]
$$

$$
(4) \qquad = \lim_{\epsilon \to 0^+} \Big[\lim_{M \to +\infty} \int_{a+\epsilon}^M \frac{f(x)}{x-a}\, \mathrm{d}x + f(a) \ln(\epsilon) \Big].
$$

Note that for the standard change of variable $s = 1 - \frac{a}{x}$ it holds:

$$
(5) \qquad =\!\!\!\!\!\!\!\int_a^{+\infty} \frac{f(x)}{x-a}\, \mathrm{d}x = =\!\!\!\!\!\!\!\int_0^1 \frac{1}{s} \Big[\frac{1}{1-s} f(\frac{a}{1-s}) \Big]\, \mathrm{d}s + f(a) \ln(a).
$$

We observe that for $x \in (-\infty, b)$, $b \in \mathbb{R}^-$, definitions analogous to $(1), (3), (4)$ can be introduced.

To compute integrals (11) required by Petrov-Galerkin method, we will use certain product quadrature rules of interpolatory type, based on the zeros of Legendre polynomials. They are of the form:

$$
(6) \qquad \int_{-1}^{+1} k(s,t) f(t)\mathrm{d}t = \sum_{i=1}^n w_i^I(s) f(t_i) + R_n^I(f; s)
$$

and are obtained by replacing $f(t)$ by its Lagrangian interpolation polynomial associated with the nodes $\{t_i\}_{i=1}^n$. These are the very familiar zeros of Legendre polynomial $L_n(t)$ of degree n. The weights $\{w_i^I(s)\}_{i=1}^n$ have been obtained in Ref. 9 for many kernels of interest in the applications: in particular, for kernels of type $\frac{1}{t-a_s}$, and $\frac{1}{(t-a_s)^2+b_s^2}$ which will arise in the following Section. Convergence results are reported in Ref. 15.

A second rule we shall need is a formula of Gauss-Radau type for Hadamard finite-part (HFP) integrals, proposed in Ref. 16:

$$
(7) \qquad =\!\!\!\!\!\!\!\int_0^1 \frac{f(s)}{s}\mathrm{d}s = w_0^{GR} f(0) + \sum_{i=1}^n w_i^{GR} f(s_i^{GR}) + R_n^{GR}(f)
$$

where $\{s_i^{GR}\}_{i=1}^n$ denote the Legendre zeros mapped onto $(0,1)$, $w_i^{GR} = \frac{\lambda_i}{2 s_i^{GR}}$, $i = 1, \cdots, n$ and $w_0^{GR} = -\sum_{i=1}^n w_i^{GR}$. This quadrature formula is obtained by replacing $f(s)$ by its n-th degree interpolation polynomial associated with the nodes $\{0, s_1^{GR}, \cdots, s_n^{GR}\}$ and it is exact whenever $f(s)$ is a polynomial of degree $\leq 2n$. Convergence results are proved in

Ref. 16.

Further, the outer integral in (11) often presents integrand functions having at most logarithmic endpoint singularities. Hence, considering for $\int_0^1 f(s)\,\mathrm{d}s$, where $f(s)$ is smooth except in $0, 1$, the change of variable $s = \phi(\tilde{s})$, with $\phi : (0,1) \to (0,1)$, $\phi'(\tilde{s}) \geq 0$, one obtains:

$$(8) \qquad \int_0^1 f(s)\mathrm{d}s = \int_0^1 f(\phi(s))\phi'(s)\mathrm{d}s.$$

If furthermore $\phi^{(i)}(0) = 0$, $i = 1, \cdots, p-1$, $p \geq 1$ and $\phi^{(i)}(1) = 0$, $i = 1, \cdots, q-1$, $q \geq 1$, we can make the integrand in the right-end side of (8) as smooth as we like simply by taking integers p, q sufficiently large and numerically evaluate this integral by means of a standard Gaussian rule. We have used the following transformation, proposed in Ref. 17:

$$(9) \qquad \phi(s) = \frac{(p+q-1)!}{(p-1)!(q-1)!} \int_0^s u^{p-1}(1-u)^{q-1}\mathrm{d}u$$

where the integral can be evaluated exactly up to machine precision by means of a $\lfloor \frac{p+q}{2} \rfloor$-points Gauss-Legendre rule.

4. Evaluation of Infinite PGBEM matrix elements

Before starting this Section, we emphasize that in the sequel we will consider the hypersingular kernel $|y - x|^{-2}$, but the proposed numerical schemes can be applied in presence of more general kernels having the same type of singularity. As already stated, we will consider separately double integrals of the form:

$$(1) \qquad \mathbf{a}) \qquad ⨍_{e_{+\infty}} \tilde{\varphi}_{e_{+\infty}}(x) ⨍_{e_{+\infty}} \frac{\tilde{\psi}_{e_{+\infty}}(y)}{|y - x|^2} \,\mathrm{d}\gamma_y\,\mathrm{d}\gamma_x$$

$$(2) \qquad \mathbf{b}) \qquad ⨍_{e_{M_h}} \tilde{\varphi}_{e_{M_h}}(x) \int_{e_{+\infty}} \frac{\tilde{\psi}_{e_{+\infty}}(y)}{|y - x|^2} \,\mathrm{d}\gamma_y\,\mathrm{d}\gamma_x$$

$$(3) \qquad \mathbf{c}) \qquad ⨍_{e_{+\infty}} \tilde{\varphi}_{e_{+\infty}}(x) \int_{e_{M_h}} \frac{\tilde{\psi}_{e_{M_h}}(y)}{|y - x|^2} \,\mathrm{d}\gamma_y\,\mathrm{d}\gamma_x$$

having indicated with $\tilde{\varphi}_{e_{M_h}}$, $\tilde{\varphi}_{e_{+\infty}}$ local basis test functions defined on mesh elements e_{M_h}, $e_{+\infty}$, respectively. A similar notation is used for local basis trial functions. Finally, it is clear that the analysis can be done exactly in the same way for the elements $e_{-\infty}, e_1$.

Case **a)** Let us consider the mesh element $e_{+\infty} = (a, +\infty)$, $a > 0$. With the standard changes of variable $x = \frac{a}{1-s}$, $y = \frac{a}{1-t}$, we reduce to the evaluation of the double integral

$$(4) \qquad I^a = ⨍_0^1 \tilde{\varphi}(s) ⨍_0^1 \frac{\tilde{\psi}(t)}{(t-s)^2} \,\mathrm{d}t\,\mathrm{d}s.$$

To efficiently evaluate (4), we rewrite it in the form:

$$I^a = \int_0^1 \tilde{\varphi}(s) ⨍_0^1 \frac{1}{t-s} \frac{\tilde{\psi}(t) - \tilde{\psi}(s)}{t-s} \,\mathrm{d}t\,\mathrm{d}s - \int_0^1 \frac{\tilde{\varphi}(s)}{1-s} \tilde{\psi}(s)\,\mathrm{d}s$$

$$- ⨍_0^1 \frac{1}{s} \tilde{\varphi}(s)\tilde{\psi}(s)\,\mathrm{d}s =: I_1^a + I_2^a + I_3^a.$$

Recalling (7), if $\tilde{\psi}(t) = 1$, integral $I_1^a \equiv 0$, otherwise it could be evaluated using product rule (6) with kernel $k(s,t) = \frac{1}{t-s}$, but since the remaining integrand function $\frac{\tilde{\psi}(t)-\tilde{\psi}(s)}{t-s}$ presents a logarithmic singularity at $t = 1$, the number of nodes to reach the required accuracy would be too high. In this case, we introduce the change of variable $t = 1 - z^2$ to eliminate the weak singularity in $t = 1$ and, having set $a_s = \sqrt{1-s}$, we obtain:

$$(5) \qquad I_1^a = 2 \int_0^1 \tilde{\varphi}(s) \int_0^1 \frac{1}{z - a_s} \frac{z[\tilde{\psi}(1-z^2) - \tilde{\psi}(s)]}{(z^2 - a_s^2)(z + a_s)} \, dz \, ds.$$

The inner integral can be evaluated using product rule (6) with kernel $k(s,z) = \frac{1}{z-a_s}$ and a number of nodes such that the required accuracy is achieved. For the outer integral, which presents an integrand with a log-singularity for $s = 0$ and a behavior of type $(1-s)\log(1-s)$ in $s = 1$, first we perform the change of variable (8), then we use Gauss-Legendre rule.

Recalling that $\tilde{\varphi}(1) = 0$ (see (8)), integrand in I_2^a presents at most a log-singularity for $s = 1$, so we can use the smoothing change of variable (8) followed by Gauss-Legendre rule. Integral I_3^a can be evaluated with the HFP quadrature formula (7), with a number of nodes such that the exact result (up to machine precision) is obtain in the case of constant trial function or such that the required accuracy is achieved otherwise.

Case **b)** We consider the aligned consecutive elements $e_{M_h} = (a - h, a)$, $e_{+\infty} = (a, +\infty)$, with $a > 0$. With the standard changes of variable $x = a - hs$, $y = \frac{a}{1-t}$, we reduce to the evaluation of the double integral:

$$(6) \qquad I^b = a \, h \fint_0^1 \frac{\tilde{\varphi}^r(1-s)}{(a - s \, h)^2} \int_0^1 \frac{\tilde{\psi}(t)}{(t - a_s)^2} \, dt \, ds,$$

having set $a_s = -\frac{s}{s-a/h}$ and having indicated with $\tilde{\varphi}^r(s)$ the generic lagrangian polynomial of degree r defined on the reference interval $(0,1)$. To efficiently evaluate (6), we rewrite it in the form:

$$I^b = \int_0^1 \frac{a \, h \, \tilde{\varphi}^r(1-s)}{a - s \, h} \int_0^1 \frac{1}{t - a_s} \frac{\tilde{\psi}(t) - \tilde{\psi}(a_s)}{(a - s \, h)t + h \, s} \, dt \, ds$$
$$+ \fint_0^1 \frac{1}{s} \tilde{\varphi}^r(1-s) \, \tilde{\psi}(a_s) \, ds =: I_1^b + I_2^b.$$

Recalling (7), if $\tilde{\psi}(t) = 1$, integral $I_1^b \equiv 0$, otherwise it could be evaluated using product rule (6) with kernel $k(s,t) = \frac{1}{t-a_s}$, but since the remaining integrand function $\frac{\tilde{\psi}(t)-\tilde{\psi}(a_s)}{(a-s\,h)t+h\,s}$ presents a logarithmic singularity at $t = 1$, we can operate as in the case **a)**. Introducing the change of variable $t = 1 - z^2$ to eliminate the weak singularity in $t = 1$ and having set $b_s = \sqrt{1 - a_s}$, we obtain for I_1^b the expression:

$$-2 \, a \, h \int_0^1 \frac{\tilde{\varphi}^r(1-s)}{a - s \, h} \int_0^1 \frac{1}{z - b_s} \frac{z[\tilde{\psi}(1-z^2) - \tilde{\psi}(a_s)] \, dz \, ds}{((a - s \, h)(1-z^2) + h \, s)(z + b_s)}.$$

The inner integral can be evaluated using product rule (6) with kernel $k(s,z) = \frac{1}{z-b_s}$, with a number of nodes such that the required accuracy is achieved. For the outer integral, which presents a log-singularity for $s = 0$, first we perform the change of variable (8), then we use Gauss-Legendre rule. Integral I_2^b can be computed with the HFP quadrature formula (7).

Case **c)** is treated in a way similar to that one just explained in case **b)**.

5. Numerical results

At first, we show results related to the numerical evaluation of the integral

$$\fint_1^{+\infty} \frac{1}{x} \fint_1^{+\infty} \frac{1}{(y-x)^2} \left(1 + \ln(y)\right) \mathrm{d}y \, \mathrm{d}x,$$

which is converted in the form (4) and then treated as explained in case **a)**. In Tables 1, 2, we present absolute errors $E_{I_i^a}$, $i = 1, 2$, (with respect to the analytical result) produced in the numerical evaluation of I_1^a, I_2^a respectively, for different values of quadrature parameters: in particular, N_t, N_s indicate the number of nodes for the inner and outer integrals, while p, q denote the parameters chosen in (9). The computation has been carried out in double precision arithmetic and the symbol '−−' means that the single precision accuracy has been achieved. For I_3^a the single precision accuracy has been obtained with 32 nodes.

Table 1.   Absolute errors produced in the numerical evaluation of I_1^a, for different adopted kernels and various quadrature parameters.

N_t	N_s	p	$E_{I_1^a}(\frac{1}{t-s})$	$E_{I_1^a}(\frac{1}{z-a_s})$	N_t	N_s	p	$E_{I_1^a}(\frac{1}{t-s})$	$E_{I_1^a}(\frac{1}{z-a_s})$
			$q = 1$					$q = 2$	
32	16	2	$3.1727\mathrm{E}-3$	$4.8310\mathrm{E}-5$	32	16	2	$6.7868\mathrm{E}-3$	$2.6006\mathrm{E}-5$
		3	$4.8968\mathrm{E}-3$	$1.9297\mathrm{E}-5$			3	$3.3381\mathrm{E}-3$	$2.6743\mathrm{E}-5$
64	16	2	$7.1697\mathrm{E}-4$	$2.7347\mathrm{E}-5$	64	16	2	$1.4857\mathrm{E}-3$	$3.9307\mathrm{E}-5$
		3	$9.2381\mathrm{E}-4$	$3.1624\mathrm{E}-5$			3	$2.6921\mathrm{E}-3$	$4.3725\mathrm{E}-6$
			$q = 3$					$q = 4$	
32	16	2	$4.2954\mathrm{E}-3$	$3.9176\mathrm{E}-5$	32	16	2	$5.3779\mathrm{E}-3$	$9.7830\mathrm{E}-5$
		3	$7.7027\mathrm{E}-3$	$4.7108\mathrm{E}-5$			3	$8.0173\mathrm{E}-3$	$5.8219\mathrm{E}-5$
64	16	2	$1.1645\mathrm{E}-3$	$8.0897\mathrm{E}-5$	64	16	2	$1.1501\mathrm{E}-3$	$1.3540\mathrm{E}-4$
		3	$1.8363\mathrm{E}-3$	$--$			3	$1.8796\mathrm{E}-3$	$2.8048\mathrm{E}-6$

Table 2.   Absolute errors produced in the numerical evaluation of I_2^a.

		$p = 1$			
N_s	q	$E_{I_2^a}$	N_s	q	$E_{I_2^a}$
	2	$1.9554\mathrm{E}-4$		2	$1.3610\mathrm{E}-5$
8	3	$8.2795\mathrm{E}-6$	16	3	$1.4946\mathrm{E}-7$
	4	$5.8332\mathrm{E}-7$		4	$--$

Further, we show some results related to the numerical evaluation of the integral

$$\fint_{0.9}^1 \frac{x - 0.9}{0.1} \int_1^{+\infty} \frac{1}{(y-x)^2} \left(1 + \ln(y)\right) \mathrm{d}y \, \mathrm{d}x$$

which is converted in the form (6) and then treated as explained in case **b)**. In Table 3 we present absolute errors (with respect to the analytical result) produced in the numerical evaluation of I_1^b, for different values of quadrature parameters; in particular: N_t, N_s indicate the number of nodes for the inner and outer integrals, respectively, while p, q denote the parameters chosen in (9).

Table 3.   Absolute errors for I_1^b.

			$q = 1$	
N_t	N_s	p	$E_{I_1^b}(\frac{1}{t-a_s})$	$E_{I_1^b}(\frac{1}{z-b_s})$
32	16	2	$2.4986\mathrm{E}-5$	$1.4294\mathrm{E}-6$
		3	$2.6613\mathrm{E}-5$	$--$
64	16	2	$5.0880\mathrm{E}-6$	$1.3642\mathrm{E}-6$
		3	$6.5301\mathrm{E}-6$	$--$

Finally, as test problem we consider a Neumann potential problem, defined on the upper half plane, deduced from Ref. 18, where the boundary datum given on $\Gamma = \mathbb{R}$ is:

$$(1) \qquad \overline{p}(x) = \begin{cases} 0 & \text{for} \quad x < -1 \\ 1 & \text{for} -1 < x < 0 \\ \frac{2}{\pi}(-\frac{1}{\sqrt{x}} + \arctan[\frac{1}{\sqrt{x}}]) & \text{for} \quad x > 0 \end{cases}.$$

The analytical solution of the problem, up to an additive constant C^*, is

$$(2) \qquad u(x) = \begin{cases} \frac{1}{2\pi}\{4\sqrt{-x} + (1+x)\log[\frac{(1+\sqrt{-x})^2}{(1-\sqrt{-x})^2}]\} & \text{for } x \le 0 \\ 0 & \text{for } x > 0 \end{cases}$$

and presents a sudden growth around $x = 0$. For the numerical solution we have considered $\Gamma_B = [-5, 5]$, equipped by a mesh made up by 75 nodes, geometrically graded towards zero where the boundary datum has a square root singularity, and related linear test and shape functions. On infinite elements we have chosen constant shape and decaying test functions as given in (7), (8), respectively. With infinite PGBEM, we have generated a linear system of order 75, regularized with the procedure mentioned at the end of Sect. 2. In Fig. 1 we show the approximate solution, related to $C^* = -0.181978$, together with the corresponding absolute error.

Fig. 1.   Numerical solution of the test problem and corresponding absolute error

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