

Multiplicative Schwarz Methods for Discontinuous Galerkin Approximations of Elliptic Problems

Paola F. Antonietti

*Dipartimento di Matematica, Università degli Studi di Pavia
via Ferrata 1, 27100 Pavia, Italy
paola.antonietti@unipv.it*

Blanca Ayuso

*Istituto di Matematica Applicata e Tecnologie Informatiche - CNR
via Ferrata 1, 27100 Pavia, Italy
blanca@imati.cnr.it*

1 Introduction

Over the last twenty years, there has been an active development of domain decomposition (DD) techniques for the solution of large algebraic systems arising from the numerical approximation of partial differential equations. Although the theory of DD techniques for finite element (FE) methods (conforming, non conforming and mixed) is by now well understood and developed (see, e.g., [8]), only few results can be found in the literature for discontinuous Galerkin (DG) approximations (see [1, 2, 5, 6]).

Based on discontinuous FE spaces, DG methods have deserved a substantial attention due to their flexibility in handling meshes with hanging nodes and their high degree of locality.

In this talk we present and analyse in a unified framework a multiplicative Schwarz preconditioner for the linear systems obtained from all the stable and consistent (in the sense of [3]) discontinuous Galerkin approximations of second order elliptic problems that have been proposed up to the date. We present the convergence analysis for the iterative methods that result by accelerating the proposed multiplicative preconditioner with a Krylov-type method. We cover the cases of symmetric DG approximations and non-symmetric ones, i.e., the incomplete interior penalty (IIP) [4] and the non-symmetric interior penalty (NIP) [7] methods. Extensive numerical experiments to verify the performance of the proposed preconditioner and to validate the theory are presented.

2 DG Approximation of Elliptic Problems

For the sake of simplicity we restrict ourselves to the model problem

$$(2.1) \quad -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is assumed to be a smooth convex domain and $f \in L^2(\Omega)$. By introducing the auxiliary variable $\boldsymbol{\sigma} = \nabla u$, the second order problem (2.1) can be rewritten in mixed form as the following first order system of equations

$$(2.2) \quad \boldsymbol{\sigma} = \nabla u \text{ in } \Omega, \quad -\operatorname{div}(\boldsymbol{\sigma}) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Let \mathcal{T}_h be a shape regular and locally quasi-uniform partition of the domain Ω , with possible hanging nodes. We denote by \mathcal{E}^I and \mathcal{E}^B the sets of all interior and boundary

faces of \mathcal{T}_h , respectively, and set $\mathcal{E} := \mathcal{E}^I \cup \mathcal{E}^B$. For $\ell_h \geq 1$, we define the discontinuous finite element spaces $V_h := \{v \in L^2(\Omega) : v|_T \in \mathcal{M}^{\ell_h}(T), \forall T \in \mathcal{T}_h\}$, $\Sigma_h := [V_h]^d$, $\mathcal{M}^{\ell_h}(T)$ being a suitable space of polynomials depending on the type of the triangulation considered. By considering the standard definitions of the average and jump trace operators, we consider the unified DG approximation based of the *flux formulation* of problem (2.1) proposed in [3]: find $(u, \sigma) \in V_h \times \Sigma_h$ such that, $\forall v \in V_h$, $\mathcal{A}(u, v) = (f, v)$, with

$$(2.3) \quad \mathcal{A}(u, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v + \int_{\mathcal{E}} [\widehat{u}(u) - u] \cdot \{\{\nabla v\}\} + \int_{\mathcal{E}^I} \{\{\widehat{u}(u) - u\}\} [\nabla v] \\ - \int_{\mathcal{E}} \{\{\widehat{\sigma}(\sigma, u)\}\} \cdot [v] - \int_{\mathcal{E}^I} [\widehat{\sigma}(\sigma, u)] \{\{v\}\},$$

where \widehat{u} , $\widehat{\sigma}$ are the *numerical fluxes* that determine the DG method (see [3]).

3 Multiplicative Schwarz Methods

We consider three level of nested refinements, a subdomain partition \mathcal{T}_{N_s} made of N_s non-overlapping subdomains, a coarse partition \mathcal{T}_H and a fine partition \mathcal{T}_h . Let \mathcal{E} be the set of all faces of the fine triangulation \mathcal{T}_h . For each subdomain Ω_i of \mathcal{T}_{N_s} we denote by \mathcal{E}_i the set of all faces of \mathcal{E} belonging to $\overline{\Omega}_i$. Three main ingredients are required for the construction of the Schwarz methods:

- (a) The *local spaces*: for each subdomain Ω_i , $i = 1, \dots, N_s$, we define $V_h^i := \{v \in V_h : v \equiv 0 \text{ in } \Omega \setminus \overline{\Omega}_i\}$ and $\Sigma_h^i := [V_h^i]^d$.
- (b) The *local solvers*: for each subdomain Ω_i , $i = 1, \dots, N_s$, we consider the DG approximation of problem (2.1) but restricted to Ω_i . Then, the local bilinear forms are defined as

$$(3.1) \quad \mathcal{A}_i(u_i, v_i) := \sum_{T \in \mathcal{T}_h} \int_T \nabla u_i \cdot \nabla v_i + \int_{\mathcal{E}_i} [\widehat{u}_i(u_i) - u_i] \cdot \{\{\nabla v_i\}\} + \int_{\mathcal{E}_i^I} \{\{\widehat{u}_i(u_i) - u_i\}\} [\nabla v_i] \\ - \int_{\mathcal{E}_i} \{\{\widehat{\sigma}_i(\sigma_i, u_i)\}\} \cdot [v_i] - \int_{\mathcal{E}_i^I} [\widehat{\sigma}_i(\sigma_i, u_i)] \{\{v_i\}\},$$

where \widehat{u}_i , $\widehat{\sigma}_i$ are the *local numerical fluxes*. We will show how they are related to the global numerical fluxes. For that purpose we will introduce some appropriate prolongation operators $R_i^T : V_h^i \longrightarrow V_h$ and $\mathcal{D}_i^T : \Sigma_h^i \longrightarrow \Sigma_h$.

- (c) The *coarse solver*: we define the coarse solver by restricting the global bilinear form (2.3) to the coarse spaces $V_H = V_h^0 := \{v_H \in L^2(\Omega) : v_H|_T \in \mathcal{M}^{\ell_H}(T), \forall T \in \mathcal{T}_H\}$ and $\Sigma_H = \Sigma_h^0 := [V_h^0]^d$, with $0 \leq \ell_H \leq \ell_h$.

Following [1], we introduce the multiplicative Schwarz operator:

$$(3.2) \quad P_{mu} := I - (I - P_{N_s})(I - P_{N_s-1}) \cdots (I - P_0),$$

where, for $i = 0, \dots, N_s$, $P_i : V_h \longrightarrow V_h$ is the \mathcal{A} -projection like operator defined by $\mathcal{A}(P_i u, R_i^T v_i) = \mathcal{A}(u, R_i^T v_i)$, $\forall v_i \in V_h^i$. The multiplicative Schwarz method consists in replacing the discrete problem $\mathcal{A}u = f$ by the equation $P_{mu}u = g$, with an appropriate right hand side g .

We also present a *symmetrized* version of (3.2) defined as

$$P_{mu}^{sym} := I - (I - P_0^*) \cdots (I - P_{N_s-1}^*)(I - P_{N_s}^*)(I - P_{N_s})(I - P_{N_s-1}) \cdots (I - P_0),$$

where, for $i = 0, \dots, N_s$, P_i^* is the adjoint operator of P_i with respect to the inner product induced by the symmetric part of $\mathcal{A}(\cdot, \cdot)$.

4 Convergence Analysis

We present the convergence analysis for the iterative methods resulting by considering the multiplicative preconditioners P_{mu} and P_{mu}^{sym} accelerated by the GMRES and CG linear solvers, respectively. The analysis of the multiplicative Schwarz methods is presented by distinguishing between symmetric and non symmetric DG approximations. In both cases, we show that the resulting preconditioners are scalable in the sense that the number of subdomains does not affect the convergence behaviour. While for symmetric DG approximations the convergence analysis follows the general abstract framework [8], for non-symmetric DG approximations additional difficulties arise when developing the convergence analysis. Furthermore, the theoretical results we obtain are less satisfactory than those shown for symmetric methods, mainly since the proofs of convergence require a technical assumption on the size of the penalty parameter which enters into the definition of the numerical fluxes. Nevertheless, the numerical experiments seem to indicate that such restriction on the penalty parameter is not required in practice.

REFERENCES

1. P.F. Antonietti and B. Ayuso, Multiplicative Schwarz Methods for Discontinuous Galerkin Approximations of Elliptic Problems, *Technical Report IMATI-CNR PV-10*, 2006.
2. P.F. Antonietti and B. Ayuso, Schwarz Domain Decomposition Preconditioners for Discontinuous Galerkin Approximations of Elliptic Problems: Non-Overlapping Case, *Technical Report IMATI-CNR PV-20*, 2005. Submitted to *Math. Model. Numer. Anal.*.
3. D.A. Arnold and F. Brezzi and B. Cockburn and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.*, vol. 39, 2001/02, 1749–1779 (electronic).
4. C. Dawson and S. Sun and M.F. Wheeler, Compatible algorithms for coupled flow and transport, *Comput. Methods Appl. Mech. Engrg.*, vol. 193, 2004, 2565–2580.
5. X. Feng and O.A. Karakashian, Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems, *SIAM J. Numer. Anal.*, vol. 39, 2001, 1343–1365 (electronic).
6. C. Lasser and A. Toselli, An overlapping domain decomposition preconditioner for a class of discontinuous Galerkin approximations of advection-diffusion problems, *Math. Comp.*, vol. 72, 2003, 1215–1238 (electronic).

7. B. Rivière, and M.F. Wheeler and V. Girault, Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems I, *Comput. Geosci.*, vol. 3, 1999, 337–360.
8. B.F. Smith and P.E. Bjørstad and W.D. Gropp, Domain decomposition. Parallel multilevel methods for elliptic partial differential equations, *Cambridge University Press*, Cambridge, 1996.