Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 1 (2006)

DOI: 10.1685/CSC06068

Discontinuous Galerkin Methods for Diffusion-Advection-Reaction Problems With Discontinuous Diffusivity

Daniele A. Di Pietro
CERMICS, École nationale des ponts et chaussées
Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
dipietro@cermics.enpc.fr

Alexandre Ern
CERMICS, École nationale des ponts et chaussées
Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
ern@cermics.enpc.fr

Jean-Luc Guermond
Dept. Math, Texas A&M, College Station, TX
77843-3368, USA and LIMSI (CNRS-UPR 3251), BP 133, 91403, Orsay, France
guermond@math.tamu.edu

1 Introduction

Inspired by the abstract framework presented in [4, 2, 3], we analyze two classes of discontinuous Galerkin (DG) methods for diffusion-advection-reaction problems with discontinuous diffusivity. DG methods for this kind of problems has already been presented in [1,5].

The two strategies considered here correspond to different ways of recasting the original problem in mixed form. In both cases the weak formulations with boundary conditions weakly enforced serve as a base for the design of the DG method.

For the first class of methods, the analysis proposed in [3] can be directly used. Our main concern will then be to point out the analogies and differences with other methods presented in the literature, paying special attention to the strategies for the local elimination of the auxiliary unknowns.

The second class of methods was considered to account for the vanishing-diffusivity limit. The lower regularity of the coefficients doesn't allow to use the theory developed in [2, 3]. After stating well-posedness at the continuous level, we propose and analyse an extended two-field Friedrichs discretization. A singular perturbation analysis is also carried out to show how the boundary and interface operators should be selected in the vanishing diffusivity case.

2 The Continuous Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and connected Lipschitz domain. We consider the diffusion-advection-reaction problem

$$-\nabla \cdot (\nu \nabla u) + \beta \cdot \nabla u + \mu u = f, \tag{1}$$

where $\nu \in [L^\infty(\Omega)]^{d,d}$ is a positive definite matrix-valued field defined on Ω whose lowest eigenvalue is uniformly bounded away from zero, $\beta \in [L^\infty(\Omega)]^d$ is such that $\nabla \cdot \beta \in L^\infty(\Omega)$ and $\mu \in L^\infty(\Omega)$ is such that $\mu - \nabla \cdot \beta \geq \mu_0 > 0$. The first mixed formulation reads

$$\begin{cases} \nu^{-1}\sigma + \nabla u = 0, \\ \nabla \cdot \sigma + \beta \cdot \nabla u + \mu u = f. \end{cases} \tag{2}$$

The above problem is indeed a Friedrichs' system belonging to the class considered in [3], to which we refer for the well-posedness analysis.

An alternative formulation can be obtained by introducing the square root of the diffusivity, i.e. the tensor κ such that $\nu = \kappa\kappa$. The second mixed formulation reads

$$\begin{cases} \sigma + \kappa \nabla u = 0, \\ \nabla \cdot (\kappa \sigma) + \beta \cdot \nabla u + \mu u = f. \end{cases} \tag{3}$$

Problem (3) doesn't satisfy the assumptions of [2, 3], and it requires a specific analysis. Let $H_\kappa(\mathrm{div};\Omega) = \{\tau \in L_\sigma;\ \nabla \cdot (\kappa\tau) \in L_u\}$, $W = H_\kappa(\mathrm{div};\Omega) \times H^1(\Omega)$ and

$$\mathcal{L}(W;L) \ni T : (\tau, v) \mapsto (\tau + \kappa \nabla v, \mu v + \nabla \cdot (\kappa \tau) + \beta \cdot \nabla v),$$

where we have set $L = [L^2(\Omega)]^{d+1}$. The well-posedness of the continuous problem (3) supplemented with suitable boundary conditions can be proved applying the results presented in [4] to the operator T.

3 Discontinuous Galerkin Approximation

Let $\mathcal{T}_h$ be a triangulation of Ω and define the space W_h of piecewise polynomial $(d+1)$-vector functions on $\mathcal{T}_h$ possibly discontinuous on elementary interfaces. Assume, moreover, that the discontinuities of the diffusivity ν are aligned with the mesh.

3.1 A Class of DG Methods for Problem (2)

We introduce the bilinear form

$$\begin{aligned} a_h((\sigma,u),(\tau,v)) = &\sum_{K\in\mathcal{T}_h} \left[(\nu^{-1}\sigma,\tau)_K + (\mu u, v)_K\right] \\ &+ \sum_{K\in\mathcal{T}_h} [(\nabla u,\tau)_K + (\nabla\cdot\sigma + \beta\cdot\nabla u, v)_K] \\ &- \sum_{F\in\mathcal{F}_h^i} [(\llbracket u\rrbracket n_F, \{\tau\})_F + (\llbracket\sigma\rrbracket\cdot n_F + (\beta\cdot n_F)\llbracket u\rrbracket, \{v\})_F] \\ &+ \tfrac{1}{2}\sum_{F\in\mathcal{F}_h^\partial} (M_F(v) - \mathcal{D}_{\partial\Omega}v, w)_F \\ &+ \sum_{F\in\mathcal{F}_h^i} (S_F^{uu}(\llbracket u\rrbracket), \llbracket v\rrbracket)_F + r_h(u,v), \end{aligned} \tag{4}$$

where $\{\cdot\}$ and $\llbracket\cdot\rrbracket$ denote the average and jump operators defined as in [3], M_F, S_F^{uu} are meant to weakly impose boundary conditions and continuity, $\mathcal{D}_{\partial\Omega}$ collects boundary terms that are also present in the continuous problem with boundary conditions weakly enforced and r_h is needed to recover IP-like methods. Consider the discrete problem

$$\begin{cases} \text{Seek } u_h \in W_h \text{ such that} \\ a_h(u_h, v_h) = (f, v_h)_L, \quad \forall v_h \in W_h. \end{cases} \tag{5}$$

The convergence analysis follows the guidelines presented in [3]. In the present work we show that some of the methods for the problem with discontinuous viscosity can be recovered on a suitable choice of M_F, S_F^u and r_h provided ν is piecewise constant or linear. Moreover, under the same assumptions, the local elimination procedure for the σ-component can be simplified so that it comes at no computational effort. The natural extension of methods originally presented for the problem with continuous diffusivity is also pointed out. Finally, in the general case, we prove that the same computational gain can be obtained by substituting ν^{-1} with a suitable projection. The order of convergence of the resulting method is the same as the original for piecewise linear and quadratic approximation.

3.2 A Class of DG Methods for Problem (3)

Consider problem (5) with the bilinear form a_h replaced by

$$\begin{aligned} a_h((\sigma,u),(\tau,v)) = & \sum_{K\in\mathcal{T}_h} (T(\sigma,u),(\tau,v))_{L,K} \\ & + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F((\sigma,u) - \mathcal{D}_{\partial\Omega}(\sigma,u),(\tau,v))_{L,F} \\ & - 2\sum_{F\in\mathcal{F}_h^i} \left[(\{\kappa\sigma\cdot n\},\{v\})_{L_u,F} + \tfrac{1}{4}([\![u]\!],[\![\kappa\tau\cdot n]\!])_{L_u,F}\right] \\ & + \sum_{F\in\mathcal{F}_h^i} (S_F^{uu}([\![u]\!]),[\![u]\!])_{L_u,F} + r_h(u,v). \end{aligned} \tag{6}$$

According to the choice of S_F^{uu}, different methods are obtained. The resulting method can be analysed extending the techniques presented in [3], although the theory therein doesn't directly apply because of the lower regularity of the diffusivity. The convergence estimate obtained by the direct argument can be improved by a standard duality argument provided elliptic regularity holds. Also in this case we provide comparison with other methods proposed in the literature. Finally, a finer analysis is devoted to the vanishing viscosity case and the modifications of the design constraint on the stabilizing operators are clearly stated.

REFERENCES

1. M. Dryja, On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients, Comp. Meth. Appl. Math., Vol. 3, 2003, 76–85.

2. A. Ern and J.-L. Guermond, Discontinuous Galerkin methods for Friedrichs systems. I. General theory, SIAM J. Numer. Anal., 2005, to appear.

3. A. Ern and J.-L. Guermond, Discontinuous Galerkin methods for Friedrichs systems. II. Second order PDE's, SIAM J. Numer. Anal., 2006, submitted.

4. A. Ern, J.-L. Guermond and G. Caplain, an intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs systems, Comm. Pure Appl. Math, 2005, to appear.

5. B. Riviére and M. F. Wheeler, *A posteriori* error estimates for a discontinuous Galerkin method applied to elliptic problems, Comp. and Math. with App., Vol. 46, 2003, 141–164.