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DG method for Stokes problem with variable viscosity

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We study the stationary Stokes problem with varying viscosity and density on a convex polygonal domain Ω in $\mathbb{R}^2$:

$$\begin{aligned} -\nabla\cdot(\nu\nabla\mathbf{u})+\nabla p+\alpha\mathbf{u} &= \rho\mathbf{f} && \text{in } \Omega\,,\\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega\,,\\ \mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega\,, \end{aligned} \tag{0.1}$$

where the Dirichlet datum $\mathbf{g}\in\left[L^2(\partial\Omega)\right]^2$ satisfies the compatibility condition $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}\,\mathrm{d}\gamma=0$, $\mathbf{n}$ being the outward unit normal to $\partial\Omega$. We assume that $\rho\mathbf{f}\in\left[L^2(\Omega)\right]^2$ and $\alpha,\nu\in L^\infty(\Omega)$. In particular, there exist four constants ν_0, ν_1, α_0 and α_1 such that $0<\nu_0<\nu(\mathbf{x})<\nu_1$ and $0\le\alpha_0<\alpha(\mathbf{x})<\alpha_1$ a.e. in Ω.

We introduce a general non conforming discretization for problem (0.1). Denoting by $\mathcal{T}_h$ a triangulation of the domain Ω, we define $\left[H^r(\mathcal{T}_h)\right]^2$ as the set of $\mathbb{R}^2$ valued functions, whose components are in $H^r(K)$ for each mesh element K. On this space, we introduce the elementwise norm $||\cdot||_{r,\mathcal{T}_h}$, defined as the sum over each element of the H^r-norm. We choose as velocity finite elements space the space $\mathbf{V}_h$ of $\mathbb{R}^2$ valued functions, whose components are polynomial with degree at most p over each mesh element. On $\mathbf{V}_h$,we define a two terms norm: the first term is an elementwise H^1-norm, while the second one penalizes the interelement discontinuities. We choose as pressure finite elements space, the set $Q_h\subset Q=L^2_0(\Omega)$ of the functions which are polynomial with degree at most q over each element, endowed with the L^2-norm. We consider a non conforming discretization of the Stokes problem of the form:

Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ so that:

$$\begin{aligned} a_\nu(\mathbf{u}_h,v)+b(\mathbf{v},p_h)&=F(\mathbf{v})\ ,\\ b(\mathbf{u}_h,q)&=G(q)\ , \end{aligned} \tag{0.2}$$

for all $(\mathbf{v},q)\in\mathbf{V}_h\times Q_h$.

The bilinear form $a_\nu:\mathbf{W}\times\mathbf{W}\to\mathbb{R}$ is a discretization of the elliptic term, while the bilinear form $b:\mathbf{W}\times Q\to\mathbb{R}$ takes into account the divergence term. The two linear functionals $F:\mathbf{W}\to\mathbb{R}$ and $G:Q\to\mathbb{R}$ take into account the right hand sides respectively of the first and second equation and, eventually, the boundary condition. We have denoted by $\mathbf{W}$ either the continuous space $\mathbf{V}=\left[H^1(\Omega)\right]^2$ or the discrete space $\mathbf{V}_h$.
We assume that:

- the bilinear form a_ν is continuous and coercive on $\mathbf{V}_h\times\mathbf{V}_h$;
- the bilinear form a_ν is in some sense continuous on $\mathbf{V}_h\times\left[H^2(\mathcal{T}_h)\cap H^1(\Omega)\right]^2$;
- the bilinear form b is continuous on $\mathbf{V}\times Q$;

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- the bilinear form b satisfies the inf-sup condition on $\mathbf{V}_h \times Q_h$;
- the functional F and G are continuous respectively on $\mathbf{V}_h$ and Q_h;
- the discretization is consistent.

We can thus prove the following theorem:

Theorem The discrete problem (0.2) admits an unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$. Moreover, assuming the continuous solution $(\mathbf{u}, p) \in [H^{p+1}(\mathcal{T}_h)]^2 \times H^{q+1}(\mathcal{T}_h)$, the following error estimates holds:

$$|| \mathbf{u} - \mathbf{u}_h ||_h \leq C_1 \left(h^p || \mathbf{u}||_{p+1,\mathcal{T}_h} + h^{q+1} || p||_{q+1,\mathcal{T}_h} \right) , \tag{0.3}$$

$$|| p - p_h ||_0 \leq C_2 \left(h^p || \mathbf{u}||_{p+1,\mathcal{T}_h} + h^{q+1} || p||_{q+1,\mathcal{T}_h} \right) \tag{0.4}$$

where C_1 and C_2 are two positive constants independent from h.

Under our assumptions, only existence and uniqueness proof can be done using classical techniques, as the one showed in[3].

We then present a discontinuous Galerkin (DG) discretization which can be cast in the previous settings. Let ∇_h be the gradient performed elementwise. Following [2], we introduce suitable jump and mean operators and define the two bilinear form:

$$\begin{aligned} a_\nu(\mathbf{u}, \mathbf{v}) = & \int_\Omega \alpha \mathbf{u} \cdot \mathbf{v} \mathrm{d}\mathbf{x} + \int_\Omega \nu \nabla_h \mathbf{u} : \nabla_h \mathbf{v} \mathrm{d}\mathbf{x} - \sum_{e \in \mathcal{E}} \int_e \Big(\{\!\{ \nu \nabla_h \mathbf{u} \}\!\} : \underline{[\![\mathbf{v}]\!]} \pm \\ & \pm \{\!\{ \nu \nabla_h \mathbf{v} \}\!\} : \underline{[\![\mathbf{u}]\!]} \Big) \mathrm{d}\gamma + \sum_{e \in \mathcal{E}} h_e^{-1} \int_e \sigma \underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]} \mathrm{d}\gamma , \\ b(\mathbf{v}, q) = & - \int_\Omega \left(q \nabla_h \cdot \mathbf{v} - \mathcal{M}(\mathbf{v}) q \right) \mathrm{d}\mathbf{x} , \end{aligned}$$

where, for each $e \in \mathcal{E}$ edge of the triangulation, h_e is the edge length, and $\mathcal{M}_e$ is the lifting operator from $\mathbf{W}$ to Q_h defined through:

$$\int_\Omega \mathcal{M}_e(\mathbf{v}) q \mathrm{d}\mathbf{x} = \int_e [\![\mathbf{v}]\!] \{\!\{ q \}\!\} \mathrm{d}\gamma, \quad \forall q \in Q_h .$$

The bilinear form a_ν is a IP (plus) or NIP (minus) discretization of the elliptic constrain. Proceeding as in [1], we can easily show that such discretization satisfies the above reported assumptions. The bilinear form b is the one presented in [4], where the properties proofs can be found. We have to set $q = p - 1$ to obtain the inf-sup condition. The definitions of the two linear functionals are a straightforward consequence of the chosen bilinear forms and of the consistency requirement:

$$\begin{aligned} F_{\mathbf{g}}(\mathbf{v}) = & \int_\Omega \rho \mathbf{f} \cdot \mathbf{v} \mathrm{d}\mathbf{x} \mp \sum_{e \in \mathcal{E}_\partial} \int_e \nu \mathbf{g} \cdot (\nabla_h \mathbf{v}\, \mathbf{n}) \mathrm{d}\gamma + \sum_{e \in \mathcal{E}_\partial} h_e^{-1} \int_e \sigma \mathbf{g} \cdot \mathbf{v} \mathrm{d}\gamma , \\ G(q) = & \sum_{e \in \mathcal{E}_\partial} \int_e q \mathbf{g} \cdot \mathbf{n} \mathrm{d}\gamma , \end{aligned}$$

where $\mathcal{E}_\partial$ is the set of all boundary edges. To prove the discretization consistency we need to suppose some regularity on the continuous solution. Such assumptions are proved in

case of constant viscosity, but at the moment we do not know any regularity results on the solution of such problem with non constant coefficients (as pointed out in [6]). We also need to suppose that the stabilization parameter is a positive function of L^∞ (big enough in the IP case) and that the possible discontinuities of the viscosity parameter ν are aligned with the elements edges, so that we can univocally define the trace from inside each mesh element. We remark that, since the estimates need only a discrete and a piecewise H^2 continuity, we can use the results for elliptic problems, which are usually obtained using a norm which contains a H^2-term (see e.g [2]). We also observe that, as a consequence of the chosen norm, we get an estimate also for the L^2-norm of the velocity error.

We present some numerical test obtained using a NIP discretization. The numerical tests show that, even using an elliptic non-symmetric discretization, we gain in the Stokes problem an order of convergence in the L^2-norm of the velocity error (see figure 1). We note that good performances are archived also in case of vanishing viscosity. Thus we try to approximate the Darcy law:

$$\begin{aligned} \mathbf{u} &= \overline{K}\,\nabla p && \text{in } \Omega\,, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega\,, \\ \mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega\,, \end{aligned}$$

with the generalized Stokes problem with small viscosity ϵ and source term $\alpha = \overline{K}^{-1}$. We also try to solve the Darcy-Stokes problem, approximating the first with a generalized Stokes problem. When the porous media becomes impermeable, the performed test show that problems in the resolution of the pressure can arise around the boundary Γ between the porus media and the fluid (see figure 2). Since, as pointed out in [5], in the first approximation, the suitable boundary condition at Γ is the continuity of the pressure, we add to the second equation a term of the form:

$$\sum_{e\in\mathcal{E}_i} \frac{\delta}{h_e} \int_e [\![p_h]\!] \cdot [\![q]\!] \, \mathrm{d}\gamma\,,$$

with δ a positive constant and $\mathcal{E}_i$ the set of all internal mesh edges. Such term does not compromise the consistency, if we assume the exact pressure solution in $H^1(\Omega)$. We suppose that it should decrease of one order the convergence in pressure, nevertheless the performed numerical test show that it stabilizes the pressure around the front (see figure 2), without modifying significantly the velocity.

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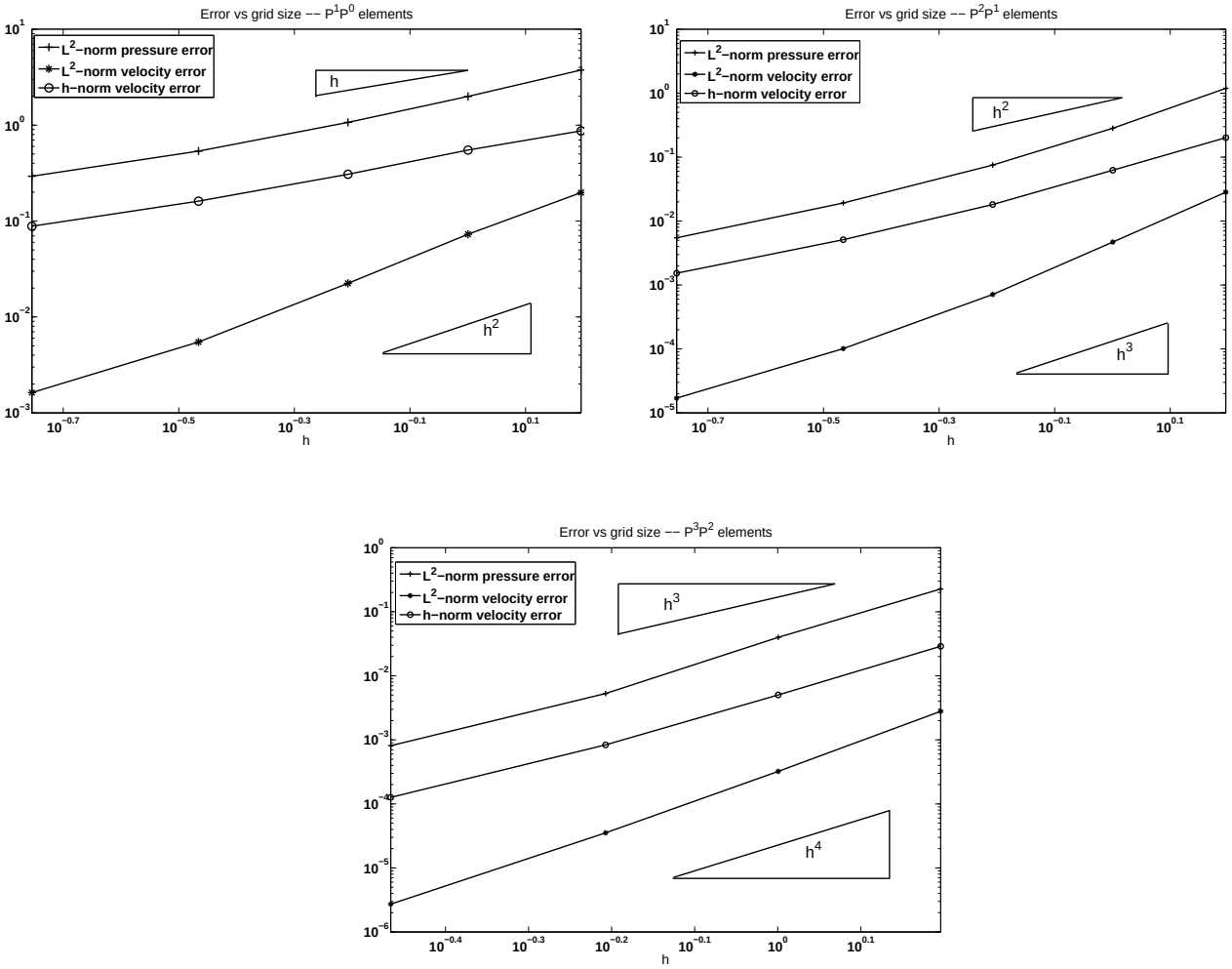

Figure 0.1: Errors plots for P^1P^0 elements (left), for P^2P^1 elements (right) end for P^3P^2 elements (center)

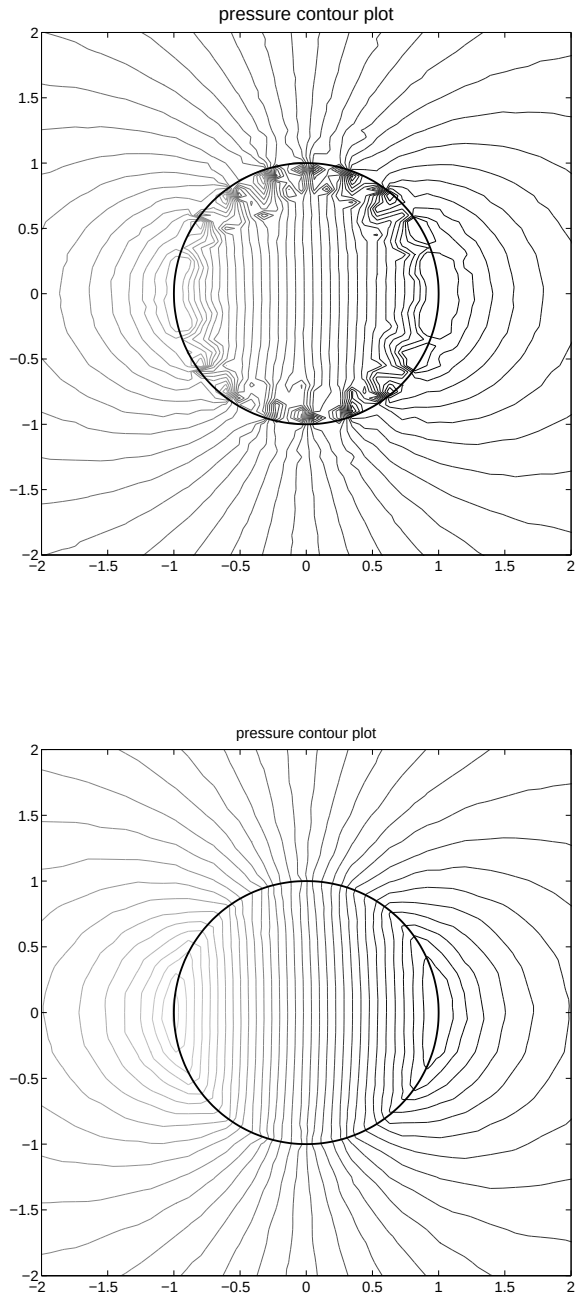

Figure 0.2: Pressure contour plots: non-stabilized and stabilized case