Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 2 (2007)

DOI: 10.1685/CSC06108

MIXED DISCONTINUOUS GALERKIN METHODS WITH MINIMAL STABILIZATION

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Abstract.

We will address the problem of finding the minimal necessary stabilization for a class of Discontinuous Galerkin (DG) methods in mixed form. In particular, we will present a new stabilized formulation of the Bassi–Rebay method (see Ref. 1 for the original unstable method) and a new formulation of the Local Discontinuous Galerkin method (LDG), introduced in 1998 by Cockburn and Shu.[2] It will be shown that, in order to reach stability, it is enough to add jump terms only over a part of the boundary of the domain, instead of over all the skeleton of the mesh, as it is usually done (see Ref. 3, for instance).

Keywords: Bassi–Rebay method, discontinuous Galerkin finite element, LDG method, stability.

1. Introduction

The first *Discontinuous Galerkin* (DG) method was introduced by Reed and Hill[4] in 1973 for hyperbolic equations; since then a great number of DG methods have been studied; in particular, there has been an active development of DG methods for elliptic equations: some examples are the Bassi–Rebay method, presented in 1997 in Refs. 1 and 5, and the Local Discontinuous Galerkin method (LDG), introduced in 1998 by Cockburn and Shu.[2] For other details on DG methods for elliptic problems see for example Ref. 3.

We will introduce the method for the two-dimensional model problem in the u unknown

$$-\operatorname{div}(\nabla u) = f \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = g \quad \text{on } \partial\Omega,$$

whose associated mixed formulation is

$$\boldsymbol{\sigma} = -\nabla u \quad \text{in } \Omega, \qquad \operatorname{div} \boldsymbol{\sigma} = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

Let $\boldsymbol{\Sigma}_h$ and V_h be the discontinuous spaces in which we look for the discrete approximation of $\boldsymbol{\sigma}$ and u, respectively, and let $\mathcal{E}$ be the skeleton of our mesh. If we define the lifting operator $R : L^1(\mathcal{E})^2 \to \boldsymbol{\Sigma}_h$ by

$$\int_\Omega R(\boldsymbol{\varphi}) \cdot \boldsymbol{\tau} \, dx = -\int_{\mathcal{E}} \boldsymbol{\varphi} \cdot \{\{\boldsymbol{\tau}\}\} \, ds \qquad \forall \, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,$$

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where $[[\cdot]]$ and $\{\{\cdot\}\}$ are the jump and the average operators, respectively (see Ref. 3), the (unstable) Bassi–Rebay formulation reads as follows

$$\text{find } u_h \in V_h \text{ s.t. } \forall\ v_h \in V_h \ \int_\Omega (\nabla u_h + R([[u_h]])) \cdot (\nabla v_h + R([[v_h]]))\ dx = 0.$$

We will show that the formulation

$$\begin{aligned}&\text{find } u_h \in V_h \text{ s.t. } \forall\ v_h \in V_h\\ &\int_\Omega (\nabla u_h + R([[u_h]])) \cdot (\nabla v_h + R([[v_h]]))\ dx \ + \ \alpha \int_\Gamma [[u_h]] \cdot [[v_h]]\ ds \ = \ 0,\end{aligned}$$

where α and $\Gamma \subseteq \partial\Omega$ are suitably chosen, is stable.
Convergence results are proved limitedly to Cartesian grids: in order to prove these estimates, we need to choose Γ as the union of two edges of the domain Ω (which is supposed to be a rectangle).
Analogous results are presented for the LDG method, where we show that the choice of Γ in order to have stability depends on a parameter which defines the method.

2. Model Problem and DG Formulation

We consider the two-dimensional model problem

$$-\operatorname{div}(\nabla u) = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega, \tag{1}$$

where Ω is assumed to be a convex polygonal domain, $\Omega \subset \mathbb{R}^2$, u is the unknown and $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ are given. Introducing $\boldsymbol{\sigma} = \nabla u$, Eq. (1) can be rewritten as

$$\boldsymbol{\sigma} = \nabla u \text{ in } \Omega, \quad -\operatorname{div}(\boldsymbol{\sigma}) = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{2}$$

Let $\mathcal{T}_h$ be a shape regular triangulation of the domain Ω whose elements are triangles or squares. If we define the spaces

$$V = \{v \in L^2(\Omega) : v|_T \in H^1(T)\ \forall\ T \in \mathcal{T}_h\}, \qquad \boldsymbol{\Sigma} = [V]^2,$$

the weak formulation of Eq. (2) is

$$\begin{aligned}&\text{find } (\boldsymbol{\sigma}, u) \in \boldsymbol{\Sigma} \times V \text{ s.t. } \forall\ T \in \mathcal{T}_h, \quad \forall\ (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma} \times V\\ &\int_T \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\ dx \ + \ \int_T u \operatorname{div}(\boldsymbol{\tau})\ dx \ - \ \int_{\partial T} u\ \boldsymbol{\tau} \cdot \boldsymbol{n}\ ds = 0,\\ &\int_T \boldsymbol{\sigma} \cdot \nabla v\ dx \ - \ \int_{\partial T} \boldsymbol{\sigma} \cdot v\ \boldsymbol{n}\ ds \ = \ \int_T f\ v\ dx.\end{aligned} \tag{3}$$

2.1. *Numerical Fluxes*

In order to introduce the Bassi–Rebay and the LDG methods we define the so-called *numerical fluxes* $\widehat{\boldsymbol{\sigma}}$ and $\widehat{u}$, which are discrete approximations of $\boldsymbol{\sigma}$ and u on the interelement boundaries of $\mathcal{T}_h$.
Let $\mathcal{E} := \bigcup_{T \in \mathcal{T}_h} \partial T$ be the skeleton of our mesh (i.e., the union of all the subdivision edges of our mesh), $\mathcal{E}^0 = \mathcal{E} \setminus \partial\Omega$ and $\mathcal{E}^\partial = \mathcal{E} \setminus \mathcal{E}^0$; we define the average $\{\{\cdot\}\}$ and the jump $[[\cdot]]$ operators on $\mathcal{E}^0$ as follows: let e be an interior edge shared by elements T_1 and T_2 and $\boldsymbol{n}_i$

be the normal unit vector pointing exterior to T_i, $i = 1, 2$; if $\boldsymbol{q}$ is a smooth vector-valued function, with $\boldsymbol{q}_i = \boldsymbol{q}_{|\partial T_i}$, $i = 1, 2$, we set

$$\{\{\boldsymbol{q}\}\} = \frac{\boldsymbol{q}_1 + \boldsymbol{q}_2}{2} \quad , \quad [[\boldsymbol{q}]] = \boldsymbol{q}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{q}_2 \cdot \boldsymbol{n}_2 \qquad \text{on } e,$$

whereas if q is a smooth scalar-valued function, with $q_i = q_{|\partial T_i}$, $i = 1, 2$, we set

$$\{\{q\}\} = \frac{q_1 + q_2}{2} \quad , \quad [[q]] = q_1\, \boldsymbol{n}_1 + q_2\, \boldsymbol{n}_2 \qquad \text{on } e.$$

We are now ready to define the numerical fluxes on $e \in \mathcal{E}^0$:

$$\begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} \{\{\boldsymbol{\sigma}\}\} \\ \{\{u\}\} \end{pmatrix} - \begin{pmatrix} \alpha & -\boldsymbol{\beta} \\ \boldsymbol{\beta} & \mathbf{0} \end{pmatrix} \begin{pmatrix} [[u]] \\ [[\boldsymbol{\sigma}]] \end{pmatrix}, \tag{4}$$

with $\alpha \in \mathbb{R}$ and $\boldsymbol{\beta} \in \mathbb{R}^2$ (see Ref. 6).
In order to deal with the inhomogeneous Dirichlet boundary condition $u = g$ on $\partial\Omega$, we need to define the average and jump operators on $\mathcal{E}^\partial$: if v is a scalar-valued test function and w is either u, the solution to Eq. (1), or an approximation in V of u, we set

$$[[v]] = v\, \boldsymbol{n} \;\text{ and }\; [[w]] = (w - g)\, \boldsymbol{n} \;\text{ on } e \;\in\; \mathcal{E}^\partial,$$

whereas if $\boldsymbol{w}$ is a vector-valued function, we set

$$\{\{\boldsymbol{w}\}\} = \boldsymbol{w} \;\text{ and }\; [[\boldsymbol{w}]] = \boldsymbol{w} \cdot \boldsymbol{n} \quad \text{on } e \;\in\; \mathcal{E}^\partial,$$

where $\boldsymbol{n}$ is the normal unit vector pointing exterior to Ω. Therefore, the numerical fluxes on $e \in \mathcal{E}^\partial$ are chosen as follows:

$$\widehat{\boldsymbol{\sigma}} = \{\{\boldsymbol{\sigma}\}\} - \alpha[[u]], \quad \widehat{u} = g. \tag{5}$$

The Bassi–Rebay method is defined by setting in Eq. (4) and in Eq. (5)

$$\alpha = 0; \quad \boldsymbol{\beta} = \mathbf{0}, \tag{6}$$

while for the LDG method the coefficients are chosen according to the following conditions:

$$\alpha = \alpha_0 h^r, \alpha_0 \in \mathbb{R}^+, r \in \{0, -1\}\,; \quad \boldsymbol{\beta} \in \mathbb{R}^2, \text{independent of } h. \tag{7}$$

2.2. *The Discrete Formulation*

If we replace the traces on ∂T by the numerical fluxes and we introduce the discontinuous finite element spaces

$$V_h = \Big\{\; v_h \;\in\; L^2(\Omega) \text{ s. t. } v_{h|_T} \in S^k(T) \;\; \forall\, T \in \mathcal{T}_h \Big\}, \quad \boldsymbol{\Sigma}_h \;=\; V_h^2,$$

with S^k equal to P^k (i.e., polynomial functions of degree at most k) for triangular meshes, or S^k equal to Q^k (i.e., polynomial functions of degree at most k in each variable) for Cartesian grids, for an integer $k \geq 1$, the DG discretization of Eq. (3) is:

$$\begin{aligned} &\text{find } (\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h \text{ s.t. } \forall\, T \in \mathcal{T}_h, \quad \forall\, (\boldsymbol{\tau}_h, v_h) \in \boldsymbol{\Sigma}_h \times V_h \\ &\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h dx - \sum_{T \in \mathcal{T}_h} \int_T u_h \mathrm{div}(\boldsymbol{\tau}_h) dx - \int_{\mathcal{E}} \widehat{u}_h\, [[\boldsymbol{\tau}_h]]\, ds = 0, \\ &\int_\Omega \boldsymbol{\sigma} \cdot \nabla_h(v)_h dx - \int_{\mathcal{E}} \widehat{\boldsymbol{\sigma}}_h\, [[v_h]]\, ds = \int_\Omega f\, v_h dx, \end{aligned} \tag{8}$$

where we denote by ∇_h the elementwise gradient. Introducing the *lifting operators* $R : [L^1(\mathcal{E})]^2 \to \boldsymbol{\Sigma}_h$ and $L_{\boldsymbol{\beta}} : [L^1(\mathcal{E})]^2 \to \boldsymbol{\Sigma}_h$ defined by

$$\int_\Omega R([[u_h]]) \cdot \boldsymbol{\tau_h} dx = -\sum_{e\in\mathcal{E}} \int_e [[u_h]] \cdot \{\{\boldsymbol{\tau_h}\}\} ds \quad \forall \boldsymbol{\tau_h} \in \boldsymbol{\Sigma}_h,$$
$$\int_\Omega L_{\boldsymbol{\beta}}([[u_h]]) \cdot \boldsymbol{\tau_h} dx = -\sum_{e\in\mathcal{E}^0} \int_e \boldsymbol{\beta} \cdot [[u_h]]\, [[\boldsymbol{\tau_h}]] ds \quad \forall \boldsymbol{\tau_h} \in \boldsymbol{\Sigma}_h,$$

from the first equation in Eq. (8) we have $\boldsymbol{\sigma}_h = -\nabla_h u_h - R([[u_h]]) - L_{\boldsymbol{\beta}}([[u_h]])$, which, substituted in the second, gives the so called Primal Formulation

$$\begin{aligned}
&\text{find} \quad u_h \in V_h \quad \text{s.t.} \quad \forall\, v_h \in V_h \\
&\int_\Omega (\nabla_h u_h + R([[u_h]]) + L_{\boldsymbol{\beta}}([[u_h]])) \cdot (\nabla_h v_h + R([[v_h]]) + L_{\boldsymbol{\beta}}([[v_h]]))\, dx \\
&\quad + \int_{\mathcal{E}} \alpha\, [[u_h]] \cdot [[v_h]] ds = \int_\Omega f v_h dx,
\end{aligned}$$

with α and $\boldsymbol{\beta}$ as in Eq. (6) and Eq. (7) for the Bassi-Rebay method and for the LDG method, respectively (see Ref. 3 for further details).

3. Minimal Stabilization

In this section we show a recipe for choosing $\Gamma \subseteq \partial\Omega$ such that by defining the stability parameter α in Eq. (4) and in Eq. (5) as $\alpha > 0$ on Γ and $\alpha = 0$ on the remaining part of the skeleton of the mesh, we obtain a method which is stable for any value of the coefficient $\boldsymbol{\beta}$. Thus the new method with Minimal Stabilization reads as follows:

$$\begin{aligned}
&\text{find} \quad u_h \in V_h \quad \text{s.t.} \quad \forall\, v_h \in V_h \\
&B(u_h, v_h) = \int_\Omega f v_h\, dx,
\end{aligned} \tag{1}$$

where the bilinear form $B(\cdot,\cdot) : V \times V \to \mathbb{R}$ is defined by

$$\begin{aligned}
B(\, u,\ v\,) = &\int_\Omega (\nabla_h u + R([[u]]) + L_{\boldsymbol{\beta}}([[u]])) \cdot (\nabla_h v + R([[v]]) + L_{\boldsymbol{\beta}}([[v]]))dx \\
&+ \int_\Gamma \alpha\, [[u]] \cdot [[v]]\, ds.
\end{aligned}$$

3.1. *The Case* $\boldsymbol{\beta} = \mathbf{0}$

In order to study the coercivity of the new method we define for any $u \in V$

$$||u||_B = \sqrt{B(\, u,\, u\,)}. \tag{2}$$

For the case $\boldsymbol{\beta} = \mathbf{0}$ we have the following result.

Theorem 3.1. *Provide that $\alpha > 0$ on Γ, if we choose $\Gamma = \{e\}$, where e is an arbitrary edge in $\mathcal{E}^\partial$, then Eq. (2) is a norm in V_h and thus the bilinear form $B(\cdot,\cdot)$ is coercive in V_h with respect to this norm.*

Proof. If $\alpha \geq 0$, Equation (2) clearly defines a seminorm in V. In order to prove that, for $\alpha > 0$, Eq. (2) is actually a norm in V_h, it is enough to show that the kernel of the bilinear form $B(\cdot,\cdot)$ is trivial.
If we define the kernel of $B(\cdot,\cdot)$ as

$$Ker(B) = \{v_h \in V_h \text{ s. t. } B(\, v_h, z_h\,) = 0 \;\; \forall\, z_h \in V_h\}\,,$$

then, considering suitable choices of the test function z_h, we have that $v_h \in Ker(B)$ if and only if

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \int_T v_h \mathrm{div}\boldsymbol{\tau}_h \; dx \;+\; \int_{\mathcal{E}^0} \{\{v_h\}\}\; [[\boldsymbol{\tau}_h]] \; ds \;= 0 \quad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h,\\
&\int_\Gamma \alpha\, v_h\, t_h\, ds \;=\; 0 \quad \forall\, t_h \in P^k(\Gamma).
\end{aligned}$$

Thus, following Ref. 7, we obtain $v_h \equiv 0$ and the proof is complete. □

3.2. *The Case* $\boldsymbol{\beta} \neq \mathbf{0}$

Reasoning as in the proof of Theorem 3.1, we have that $v_h \in Ker(B)$ if and only if

$$\begin{aligned}
&\int_T v_h q \; dx = 0 \quad \forall\, q \in \widetilde{P}(T), \quad \forall\, T \in \mathcal{T}_h,\\
&\{\{v_h\}\} - \boldsymbol{\beta}\cdot[[v_h]] = 0 \;\; \text{on } \mathcal{E}^0,\\
&v_h = 0 \;\; \text{on } \Gamma,
\end{aligned} \tag{3}$$

where $\widetilde{P}(T) = \left\{v \in L^2(T) \text{ s.t. } \exists\, \boldsymbol{w} \in \boldsymbol{\Sigma}_h :\; v = \mathrm{div}\boldsymbol{w}_{|_T}\right\}$ (for further details, see Refs. 7, 8 and 9). Let $e \in \mathcal{E}$ and $T_i, T_j \in \mathcal{T}_h$ such that $\partial T_i \cap \partial T_j = e$, then Eq. (3) implies

$$\left(\boldsymbol{\beta}\cdot\boldsymbol{n} - \frac{1}{2}\right) v_h^i(x) \;=\; \left(\boldsymbol{\beta}\cdot\boldsymbol{n} + \frac{1}{2}\right) v_h^j(x) \quad \forall\, x \in e, \tag{4}$$

where v_h^i and v_h^j are the restrictions of v_h to T_i and T_j, respectively.

Numerical experiments have suggested that, if $\boldsymbol{\beta} \neq \mathbf{0}$, a possible choice of Γ in order to have stability is the following: we define

$$G \;=\; \left\{\, T \in \mathcal{T}_h \text{ s.t. } \;\mathcal{E}^\partial \cap \partial T \neq \{\emptyset\} \;\text{ and } \boldsymbol{\beta}\cdot\boldsymbol{n}_e > 0 \;\; \forall\, e \in \mathcal{E}^0 \cap \partial T\right\},$$

where $\boldsymbol{n}_e$ is the normal unit vector to e pointing outside T, and we select Γ as the union of one arbitrary $e \in \mathcal{E}^\partial \cap \partial T$ for each $T \in G$. A motivation of this, considering Eq. (4), is that $T_i \in G$ if and only if $|\boldsymbol{\beta}\cdot\boldsymbol{n} - 1/2| < |\boldsymbol{\beta}\cdot\boldsymbol{n} + 1/2|$; this implies that if $|v^i_{h|_e}| < \epsilon$, then $|v^j_{h|_e}| < \epsilon$, while it does not imply the converse. In other words, the stabilization, in this case, propagates from the element T_i to the adjacent element T_j and not the other way around (*outflow stabilization*), therefore we need to stabilize on $e \in \mathcal{E}^\partial \cap \partial T$ for each element $T \in G$.

Whenever G is empty, we define

$$\widehat{G} \;=\; \left\{\, T \in \mathcal{T}_h \text{ s.t. } \;\mathcal{E}^\partial \cap \partial T \neq \{\emptyset\} \;\text{ and } \boldsymbol{\beta}\cdot\boldsymbol{n} \geq 0 \;\; \forall\, e \in \mathcal{E}^0 \cap \partial T\right\}$$

and select Γ as one arbitrary edge of one arbitrary element of $\widehat{G}$; in this case it is enough to consider only one element because every element $T \in \widehat{G}$ has an edge e for which $\boldsymbol{\beta}\cdot\boldsymbol{n}_e = 0$,

i.e., an edge with good propagation of the stability both from the interior of T to the exterior and from the exterior of T to the interior.

Remark 3.1. In Figure 1 we present an example in which we set $\Omega := [-1,1] \times [-1,1]$ and we consider the model problem Eq. (1) with f and g such that the solution is

$$u(x,y) \;=\; \sin\left(\frac{\pi}{2}(x+1)\right)\sin\left(\frac{\pi}{2}(y+1)\right).$$

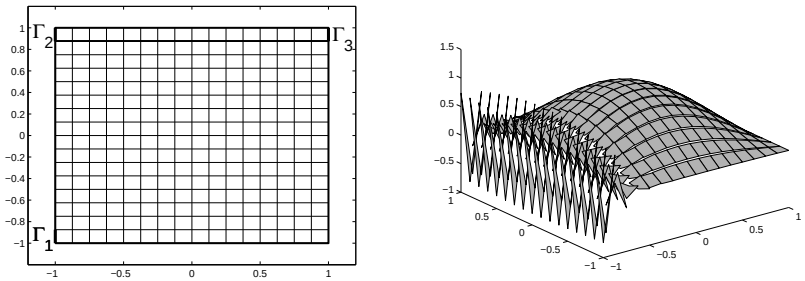

Fig. 1. Mesh and discrete solution considering $\Gamma = \Gamma_3$, $\alpha = 1$ on Γ and $\boldsymbol{\beta} = (1,0)$.

We consider $\boldsymbol{\beta} = (1,0)$ and a Cartesian grid; in this case the set G is empty, thus we consider the set $\widehat{G}$. In Fig. 1 we show the mesh and three possible choices of Γ: Γ_1 and Γ_2 are edges of elements which belong to $\widehat{G}$, while this does not hold for Γ_3 and in Fig. 1 we see that the solution related to the choice $\Gamma = \Gamma_3$ is unstable as we do not have the control of the elements belonging to $\widehat{G}$.

4. *A Priori* Error Analysis

In this section we present *A Priori* error estimates limitedly to Cartesian grids. In order to derive these error estimates, we need to choose a new Γ that includes the one presented in the previous section for any $\boldsymbol{\beta} \in \mathbb{R}^2$ (thus the coercivity of the bilinear form $B(\cdot,\cdot)$ holds also for this new choice).

Definition 4.1 (Choice of Γ).

$$\Gamma \;=\; \Gamma(\boldsymbol{\beta}) \;=\; \{(x,y) \in \partial\Omega \;\text{ s.t. }\; \boldsymbol{\beta}\cdot\boldsymbol{n} \le 0\},$$

where $\boldsymbol{n}$ is the normal unit vector pointing exterior to Ω. Moreover if $\boldsymbol{\beta}\cdot\boldsymbol{n} = 0$ on two parallel edges, we can keep in Γ only one of them.

We denote by $||\cdot||_{s,A}$ the usual norm of the Sobolev space $H^s(A)$, $s \in \mathbb{N}$, and we set $||\cdot||_s := ||\cdot||_{s,\Omega}$; we also define the following norm

$$|||v|||^2 = \sum_{T\in\mathcal{T}_h} ||\nabla v||^2_{0,T} + \frac{1}{h}\int_e [[v]]^2\,(x)\,dx,$$

where h_i is the diameter of the element $T_i \in \mathcal{T}_h$ and $h = \max_{T_i\in\mathcal{T}_h} h_i$. From now on, we assume that the following hypothesis of quasi-uniformity of the mesh holds:

$$\exists\; L_1 > 0,\; L_2 > 0 \;\text{ s. t. }\; L_1 h_i \le h_j \le L_2 h_i, \;\; \forall\; T_i,\; T_j \in \mathcal{T}_h.$$

We have the following result.

Theorem 4.1. *Let $u \in H^s(\Omega)$, $s \geq 2$, $\boldsymbol{\beta} = (b_1, b_2)$ and Γ as in Definition 4.1. Then*

$$||u - u_h||_0 \leq Ch^{\min(k+1,s-a)-a}||u||_s,$$
$$|||u - u_h||| \leq Ch^{\min(k+1,s-a)-a-1}||u||_s,$$

where

$$\begin{cases} a = 1 & \text{if } b_1 = 0 \text{ and/or } b_2 = 0 \text{ and either } \alpha = O(1) \text{ or } \alpha = O(h^{-1}), \\ a = \frac{1}{2} & \text{if } b_1 \neq 0,\ b_2 \neq 0 \text{ and } \alpha = O(1), \\ a = 0 & \text{if } b_1 \neq 0,\ b_2 \neq 0 \text{ and } \alpha = O(h^{-1}), \end{cases}$$

u is the analytical solution to Eq. (1), u_h is the solution to the discrete problem Eq. (1) and $C > 0$ is independent of h.

For the proof of this result see Ref. 7.

5. Numerical Experiments: Cartesian Grids

The aim of this section is to numerically validate the results presented in the previous section. We consider the model problem Eq. (1) with $\Omega := [-1, 1] \times [-1, 1]$, f and g such that the analytical solution is

$$u(x, y) = \sin\left(\frac{\pi}{2}(x + 1)\right) \sin\left(\frac{\pi}{2}(y + 1)\right) + 1,$$

and Cartesian grids with hanging nodes (see Fig. 1).

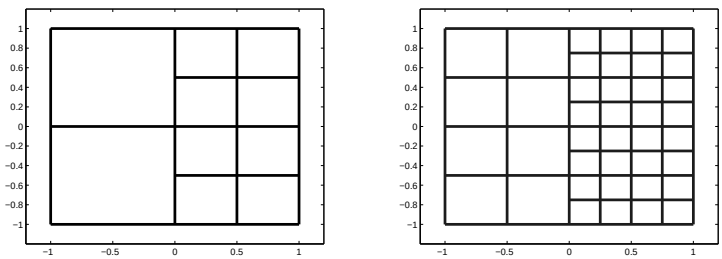

Fig. 1. First and second mesh.

First of all we consider the case $\boldsymbol{\beta} = 0$. In Table 1 we report the errors and the orders of convergence for both the L^2 and the discrete norm $|||\cdot|||$ for the choice $\Gamma = ([-1, 1] \times \{1\}) \cup (\{1\} \times [-1, 1])$ (according to Definition 4.1) and taking $\alpha = 1$ on Γ
Finally we consider $\boldsymbol{\beta} = (-1, 1)$, $\Gamma = ([-1, 1] \times \{-1\}) \cup (\{1\} \times [-1, 1])$ and $\alpha = h^{-1}$ on Γ; in Table 2 we report the same results as above.

Numerical experiments here not reported show that the same orders of convergence are achieved if we consider Cartesian grids without hanging nodes (see Refs. 7, 8 and 9). From these numerical results it is clear that the *A Priori* error estimates of Theorem 4.1 are sharp only in some cases. In Tables 3–4 we summarize the convergence rates obtained theoretically (see Theorem 4.1) as well as numerically, for smooth analytical solutions.

Table 1. Errors and orders of convergence (OC).

	$k=1$		$k=2$	
h	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$
0.500000	0.132480953	0.990234298	0.028541595	0.333433461
0.250000	0.053457182	0.714171582	0.002603588	0.063587652
0.125000	0.024570001	0.618305927	0.000302639	0.015105410
0.062500	0.011952906	0.588468251	0.000037149	0.003722353
0.031250	0.005925354	0.579264665	0.000004536	0.000910243
0.015625	0.002954278	0.576232685	-	-
OC	1.004095	0.007571	3.033631	2.031891

Table 2. Errors and orders of convergence (OC).

	$k=1$		$k=2$	
h	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$
0.500000	0.138077634	0.935734561	0.017174017	0.217207624
0.250000	0.034192220	0.446800980	0.002528265	0.056532052
0.125000	0.007581990	0.212897602	0.000396905	0.015809757
0.062500	0.001817563	0.104799600	0.000056106	0.004212842
0.031250	0.000446956	0.052080099	0.000007394	0.001081720
0.015625	0.000110924	0.025968834	-	-
OC	2.010553	1.003951	2.923657	1.961467

Table 3. $\boldsymbol{\beta} = (b, 0)$ or $\boldsymbol{\beta} = (0, b)$ with $b \in \mathbb{R}$.

	Error Estimates		Numerical Results	
	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$
k odd; $\alpha = O(1)$ or $\alpha = O(h^{-1})$	k	$k-1$	k	$k-1$
k even; $\alpha = O(1)$ or $\alpha = O(h^{-1})$	k	$k-1$	$k+1$	k

Table 4. $\boldsymbol{\beta} = (b_1, b_2)$ with $b_1, b_2 \in \mathbb{R} \setminus \{0\}$.

	Error Estimates		Numerical Results	
	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$
$\forall\ k;\ \alpha = O(1)$	$k+\frac{1}{2}$	$k-\frac{1}{2}$	$k+1$	k
$\forall\ k;\ \alpha = O(h^{-1})$	$k+1$	k	$k+1$	k

6. Numerical Experiments: Triangular Mesh

In this section we present numerical results for triangular meshes. We consider the problem Eq. (1) with $\Omega := [-1, 1] \times [-1, 1]$ and analytical solution

$$u(x, y) = \sin\left(\frac{\pi}{2}(x+1)\right) \sin\left(\frac{\pi}{2}(y+1)\right).$$

First of all we consider unstructured triangular meshes and we choose $\alpha = 1$ on $\Gamma := ([-1,1]\times\{1\})\cup(\{1\}\times[-1,1])$ and $\boldsymbol{\beta} = (0,0)$; in Table 1 we report the errors and the orders of convergence for $k = 1$ and $k = 2$.

Table 1. Errors and orders of convergence.

$k=1$		$k=2$	
$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$
0.213372	0.966852	0.102715	0.871829
0.092759	1.100663	0.016109	0.324003
0.006115	0.646673	0.001865	0.077093
0.001520	0.293431	0.000234	0.019572
0.000381	0.147094	0.000030	0.005044
1.995826	0.991265	2.953573	1.956116

Finally we consider structured meshes and set $\alpha = 1$ on $\Gamma := ([-1,1]\times\{1\})\cup(\{1\}\times[-1,1])$ and $\boldsymbol{\beta} = (-1,-2)$; in Table 2 we report the same data as above. Notice that Γ is chosen according to Definition 4.1.

Table 2. Errors and orders of convergence.

$k=1$		$k=2$	
$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$	$\|\|\cdot\|\|_0$	$\|\|\|\cdot\|\|\|$
0.361955	2.575249	0.074518	0.859388
0.071594	1.147721	0.008084	0.184174
0.015751	0.518565	0.001008	0.043099
0.003728	0.244943	0.000129	0.010595
0.000911	0.118789	0.000016	0.002605
2.033497	1.044046	2.989265	2.024366

From these numerical results and from others here not reported, we see that the orders of convergence for triangular meshes are optimal for any $\boldsymbol{\beta} \in \mathbb{R}^2$. Therefore there is an important difference between triangular meshes and Cartesian grids: if Γ is chosen according to Definition 4.1 and we consider either structured or unstructured triangular meshes, we always have optimal orders of convergence for both the L^2 and the discrete norm, while for Cartesian grids we have sub-optimal orders of convergence if $\boldsymbol{\beta} = (b,0)$ or $\boldsymbol{\beta} = (0,b)$, $b \in \mathbb{R}$.

7. Conclusions

In this paper we have showed that it is possible to reduce the stabilization term of the original LDG method keeping it stable and we have proposed a new stable formulation for the original (unstable) Bassi–Rebay method; we have therefore proved that in order to stabilize a DG method it might be enough to consider a penalization term involving jumps only over a part of the boundary Γ of the domain. In Sec. 3 and 4 we have found the minimal Γ in order to obtain stable methods and both stable and quasi-optimally

convergent methods (limitedly to Cartesian grids), respectively.
The error estimates we have presented in Sec. 4, limitedly to Cartesian grids, are optimal and sharp only in some cases. In Tables 3–4 we have summarized the convergence rates obtained theoretically as well as numerically. In Section 6 we have also considered triangular meshes: numerical experiments show optimal orders of convergence for any $\boldsymbol{\beta} \in \mathbb{R}^2$. *A Priori* error estimates for this kind of mesh will be subject of further investigation.
Numerical results here not reported (see Refs. 7, 8 and 9) show that:

(i) in the one-dimensional case we obtain the same orders of convergence as in the two-dimensional case with Cartesian grids;
(ii) if we consider Cartesian grids and Γ chosen according to Sec. 3, we do not obtain the optimal orders of convergence proved in Theorem 4.1;
(iii) if we consider triangular meshes, we obtain optimal orders of convergence for any value of $\boldsymbol{\beta} \in \mathbb{R}^2$, for either Γ chosen according to Sec. 3 or Sec. 4.

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