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GENERATORS OF GENERALIZED GRAPH IDEALS

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Abstract.

This work deals with the way to determine, in the degree $q \leqslant 6$, how many paths of length $(q-1)$ are contained in a connected graph G, using only its incidence matrix. The composition of such paths and the generators of the generalized graph ideals relative to G are studied for every degree q. An interesting application is given in tourist sphere.

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1. Introduction

To get paths of length $(q-1)$, q positive integer, in a connected graph G, algebraically means to find generators of a monomial ideal to which G can be associated, the generalized graph ideal $I_q(G)$.

The question to compute, using only the incidence matrix of G, the number and the structure of paths of G, and the generators of the relative generalized graph ideals, has been introduced in [2] and extended in [3].

In such papers, an answer to it has been given for $q \leqslant 6$. The number of paths of G having length less than 5 has been obtained in terms of multiplicity of pairs of rows in the incidence matrix of G, holding account of eventual cycle subgraphs contained in G. Their structure has been found by joining alternatively on the rows and columns of the incidence matrix specific entries 1 relative to the vertices of the paths starting from the inside.

In this work, by also introducing new strategies in the calculations, the problem to determine the structure of paths of any length in G, and the generators of the relative generalized graph ideals is fully solved.

An example of algebraic model arising from a transportation problem that can occur in tourist sphere is presented.

2. Preliminaries and paths of a graph

Definition 2.1. A graph G is said *connected* if every pair of vertices of G are joined by a path, that is a walk whose vertices are distinct.

Definition 2.2. Let G be a connected graph having vertices $v_1,\ldots,v_p$. A *generalized graph ideal* $I_q(G)$, $\mathbb{N}\ni q\leqslant p$, is an ideal of the polynomial ring $K[x_1,\ldots,x_p]$, where K is a field and each variable x_i corresponds to v_i, generated by all the square-free monomials $x_{i_1}\cdots x_{i_q}$ of degree q such that the vertex v_{i_j} is adjacent to $v_{i_{j+1}}$, for all $1\leqslant j\leqslant (q-1)$.

Remark 2.1. $I_2(G)$ is the generalized graph ideal generated by the edges of G, the so-called edge ideal. More generally, the generators of $I_q(G)$ are paths of G of length $(q-1)$, simply called $(q-1)$-paths.

Definition 2.3. A monomial ideal $L_q\subset K[x_1,\ldots,x_m;y_1,\ldots,y_n]$ is an *ideal of mixed products* if it is writable as $L_q=I_p\,J_r+I_s\,J_t$, where

$q=p+r=s+t$, for p,r,s,t non-negative integers;

I_p (resp. J_r) is an ideal of $K[x_1,\ldots,x_m;y_1,\ldots,y_n]$ generated by square-free monomials of degree p (resp. r) in the variables $x_1,\ldots,x_m$ (resp. $y_1,\ldots,y_n$).

Definition 2.4. A graph G is said *bipartite* if the vertex set V of G can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2.

For bipartite graphs, the generalized graph ideals are particular ideals of mixed products. To this proposal the following properties hold.

Property 2.1. *Let G be a complete bipartite graph, $V=\{x_1,\ldots,x_m;y_1,\ldots,y_n\}$ the vertex set of G. Then, for $2\leqslant q<(m+n)$, the generalized graph ideal $L_q(G)$ has the form*

$$L_q(G)=\begin{cases} I_h\,J_{h+1}+I_{h+1}\,J_h & \text{if } q=2h+1\\ I_h\,J_h & \text{if } q=2h\end{cases}.$$

Property 2.2. *Let $L_q\subset K[x_1,\ldots,x_m;y_1,\ldots,y_n]$, $2\leqslant q<(m+n)$, be an ideal of mixed products of the form*

a) $I_h\,J_{h+1}$, *or* $I_{h+1}\,J_h$, *or* $I_h\,J_{h+1}+I_{h+1}\,J_h$ *for* $h=\dfrac{q-1}{2}$;

b) $I_h\,J_h$ *for* $h=\dfrac{q}{2}$.

Then $L_q=L_q(G)$, G a complete bipartite graph with vertex set $V=\{x_1,\ldots,x_m;y_1,\ldots,y_n\}$.

Definition 2.5. Let G be a graph with p vertices and t edges. The *incidence matrix* M_G of G is a $(p\times t)$-matrix whose entries a_{ij} are equal to 1 if the i-th vertex of G belongs to the j-th edge, 0 otherwise.

Remark 2.2. Each row of M_G has as many entries 1 as the degree of the relative vertex of G. Each column of M_G has two entries 1 and the remaining 0.

Definition 2.6. Let G be a connected graph and M_G be the incidence matrix of G. We call *multiplicity* of a pair of rows in M_G, corresponding in G to a pair of vertices having the degrees α and β respectively, the product $(\alpha-1)(\beta-1)$, and denote it $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$.

Property 2.3. *Let G and M_G be as in the last definition. The multiplicity of a pair of rows in M_G, that correspond to a pair of vertices of G joined by a $(q-1)$-path, $q \geqslant 2$, and that have the degrees $\alpha \geqslant 2$ and $\beta \geqslant 2$, gives the number of walks of length $(q+1)$ in G containing the $(q-1)$-path inside.*

The following results give the method to calculate, considering only the incidence matrix of a connected graph G, how many $(q-1)$-paths are contained in G, for $3 \leqslant q \leqslant 6$.

Proposition 2.1. (number of 2-paths)
Let G be a connected graph with $p \geqslant 3$ vertices, and incidence matrix M_G. Let λ_i, $\lambda_i \geqslant 2$, denote the number of entries 1 in the i-th row of M_G.
Then G has $\sum_{i=1}^{p} \binom{\lambda_i}{2}$ 2-paths.

Proposition 2.2. (number of 3-paths)
Let G be a connected graph with $p \geqslant 4$ vertices, t edges, s cycle subgraphs C_3, and incidence matrix M_G.
Let $\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}$, $\alpha_j \geqslant \beta_j \geqslant 2$, denote the multiplicity of the rows of M_G on which the entries 1 of its j-th column lie.
Then G has $\sum_{j=1}^{t} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} - 3\,s$ 3-paths.

Proposition 2.3. (number of 4-paths)
Let G be a connected graph with $p \geqslant 5$ vertices $v_1, \ldots, v_p$, t edges, s cycle subgraphs C_3, r cycle subgraphs C_4, and incidence matrix M_G.
Let d_h be the number of vertices of G adjacent to C_3, $h = 1, \ldots, s$.
For every $(p \times 2)$-submatrix A_ℓ of M_G having one row of 1's, let $\begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}$, $\alpha_\ell \geqslant \beta_\ell \geqslant 2$, be the multiplicity of the rows of M_G corresponding to the rows of A_ℓ with a unique entry 1.
Then G has $\sum_{\ell} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix} - \sum_{h=1}^{s} (3 + 2\,d_h) - 4\,r$ 4-paths.

Proof. We think of a 4-path of G as a pair of distinct 3-paths having a common 2-path.
G has $\sum_{i=1}^{p} \binom{\deg v_i}{2}$ 2-paths, each characterized by a $(p \times 2)$-submatrix A_ℓ of M_G having just one row of 1's and two rows with a unique entry 1.
If m_ℓ is the multiplicity of the rows in M_G that correspond to the rows of A_ℓ with a unique entry 1, every pair of 3-paths having a common 2-path determine m_ℓ walks of length 4 in G having as internal vertices the three vertices of their common 2-path.
For $\ell = 1, \ldots, \sum_{i=1}^{p} \binom{\deg v_i}{2}$, all the walks of length 4 in G are found.
The assertion follows excluding walks having some repeated vertex. □

Proposition 2.4. (number of 5-paths)
Let G be a connected graph with $p \geqslant 6$ vertices $v_1, \ldots, v_p$, t edges, s cycle subgraphs C_3, r cycle subgraphs C_4, ρ cycle subgraphs C_5, and incidence matrix M_G.
For any C_3 in G, let d_h be the number of vertices v_{i_λ} of G adjacent to C_3, $h = 1, \ldots, s$, and $\wp$ be the number of pairs of C_3 with a common edge.

For any C_4 in G, let δ_k be the number of vertices of G, not belonging to C_4, adjacent to C_4, $k = 1, \ldots, r$.
For every $(p \times 3)$-submatrix B_ℓ of M_G having two rows each with two entries 1, let $\begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}$, $\alpha_\ell \geqslant \beta_\ell \geqslant 2$, be the multiplicity of the rows of M_G corresponding to the rows of B_ℓ with a unique entry 1.
Then G has $\sum_\ell \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix} - 2\left(\sum_{\lambda=1}^{d_h} (\deg v_{i_\lambda} - 1)\right) + 2\wp - \sum_{k=1}^{r} (4 + 2\delta_k) - 5\rho$ *5-paths.*

Proof. We think of a 5-path of G as a pair of distinct 4-paths having a common 3-path.
G has $\sum n_j - 3s$ 3-paths, where n_j is the multiplicity of the rows of M_G on which the entries 1 of its j-th column lie, $j = 1, \ldots, t$.
Every 3-path is characterized by a $(p \times 3)$-submatrix B_ℓ of M_G having just two rows with two 1's and two rows with a unique entry 1.
If m_ℓ is the multiplicity of the rows in M_G that correspond to the rows of B_ℓ with a unique entry 1, every pair of 4-paths having a common 3-path determine m_ℓ walks of length 5 in G having as internal vertices the four vertices of their common 3-path.
For $\ell = 1, \ldots, \left(\sum n_j - 3s\right)$, all the walks of length 5 in G are found.
The assertion follows excluding walks with some repeated vertex. □

3. Main result and special cases

In this section the study of the structure of the $(q-1)$-paths contained in a connected graph G and of the generators of the generalized graph ideals $I_q(G)$ is developed for every admissible q.

To this proposal, the following two theorems give a first partial answer.

Theorem 3.1. $(q = 3)$
Let G, and M_G be as in the Proposition 2.1, and R_{i_2} be any row of M_G.
Let's consider on R_{i_2} every pair $a_{i_2 h}, a_{i_2 k}$ of entries 1, and let $a_{i_1 h}, a_{i_3 k}$ be the other entries 1 of the relative columns.
Then, for every choice of R_{i_2}, all the 2-paths of G and the generators of $I_3(G)$ are of the type $x_{i_1} x_{i_2} x_{i_3}$, $1 \leqslant i_1 \neq i_2 \neq i_3 \leqslant p$.

Theorem 3.2. $(q = 4)$
Let G, and M_G be as in the Proposition 2.2, and Γ_j be any column of M_G.
Let R_{i_2}, R_{i_3} be the rows of M_G on which the entries 1 of Γ_j, $a_{i_2 j}, a_{i_3 j}$, lie.
Let's combine the pairs of 1's on R_{i_2} that contain $a_{i_2 j}$ together with the pairs of 1's on R_{i_3} that contain $a_{i_3 j}$.
Let $a_{i_2 h} \in \Gamma_h$, $a_{i_3 k} \in \Gamma_k$ be the other entries 1 in the pairs for each of these combinations, and $a_{i_1 h} \in R_{i_1}$, $a_{i_4 k} \in R_{i_4}$ be the remaining entries 1 of the relative columns.
Then, for every choice of Γ_j, all the 3-paths of G are of the type $x_{i_1} x_{i_2} x_{i_3} x_{i_4}$, $1 \leqslant i_1 \neq i_2 \neq \neq i_3 \neq i_4 \leqslant p$, and the generators of $I_4(G)$ are the 3-paths of G different from one another for at least an index.

The next result finds a solution to the problem of determining the composition of the $(q-1)$-paths of G and the generators of $I_q(G)$.

It is essential in the proof the fact that all these paths are connected to well-defined submatrices of the incidence matrix of G.

Theorem 3.3.
Let G be a connected graph with p vertices, and incidence matrix M_G.
Let N_ℓ be any $(p\times(q-3))$-submatrix of M_G, for $5\leqslant q\leqslant p$, having $q-4$ rows each with two entries 1, say $R_{i_3},\ldots, R_{i_{q-2}}$.
Let $R_{i_2}, R_{i_{q-1}}$ be the rows of M_G relative to the rows of N_ℓ with a unique entry 1, and R_{i_1}, R_{i_q} be the rows of M_G on which lies the remaining entry 1 of the columns of M_G, not of N_ℓ, located by an entry 1 of R_{i_2} and an entry 1 of $R_{i_{q-1}}$.
Then all the $(q-1)$-paths of G are of the type $x_{i_1}\ldots x_{i_q}$, $1\leqslant i_1\neq\ldots\neq i_q\leqslant p$, and the generators of $I_q(G)$ are the $(q-1)$-paths of G different from one another for at least an index.

Proof. Every $(p\times\tau)$-submatrix of M_G, $\tau\leqslant(p-3)$, having $\tau-1$ rows each with two entries 1 and two rows with a unique entry 1, corresponds to a τ-path of G whose internal vertices are related to the rows with two 1's and whose ends to the rows with a unique 1.
Then to construct the $(q-1)$-paths contained in G, when $5\leqslant q\leqslant p$, it needs to start from any submatrix N_ℓ, that corresponds to any $(q-3)$-path whose vertices are related to the non-zero rows $R_{i_2},\ldots,R_{i_{q-1}}$ of N_ℓ and represent the internal vertices of the $(q-1)$-paths of G.
To determine the ends of the $(q-1)$-paths of G, it is necessary to consider all the entries 1 in each of the rows $R_{i_2}, R_{i_{q-1}}$ of M_G.
If one of these rows contains a unique entry 1, no $(q-1)$-path is formed.
Otherwise, let's combine every pair of 1's on R_{i_2} that contains the $1\in N_\ell$ together with every pair of 1's on $R_{i_{q-1}}$ that contains the $1\in N_\ell$.
For each of these combinations, if Γ_h and Γ_k are the columns of M_G to which the $1\notin N_\ell$ belong, let R_{i_1}, R_{i_q} be the rows of M_G on which the remaining entry 1 of Γ_h and Γ_k lies.
When R_{i_1}, R_{i_q} are different from each other and from $R_{i_2},\ldots, R_{i_{q-1}}$, they represent the ends of the $(q-1)$-paths contained in G.
Such $(q-1)$-paths, for every choice of N_ℓ, have the form $x_{i_1}\ldots x_{i_q}$, where x_{i_j} are related to R_{i_j}, $j=1,\ldots,q$.
The last assertion derives from the definition of generalized graph ideal. □

Remark 3.1. The question to calculate the number of paths of any length is fully solved for particular graphs such as cycles, complete and complete bipartite graphs.

Example 3.1. Determine the number of $(q-1)$-paths, $q\geqslant 2$, of a complete bipartite graph $K_{m,n}$, $(m+n)\geqslant 3$.

$K_{m,n}$ has m vertices of degree n and n vertices of degree m;
$K_{m,n}$ has mn edges (or 1-paths);
in $K_{m,n}$ there are $\dfrac{k!\,k!}{2k}\binom{m}{k}\binom{n}{k}$ cycle subgraphs C_{2k}, $k\geqslant 2$;
the incidence matrix of $K_{m,n}$ is a $((m+n)\times mn)$-matrix.

According to Propositions 2.1, 2.2, 2.3, 2.4, and [4] Corollary 4, in the complete bipartite graph $K_{m,n}$ there are:

$$m\binom{n}{2}+n\binom{m}{2} \quad \text{2-paths;}$$

$$m\,n\begin{bmatrix}m\\n\end{bmatrix}=m\,(m-1)\,n\,(n-1)=2\binom{m}{2}2\binom{n}{2} \quad \text{3-paths;}$$

$$\begin{aligned}
&m\binom{n}{2}\begin{bmatrix}m\\m\end{bmatrix}+n\binom{m}{2}\begin{bmatrix}n\\n\end{bmatrix}-4\,\frac{2!\,2!}{4}\binom{m}{2}\binom{n}{2}=\\
&\quad=2\binom{m}{2}2\binom{n}{2}\frac{m-1}{2}+2\binom{m}{2}2\binom{n}{2}\frac{n-1}{2}-2\binom{m}{2}2\binom{n}{2}=\\
&\quad=m\,(m-1)\,n\,(n-1)\,\frac{(m-2)+(n-2)}{2}=\\
&\quad=3!\binom{m}{3}\binom{n}{2}+\binom{m}{2}3!\binom{n}{3} \quad \text{4-paths;}
\end{aligned}$$

$$\begin{aligned}
&m\,n\begin{bmatrix}m\\n\end{bmatrix}\begin{bmatrix}m\\n\end{bmatrix}-(4+2\,(2\,(m-2)+2\,(n-2)\,)\,)\,\frac{2!\,2!}{4}\binom{m}{2}\binom{n}{2}=\\
&\quad=2\binom{m}{2}2\binom{n}{2}\big((m-1)\,(n-1)-(1+m-2+n-2\,)\big)=\\
&\quad=m\,(m-1)\,n\,(n-1)\,(\,m\,n-m-n+1-m-n+3\,)=\\
&\quad=m\,(m-1)\,(m-2)\,n\,(n-1)\,(n-2)=3!\binom{m}{3}+3!\binom{n}{3} \quad \text{5-paths;}
\end{aligned}$$

. . . .

$$\frac{h!}{2}\binom{m}{h}(h+1)!\binom{n}{h+1}+(h+1)!\binom{m}{h+1}\frac{h!}{2}\binom{n}{h} \quad (m+n-1)\text{-paths}$$

if $m+n=2\,h+1$ is odd,

$$h!\binom{m}{h}h!\binom{n}{h} \quad (m+n-1)\text{-paths}$$

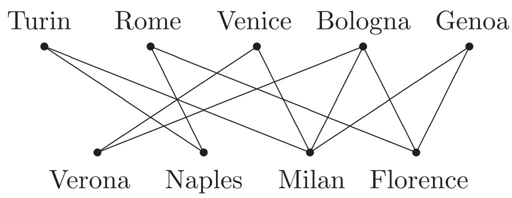
if $m+n=2\,h$ is even.

When $q\geqslant 3$, Theorem 3.3 gives the structure of $(q-1)$-paths of $K_{m,n}$ and the generators of the generalized graph ideals $L_q(K_{m,n})$.

4. An application

Let's consider an interesting application of a transportation problem that can occur in tourist sphere.

Problem 4.1. *An Italian travel agency proposes tourist packs, reserved to foreign visitors, that include sightseeing of famous Italian towns to be chosen from a group of* 9 *towns. These towns are linked up according to the following connected (bipartite not complete) graph G:*



Determine how many and which distinct tourist packs including sightseeing of 3 *towns the tour operator can propose. And of* 4, 5, *or* 6 *towns?*
Are there distinct tourist packs including sightseeing of more than 6 *towns?*

It is clear that the difficulty of computation does not depend on the total number of available towns, but only on the number of towns that each pack includes together with the connections among these towns.
The algebraic interest of the argument is evident. The generalized graph ideals are still studied by several known algebraists (see [5]).
A solution to this problem is found by applying the expounded techniques.
Towns in G are the generators of the generalized graph ideal:

$$I_1(G) = (\, x_1 = \text{Turin},\ x_2 = \text{Rome},\ x_3 = \text{Venice},\ x_4 = \text{Bologna},\ x_5 = \text{Genoa}\,; \quad y_6 = \text{Verona},\ y_7 = \text{Naples},\ y_8 = \text{Milan},\ y_9 = \text{Florence}\,)\,.$$

Note that the generalized graph ideal is an ideal of mixed products for $q \geqslant 2$.
Lines joining two towns in G generate the generalized graph ideal:

$$L_2(G) = (\, x_1 y_7,\ x_1 y_8,\ x_2 y_7,\ x_2 y_9,\ x_3 y_6,\ x_3 y_8,\ x_4 y_6,\ x_4 y_8,\ x_4 y_9,\ x_5 y_8,\ x_5 y_9\,)\,.$$

The incidence matrix of the graph is $M_G = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&1&0&1&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&1&0&1&0\\ 0&0&0&1&0&0&0&0&1&0&1 \end{pmatrix}$.

Notice that the maximal length of a path in G is 8, so the tour operator can propose tourist packs that include sightseeing of all the towns in G.

Let's first determine the number and the composition of distinct tourist packs that include sightseeing of 3 towns.

According to Proposition 2.1, the number of tourist packs is:

$$\binom{2}{2}+\binom{2}{2}+\binom{2}{2}+\binom{3}{2}+\binom{2}{2}+\binom{2}{2}+\binom{2}{2}+\binom{4}{2}+\binom{3}{2} = 18\,.$$

According to Theorem 3.1, their composition is:

$$x_1 y_7 x_2,\ x_1 y_8 x_3,\ x_1 y_8 x_4,\ x_1 y_8 x_5,\ x_2 y_9 x_4,\ x_2 y_9 x_5,$$
$$x_3 y_6 x_4,\ x_3 y_8 x_4,\ x_3 y_8 x_5,\ x_4 y_8 x_5,\ x_4 y_9 x_5,\ y_6 x_3 y_8,$$
$$y_6 x_4 y_8,\ y_6 x_4 y_9,\ y_7 x_1 y_8,\ y_7 x_2 y_9,\ y_8 x_4 y_9,\ y_8 x_5 y_9\,.$$

They are different from one another at least in a town, so

$$L_3(G) = (\, x_1 y_7 x_2,\ x_1 y_8 x_3,\ x_1 y_8 x_4,\ x_1 y_8 x_5,\ x_2 y_9 x_4,\ x_2 y_9 x_5,$$
$$x_3 y_6 x_4,\ x_3 y_8 x_4,\ x_3 y_8 x_5,\ x_4 y_8 x_5,\ x_4 y_9 x_5,\ y_6 x_3 y_8,$$
$$y_6 x_4 y_8,\ y_6 x_4 y_9,\ y_7 x_1 y_8,\ y_7 x_2 y_9,\ y_8 x_4 y_9,\ y_8 x_5 y_9)\,,$$

has 18 generators.

Let's determine the number and the composition of distinct tourist packs that include sightseeing of 4 towns.

According to Proposition 2.2, the number of tourist packs is:

$$\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix} =$$
$$= \begin{bmatrix}4\\3\end{bmatrix}+3\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+3\begin{bmatrix}3\\2\end{bmatrix}+3\begin{bmatrix}2\\2\end{bmatrix} = 6+9+4+6+3 = 28\,.$$

According to Theorem 3.2 , their composition is:

$$x_1\,y_7\,x_2\,y_9,\ x_1\,y_8\,x_3\,y_6,\ x_1\,y_8\,x_4\,y_6,\ x_1\,y_8\,x_4\,y_9,\ x_1\,y_8\,x_5\,y_9,\ x_2\,y_7\,x_1\,y_8,\ x_2\,y_9\,x_4\,y_6,$$
$$x_2\,y_9\,x_4\,y_8,\ x_2\,y_9\,x_5\,y_8,\ x_3\,y_6\,x_4\,y_8,\ x_3\,y_6\,x_4\,y_9,\ x_3\,y_8\,x_1\,y_7,\ x_3\,y_8\,x_4\,y_6,\ x_3\,y_8\,x_4\,y_9,$$
$$x_3\,y_8\,x_5\,y_9,\ x_4\,y_6\,x_3\,y_8,\ x_4\,y_8\,x_1\,y_7,\ x_4\,y_8\,x_3\,y_6,\ x_4\,y_8\,x_5\,y_9,\ x_4\,y_9\,x_2\,y_7,\ x_4\,y_9\,x_5\,y_8,$$
$$x_5\,y_8\,x_1\,y_7,\ x_5\,y_8\,x_3\,y_6,\ x_5\,y_8\,x_4\,y_6,\ x_5\,y_8\,x_4\,y_9,\ x_5\,y_9\,x_2\,y_7,\ x_5\,y_9\,x_4\,y_6,\ x_5\,y_9\,x_4\,y_8\,.$$

Those different from one another at least in a town generate

$$\begin{aligned}L_4(G) = (\,& x_1\,y_7\,x_2\,y_9,\ x_1\,y_8\,x_3\,y_6,\ x_1\,y_8\,x_4\,y_6,\ x_1\,y_8\,x_4\,y_9,\ x_1\,y_8\,x_5\,y_9,\ x_2\,y_7\,x_1\,y_8,\\ & x_2\,y_9\,x_4\,y_6,\ x_2\,y_9\,x_4\,y_8,\ x_2\,y_9\,x_5\,y_8,\ x_3\,y_6\,x_4\,y_8,\ x_3\,y_6\,x_4\,y_9,\ x_3\,y_8\,x_1\,y_7,\\ & x_3\,y_8\,x_4\,y_9,\ x_3\,y_8\,x_5\,y_9,\ x_4\,y_8\,x_1\,y_7,\ x_4\,y_8\,x_5\,y_9,\ x_4\,y_9\,x_2\,y_7,\ x_5\,y_8\,x_1\,y_7,\\ & x_5\,y_8\,x_3\,y_6,\ x_5\,y_8\,x_4\,y_6,\ x_5\,y_9\,x_2\,y_7,\ x_5\,y_9\,x_4\,y_6)\,,\end{aligned}$$

and they are 22 .

Let's determine the number and the composition of distinct tourist packs that include sightseeing of 5 towns.

Observe that there are 18 (9×2)-submatrices of M_G having just one row of 1's and two rows with a unique 1 .
These three rows correspond respectively to the internal town and to the towns at the ends of each tourist pack that includes sightseeing of 3 towns.
By using software `Mathematica`, one can write out the above submatrices.
According to Proposition 2.3 , the number of tourist packs is:

$$\begin{aligned}&\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\\ &+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}-4\cdot 2=\\ &=2\begin{bmatrix}4\\3\end{bmatrix}+3\begin{bmatrix}4\\2\end{bmatrix}+8\begin{bmatrix}3\\2\end{bmatrix}+5\begin{bmatrix}2\\2\end{bmatrix}-8=12+9+16+5-8=34\,.\end{aligned}$$

According to Theorem 3.3 , their composition is:

$$x_1\,y_7\,x_2\,y_9\,x_4,\ x_1\,y_7\,x_2\,y_9\,x_5,\ x_1\,y_8\,x_3\,y_6\,x_4,\ x_1\,y_8\,x_4\,y_6\,x_3,\ x_1\,y_8\,x_4\,y_9\,x_2,$$
$$x_1\,y_8\,x_4\,y_9\,x_5,\ x_1\,y_8\,x_5\,y_9\,x_2,\ x_1\,y_8\,x_5\,y_9\,x_4,\ x_2\,y_7\,x_1\,y_8\,x_3,\ x_2\,y_7\,x_1\,y_8\,x_4,$$
$$x_2\,y_7\,x_1\,y_8\,x_5,\ x_2\,y_9\,x_4\,y_6\,x_3,\ x_2\,y_9\,x_4\,y_8\,x_3,\ x_2\,y_9\,x_4\,y_8\,x_5,\ x_2\,y_9\,x_5\,y_8\,x_3,$$
$$x_2\,y_9\,x_5\,y_8\,x_4,\ x_3\,y_6\,x_4\,y_8\,x_5,\ x_3\,y_6\,x_4\,y_9\,x_5,\ x_3\,y_8\,x_4\,y_9\,x_5,\ x_3\,y_8\,x_5\,y_9\,x_4,$$
$$x_4\,y_6\,x_3\,y_8\,x_5,\ y_6\,x_3\,y_8\,x_1\,y_7,\ y_6\,x_3\,y_8\,x_4\,y_9,\ y_6\,x_3\,y_8\,x_5\,y_9,\ y_6\,x_4\,y_8\,x_1\,y_7,$$
$$y_6\,x_4\,y_8\,x_5\,y_9,\ y_6\,x_4\,y_9\,x_2\,y_7,\ y_6\,x_4\,y_9\,x_5\,y_8,\ y_7\,x_1\,y_8\,x_4\,y_9,\ y_7\,x_1\,y_8\,x_5\,y_9,$$
$$y_7\,x_2\,y_9\,x_4\,y_8,\ y_7\,x_2\,y_9\,x_5\,y_8,\ y_8\,x_1\,y_7\,x_2\,y_9,\ y_8\,x_3\,y_6\,x_4\,y_9\,.$$

Those different from one another at least in a town generate

$$\begin{aligned}L_5(G) = (\,& x_1\,y_7\,x_2\,y_9\,x_4,\ x_1\,y_7\,x_2\,y_9\,x_5,\ x_1\,y_8\,x_3\,y_6\,x_4,\ x_1\,y_8\,x_4\,y_9\,x_2,\\ & x_1\,y_8\,x_4\,y_9\,x_5,\ x_1\,y_8\,x_5\,y_9\,x_2,\ x_2\,y_7\,x_1\,y_8\,x_3,\ x_2\,y_7\,x_1\,y_8\,x_4,\\ & x_2\,y_7\,x_1\,y_8\,x_5,\ x_2\,y_9\,x_4\,y_6\,x_3,\ x_2\,y_9\,x_4\,y_8\,x_3,\ x_2\,y_9\,x_4\,y_8\,x_5,\\ & x_2\,y_9\,x_5\,y_8\,x_3,\ x_3\,y_6\,x_4\,y_8\,x_5,\ x_3\,y_6\,x_4\,y_9\,x_5,\ x_3\,y_8\,x_4\,y_9\,x_5,\\ & y_6\,x_3\,y_8\,x_1\,y_7,\ y_6\,x_3\,y_8\,x_4\,y_9,\ y_6\,x_3\,y_8\,x_5\,y_9,\ y_6\,x_4\,y_8\,x_1\,y_7,\\ & y_6\,x_4\,y_8\,x_5\,y_9,\ y_6\,x_4\,y_9\,x_2\,y_7,\ y_7\,x_1\,y_8\,x_4\,y_9,\ y_7\,x_1\,y_8\,x_5\,y_9,\\ & y_7\,x_2\,y_9\,x_4\,y_8,\ y_7\,x_2\,y_9\,x_5\,y_8,\ y_8\,x_1\,y_7\,x_2\,y_9)\,,\end{aligned}$$

and they are 27 .

Let's determine the number and the composition of distinct tourist packs that include sightseeing of 6 towns.

Observe that there are 28 (9×3)-submatrices of M_G having just two rows with two 1's and two rows with a unique 1 .
These four rows correspond respectively to the internal towns and to the towns at the ends of each tourist pack that includes sightseeing of 4 towns.
By using software `Mathematica`, one can write out the above submatrices.
According to Proposition 2.4 , the number of tourist packs is:

$$\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+$$
$$+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+$$
$$+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}-(4+6)-(4+8)=$$
$$=2\begin{bmatrix}4\\3\end{bmatrix}+5\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+10\begin{bmatrix}3\\2\end{bmatrix}+10\begin{bmatrix}2\\2\end{bmatrix}-10-12=$$
$$=12+15+4+20+10-22=39\,.$$

According to Theorem 3.3 , their composition is:

$$\begin{array}{lllll}
x_1\,y_7\,x_2\,y_9\,x_4\,y_6, & x_1\,y_7\,x_2\,y_9\,x_4\,y_8, & x_1\,y_7\,x_2\,y_9\,x_5\,y_8, & x_1\,y_8\,x_3\,y_6\,x_4\,y_9, & x_1\,y_8\,x_4\,y_9\,x_2\,y_7,\\
x_1\,y_8\,x_5\,y_9\,x_2\,y_7, & x_1\,y_8\,x_5\,y_9\,x_4\,y_6, & x_2\,y_7\,x_1\,y_8\,x_3\,y_6, & x_2\,y_7\,x_1\,y_8\,x_4\,y_6, & x_2\,y_7\,x_1\,y_8\,x_4\,y_9,\\
x_2\,y_7\,x_1\,y_8\,x_5\,y_9, & x_2\,y_9\,x_4\,y_6\,x_3\,y_8, & x_2\,y_9\,x_4\,y_8\,x_1\,y_7, & x_2\,y_9\,x_4\,y_8\,x_3\,y_6, & x_2\,y_9\,x_5\,y_8\,x_1\,y_7,\\
x_2\,y_9\,x_5\,y_8\,x_3\,y_6, & x_2\,y_9\,x_5\,y_8\,x_4\,y_6, & x_3\,y_6\,x_4\,y_8\,x_1\,y_7, & x_3\,y_6\,x_4\,y_8\,x_5\,y_9, & x_3\,y_6\,x_4\,y_9\,x_2\,y_7,\\
x_3\,y_6\,x_4\,y_9\,x_5\,y_8, & x_3\,y_8\,x_1\,y_7\,x_2\,y_9, & x_3\,y_8\,x_4\,y_9\,x_2\,y_7, & x_3\,y_8\,x_5\,y_9\,x_2\,y_7, & x_3\,y_8\,x_5\,y_9\,x_4\,y_6,\\
x_4\,y_6\,x_3\,y_8\,x_1\,y_7, & x_4\,y_6\,x_3\,y_8\,x_5\,y_9, & x_4\,y_8\,x_1\,y_7\,x_2\,y_9, & x_4\,y_8\,x_5\,y_9\,x_2\,y_7, & x_4\,y_9\,x_2\,y_7\,x_1\,y_8,\\
x_4\,y_9\,x_5\,y_8\,x_1\,y_7, & x_4\,y_9\,x_5\,y_8\,x_3\,y_6, & x_5\,y_8\,x_1\,y_7\,x_2\,y_9, & x_5\,y_8\,x_3\,y_6\,x_4\,y_9, & x_5\,y_8\,x_4\,y_9\,x_2\,y_7,\\
x_5\,y_9\,x_2\,y_7\,x_1\,y_8, & x_5\,y_9\,x_4\,y_6\,x_3\,y_8, & x_5\,y_9\,x_4\,y_8\,x_1\,y_7, & x_5\,y_9\,x_4\,y_8\,x_3\,y_6\,. &
\end{array}$$

Those different from one another at least in a town generate

$$\begin{aligned}
L_6(G)=(\,&x_1\,y_7\,x_2\,y_9\,x_4\,y_6,\ x_1\,y_7\,x_2\,y_9\,x_4\,y_8,\ x_1\,y_7\,x_2\,y_9\,x_5\,y_8,\ x_1\,y_8\,x_3\,y_6\,x_4\,y_9,\\
&x_1\,y_8\,x_5\,y_9\,x_4\,y_6,\ x_2\,y_7\,x_1\,y_8\,x_3\,y_6,\ x_2\,y_7\,x_1\,y_8\,x_4\,y_6,\ x_2\,y_9\,x_4\,y_6\,x_3\,y_8,\\
&x_2\,y_9\,x_5\,y_8\,x_3\,y_6,\ x_2\,y_9\,x_5\,y_8\,x_4\,y_6,\ x_3\,y_6\,x_4\,y_8\,x_1\,y_7,\ x_3\,y_6\,x_4\,y_8\,x_5\,y_9,\\
&x_3\,y_6\,x_4\,y_9\,x_2\,y_7,\ x_3\,y_8\,x_1\,y_7\,x_2\,y_9,\ x_3\,y_8\,x_4\,y_9\,x_2\,y_7,\ x_3\,y_8\,x_5\,y_9\,x_2\,y_7,\\
&x_4\,y_8\,x_5\,y_9\,x_2\,y_7,\ x_4\,y_9\,x_5\,y_8\,x_1\,y_7)\,,
\end{aligned}$$

and they are 18 .

Finally, by Theorem 3.3, we are able to know which distinct tourist packs including sightseeing of more than 6 towns the tour operator can propose.

To determine the composition of tourist packs that include sightseeing of 7 towns, let's consider the 34 $(9{\times}4)$-submatrices of M_G having just three rows with two 1's and two rows with a unique 1, that correspond to the tourist packs including sightseeing of 5 towns. So it is:

$$\begin{array}{llll}
x_1\,y_7\,x_2\,y_9\,x_4\,y_6\,x_3, & x_1\,y_7\,x_2\,y_9\,x_4\,y_8\,x_3, & x_1\,y_7\,x_2\,y_9\,x_4\,y_8\,x_5, & x_1\,y_7\,x_2\,y_9\,x_5\,y_8\,x_3,\\
x_1\,y_7\,x_2\,y_9\,x_5\,y_8\,x_4, & x_1\,y_8\,x_3\,y_6\,x_4\,y_9\,x_2, & x_1\,y_8\,x_3\,y_6\,x_4\,y_9\,x_5, & x_1\,y_8\,x_5\,y_9\,x_4\,y_6\,x_3,\\
x_2\,y_7\,x_1\,y_8\,x_3\,y_6\,x_4, & x_2\,y_7\,x_1\,y_8\,x_4\,y_6\,x_3, & x_2\,y_7\,x_1\,y_8\,x_4\,y_9\,x_5, & x_2\,y_7\,x_1\,y_8\,x_5\,y_9\,x_4,\\
x_2\,y_9\,x_4\,y_6\,x_3\,y_8\,x_5, & x_2\,y_9\,x_5\,y_8\,x_3\,y_6\,x_4, & x_2\,y_9\,x_5\,y_8\,x_4\,y_6\,x_3, & x_3\,y_8\,x_1\,y_7\,x_2\,y_9\,x_4,\\
x_3\,y_8\,x_1\,y_7\,x_2\,y_9\,x_5, & x_4\,y_8\,x_1\,y_7\,x_2\,y_9\,x_5, & x_4\,y_9\,x_2\,y_7\,x_1\,y_8\,x_5, & y_6\,x_3\,y_8\,x_1\,y_7\,x_2\,y_9,\\
y_6\,x_3\,y_8\,x_4\,y_9\,x_2\,y_7, & y_6\,x_3\,y_8\,x_5\,y_9\,x_2\,y_7, & y_6\,x_4\,y_8\,x_1\,y_7\,x_2\,y_9, & y_6\,x_4\,y_8\,x_5\,y_9\,x_2\,y_7,\\
y_6\,x_4\,y_9\,x_2\,y_7\,x_1\,y_8, & y_6\,x_4\,y_9\,x_5\,y_8\,x_1\,y_7, & y_7\,x_1\,y_8\,x_3\,y_6\,x_4\,y_9, & y_7\,x_2\,y_9\,x_4\,y_6\,x_3\,y_8\,,
\end{array}$$

and they are 28 .

Those different from one another at least in a town generate

$$L_7(G) = (\, x_1\, y_7\, x_2\, y_9\, x_4\, y_6\, x_3,\;\; x_1\, y_7\, x_2\, y_9\, x_4\, y_8\, x_3,\;\; x_1\, y_7\, x_2\, y_9\, x_4\, y_8\, x_5,\;\; x_1\, y_7\, x_2\, y_9\, x_5\, y_8\, x_3,\;\; x_1\, y_8\, x_3\, y_6\, x_4\, y_9\, x_2,\;\; x_1\, y_8\, x_3\, y_6\, x_4\, y_9\, x_5,\;\; x_2\, y_7\, x_1\, y_8\, x_3\, y_6\, x_4,\;\; x_2\, y_9\, x_5\, y_8\, x_3\, y_6\, x_4,\;\; y_6\, x_3\, y_8\, x_1\, y_7\, x_2\, y_9,\;\; y_6\, x_3\, y_8\, x_4\, y_9\, x_2\, y_7,\;\; y_6\, x_3\, y_8\, x_5\, y_9\, x_2\, y_7,\;\; y_6\, x_4\, y_8\, x_1\, y_7\, x_2\, y_9,\;\; y_6\, x_4\, y_8\, x_5\, y_9\, x_2\, y_7,\;\; y_6\, x_4\, y_9\, x_5\, y_8\, x_1\, y_7,\;\; y_7\, x_1\, y_8\, x_3\, y_6\, x_4\, y_9\,)\,,$$

and they are 15 .

In a similar way, the composition of the remaining tourist packs is easily definable. In particular the tour operator can propose 21 tourist packs that include sightseeing of 8 towns, of which 5 generate $L_8(G)$, and 8 tourist packs that include sightseeing of 9 towns.

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