Generators of Generalized Graph Ideals

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Abstract.

This work deals with the way to determine, in the degree \( q \leq 6 \), how many paths of length \((q-1)\) are contained in a connected graph \(G\), using only its incidence matrix. The composition of such paths and the generators of the generalized graph ideals relative to \(G\) are studied for every degree \(q\). An interesting application is given in tourist sphere.

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1. Introduction

To get paths of length \((q-1)\), \(q\) positive integer, in a connected graph \(G\), algebraically means to find generators of a monomial ideal to which \(G\) can be associated, the generalized graph ideal \(I_q(G)\).

The question to compute, using only the incidence matrix of \(G\), the number and the structure of paths of \(G\), and the generators of the relative generalized graph ideals, has been introduced in [2] and extended in [3].

In such papers, an answer to it has been given for \(q \leq 6\). The number of paths of \(G\) having length less than 5 has been obtained in terms of multiplicity of pairs of rows in the incidence matrix of \(G\), holding account of eventual cycle subgraphs contained in \(G\). Their structure has been found by joining alternatively on the rows and columns of the incidence matrix specific entries 1 relative to the vertices of the paths starting from the inside.

In this work, by also introducing new strategies in the calculations, the problem to determine the structure of paths of any length in \(G\), and the generators of the relative generalized graph ideals is fully solved.

An example of algebraic model arising from a transportation problem that can occur in tourist sphere is presented.
2. Preliminaries and paths of a graph

**Definition 2.1.** A graph $G$ is said connected if every pair of vertices of $G$ are joined by a path, that is a walk whose vertices are distinct.

**Definition 2.2.** Let $G$ be a connected graph having vertices $v_1, \ldots, v_p$. A generalized graph ideal $I_q(G)$, $N \geq q \leq p$, is an ideal of the polynomial ring $K[x_1, \ldots, x_p]$, where $K$ is a field and each variable $x_i$ corresponds to $v_i$, generated by all the square-free monomials $x_{i_1} \cdots x_{i_q}$ of degree $q$ such that the vertex $v_i$ is adjacent to $v_{i_j}$, for all $1 \leq j \leq (q-1)$.

**Remark 2.1.** $I_2(G)$ is the generalized graph ideal generated by the edges of $G$, the so-called edge ideal. More generally, the generators of $I_q(G)$ are paths of $G$ of length $(q-1)$, simply called $(q-1)$-paths.

**Definition 2.3.** A monomial ideal $L_q \subset K[x_1, \ldots, x_m; y_1, \ldots, y_n]$ is an ideal of mixed products if it is writable as $L_q = I_p J_r + I_s J_t$, where

$q = p + r = s + t$, for $p, r, s, t$ non-negative integers;

$I_p$ (resp. $J_r$) is an ideal of $K[x_1, \ldots, x_m; y_1, \ldots, y_n]$ generated by square-free monomials of degree $p$ (resp. $r$) in the variables $x_1, \ldots, x_m$ (resp. $y_1, \ldots, y_n$).

**Definition 2.4.** A graph $G$ is said bipartite if the vertex set $V$ of $G$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins $V_1$ with $V_2$.

For bipartite graphs, the generalized graph ideals are particular ideals of mixed products. To this proposal the following properties hold.

**Property 2.1.** Let $G$ be a complete bipartite graph, $V = \{x_1, \ldots, x_m; y_1, \ldots, y_n\}$ the vertex set of $G$. Then, for $2 \leq q < (m+n)$, the generalized graph ideal $L_q(G)$ has the form

$$L_q(G) = \begin{cases} I_h J_{h+1} + I_{h+1} J_h & \text{if } q = 2h+1 \\ I_h J_h & \text{if } q = 2h \\ \end{cases}.$$ 

**Property 2.2.** Let $L_q \subset K[x_1, \ldots, x_m; y_1, \ldots, y_n]$, $2 \leq q < (m+n)$, be an ideal of mixed products of the form

$$(a) \quad I_h J_{h+1} \quad \text{or} \quad I_{h+1} J_h \quad \text{or} \quad I_h J_{h+1} + I_{h+1} J_h \quad \text{for } h = \frac{q-1}{2};$$

$$(b) \quad I_h J_h \quad \text{for } h = \frac{q}{2}.$$ 

Then $L_q = L_q(G)$, $G$ a complete bipartite graph with vertex set $V = \{x_1, \ldots, x_m; y_1, \ldots, y_n\}$.

**Definition 2.5.** Let $G$ be a graph with $p$ vertices and $t$ edges. The incidence matrix $M_G$ of $G$ is a $(p \times t)$-matrix whose entries $a_{ij}$ are equal to 1 if the $i$-th vertex of $G$ belongs to the $j$-th edge, 0 otherwise.

**Remark 2.2.** Each row of $M_G$ has as many entries 1 as the degree of the relative vertex of $G$. Each column of $M_G$ has two entries 1 and the remaining 0.

**Definition 2.6.** Let $G$ be a connected graph and $M_G$ be the incidence matrix of $G$. We call multiplicity of a pair of rows in $M_G$, corresponding in $G$ to a pair of vertices having the degrees $\alpha$ and $\beta$ respectively, the product $(\alpha-1)(\beta-1)$, and denote it $\left[\begin{array}{c} \alpha \\ \beta \end{array}\right]$. 
Property 2.3. Let \( G \) and \( M_G \) be as in the last definition. The multiplicity of a pair of rows in \( M_G \), that correspond to a pair of vertices of \( G \) joined by a \((q-1)\)-path, \( q \geq 2 \), and that have the degrees \( \alpha \geq 2 \) and \( \beta \geq 2 \), gives the number of walks of length \((q+1)\) in \( G \) containing the \((q-1)\)-path inside.

The following results give the method to calculate, considering only the incidence matrix of a connected graph \( G \), how many \((q-1)\)-paths are contained in \( G \), for \( 3 \leq q \leq 6 \).

Proposition 2.1. (number of 2-paths)
Let \( G \) be a connected graph with \( p \geq 3 \) vertices, and incidence matrix \( M_G \). Let \( \lambda_1, \lambda_i \geq 2 \), denote the number of entries 1 in the \( i \)-th row of \( M_G \).

Then \( G \) has \( \sum_{i=1}^{p} \left( \frac{\lambda_i}{2} \right) \) 2-paths.

Proposition 2.2. (number of 3-paths)
Let \( G \) be a connected graph with \( p \geq 4 \) vertices, \( t \) edges, \( s \) cycle subgraphs \( C_3 \), and incidence matrix \( M_G \).

Let \( \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}, \alpha_j \geq \beta_j \geq 2 \), denote the multiplicity of the rows of \( M_G \) on which the entries 1 of its \( j \)-th column lie.

Then \( G \) has \( \sum_{j=1}^{t} \left( \frac{\alpha_j}{\beta_j} \right) - 3s \) 3-paths.

Proposition 2.3. (number of 4-paths)
Let \( G \) be a connected graph with \( p \geq 5 \) vertices \( v_1, \ldots, v_p \), \( t \) edges, \( s \) cycle subgraphs \( C_3 \), \( r \) cycle subgraphs \( C_4 \), and incidence matrix \( M_G \).

Let \( d_h \) be the number of vertices of \( G \) adjacent to \( C_3 \), \( h = 1, \ldots, s \).

For every \((p \times 2)\)-submatrix \( A_\ell \) of \( M_G \) having one row of 1’s, let \( \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix} \), \( \alpha_\ell \geq \beta_\ell \geq 2 \), be the multiplicity of the rows of \( M_G \) corresponding to the rows of \( A_\ell \) with a unique entry 1.

Then \( G \) has \( \sum_{\ell} \left[ \frac{\alpha_\ell}{\beta_\ell} \right] - \sum_{h=1}^{s} (3 + 2d_h) - 4r \) 4-paths.

Proof. We think of a 4-path of \( G \) as a pair of distinct 3-paths having a common 2-path.

\( G \) has \( \sum_{i=1}^{p} \left( \frac{\deg v_i}{2} \right) \) 2-paths, each characterized by a \((p \times 2)\)-submatrix \( A_\ell \) of \( M_G \) having just one row of 1’s and two rows with a unique entry 1.

If \( m_\ell \) is the multiplicity of the rows in \( M_G \) that correspond to the rows of \( A_\ell \) with a unique entry 1, every pair of 3-paths having a common 2-path determine \( m_\ell \) walks of length 4 in \( G \) having as internal vertices the three vertices of their common 2-path.

For \( \ell = 1, \ldots, \sum_{i=1}^{p} \left( \frac{\deg v_i}{2} \right) \), all the walks of length 4 in \( G \) are found.

The assertion follows excluding walks having some repeated vertex. \( \square \)

Proposition 2.4. (number of 5-paths)
Let \( G \) be a connected graph with \( p \geq 6 \) vertices \( v_1, \ldots, v_p \), \( t \) edges, \( s \) cycle subgraphs \( C_3 \), \( r \) cycle subgraphs \( C_4 \), \( \rho \) cycle subgraphs \( C_5 \), and incidence matrix \( M_G \).

For any \( C_5 \) in \( G \), let \( d_h \) be the number of vertices \( v_i \) of \( G \) adjacent to \( C_3 \), \( h = 1, \ldots, s \), and \( \varphi \) be the number of pairs of \( C_5 \) with a common edge.
For any $C_4$ in $G$, let $\delta_k$ be the number of vertices of $G$, not belonging to $C_4$, adjacent to $C_4$, $k = 1, \ldots, r$.

For every $(p \times 3)$-submatrix $B_\ell$ of $M_G$ having two rows each with two entries 1, let $[\alpha_\ell \beta_\ell]$, $\alpha_\ell \geq \beta_\ell \geq 2$, be the multiplicity of the rows of $M_G$ corresponding to the rows of $B_\ell$ with a unique entry 1.

Then $G$ has $\sum_\ell \left[ \frac{\alpha_\ell}{\beta_\ell} \right] - 2 \left( \sum_\lambda d_\lambda (\deg v_\lambda - 1) \right) + 2 \varphi - \sum_k (4 + 2 \delta_k) - 5 \rho$ 5-paths.

**Proof.** We think of a 5-path of $G$ as a pair of distinct 4-paths having a common 3-path. $G$ has $\sum n_j - 3s$ 3-paths, where $n_j$ is the multiplicity of the rows of $M_G$ on which the entries 1 of its $j$-th column lie, $j = 1, \ldots, t$.

Every 3-path is characterized by a $(p \times 3)$-submatrix $B_\ell$ of $M_G$ having just two rows with two 1’s and two rows with a unique entry 1.

If $m_\ell$ is the multiplicity of the rows in $M_G$ that correspond to the rows of $B_\ell$ with a unique entry 1, every pair of 4-paths having a common 3-path determine $m_\ell$ walks of length 5 in $G$ having as internal vertices the four vertices of their common 3-path.

For $\ell = 1, \ldots, (\sum n_j - 3 s)$, all the walks of length 5 in $G$ are found.

The assertion follows excluding walks with some repeated vertex. $\square$

### 3. Main result and special cases

In this section the study of the structure of the $(q-1)$-paths contained in a connected graph $G$ and of the generators of the generalized graph ideals $I_q(G)$ is developed for every admissible $q$.

To this proposal, the following two theorems give a first partial answer.

**Theorem 3.1.** $(q = 3)$

Let $G$, and $M_G$ be as in the Proposition 2.1, and $R_{i_3}$ be any row of $M_G$.

Let’s consider on $R_{i_3}$ every pair $a_{i_2 \cdot h}, a_{i_2 \cdot k}$ of entries 1, and let $a_{i_1 \cdot h}, a_{i_1 \cdot k}$ be the other entries 1 of the relative columns.

Then, for every choice of $R_{i_3}$, all the 2-paths of $G$ and the generators of $I_3(G)$ are of the type $x_{i_1} x_{i_2} x_{i_3}$, $1 \leq i_1 \neq i_2 \neq i_3 \leq p$.

**Theorem 3.2.** $(q = 4)$

Let $G$, and $M_G$ be as in the Proposition 2.2, and $\Gamma_j$ be any column of $M_G$.

Let $R_{i_2}, R_{i_3}$ be the rows of $M_G$ on which the entries 1 of $\Gamma_j$, $a_{i_2 \cdot j}, a_{i_3 \cdot j}$ lie.

Let’s combine the pairs of 1’s on $R_{i_2}$ that contain $a_{i_2 \cdot j}$ together with the pairs of 1’s on $R_{i_3}$ that contain $a_{i_3 \cdot j}$.

Let $a_{i_2 \cdot h} \in \Gamma_h$, $a_{i_3 \cdot k} \in \Gamma_k$ be the other entries 1 in the pairs for each of these combinations, and $a_{i_1 \cdot h} \in R_{i_1}$, $a_{i_4 \cdot k} \in R_{i_4}$ be the remaining entries 1 of the relative columns.

Then, for every choice of $\Gamma_j$, all the 3-paths of $G$ are of the type $x_{i_1} x_{i_2} x_{i_3} x_{i_4}$, $1 \leq i_1 \neq i_2 \neq i_3 \neq i_4 \leq p$, and the generators of $I_4(G)$ are the 3-paths of $G$ different from one another for at least an index.

The next result finds a solution to the problem of determining the composition of the $(q-1)$-paths of $G$ and the generators of $I_q(G)$. 
It is essential in the proof the fact that all these paths are connected to well-defined submatrices of the incidence matrix of \( G \).

**Theorem 3.3.**

Let \( G \) be a connected graph with \( p \) vertices, and incidence matrix \( M_G \).

Let \( N_\ell \) be any \((p \times (q-3))\)-submatrix of \( M_G \), for \( 5 \leq q \leq p \), having \( q-4 \) rows each with two entries \( 1 \), say \( R_{i_1}, \ldots, R_{i_{q-2}} \).

Let \( R_{i_2}, R_{i_{q-1}} \) be the rows of \( M_G \) relative to the rows of \( N_\ell \) with a unique entry \( 1 \), and \( R_{i_1}, R_{i_q} \) be the rows of \( M_G \) on which lies the remaining entry \( 1 \) of the columns of \( M_G \), not of \( N_\ell \), located by an entry \( 1 \) of \( R_{i_2} \) and an entry \( 1 \) of \( R_{i_{q-1}} \).

Then all the \((q-1)\)-paths of \( G \) are of the type \( x_{i_1} \ldots x_{i_q} \), \( 1 \leq i_1 \neq \ldots \neq i_q \leq p \), and the generators of \( I_q(G) \) are the \((q-1)\)-paths of \( G \) different from one another for at least an index.

**Proof.** Every \((p \times \tau)\)-submatrix of \( M_G \), \( \tau \leq (p-3) \), having \( \tau-1 \) rows each with two entries \( 1 \) and two rows with a unique entry \( 1 \), corresponds to a \( \tau \)-path of \( G \) whose internal vertices are related to the rows with two \( 1 \)'s and whose ends to the rows with a unique \( 1 \).

Then to construct the \((q-1)\)-paths contained in \( G \), when \( 5 \leq q \leq p \), it needs to start from any submatrix \( N_\ell \), that corresponds to any \((q-3)\)-path whose vertices are related to the non-zero rows \( R_{i_2}, \ldots, R_{i_{q-1}} \) of \( N_\ell \) and represent the internal vertices of the \((q-1)\)-paths of \( G \).

To determine the ends of the \((q-1)\)-paths of \( G \), it is necessary to consider all the entries \( 1 \) in each of the rows \( R_{i_2}, R_{i_{q-1}} \) of \( M_G \).

If one of these rows contains a unique entry \( 1 \), no \((q-1)\)-path is formed.

Otherwise, let’s combine every pair of \( 1 \)'s on \( R_{i_2} \) that contains the \( 1 \in N_\ell \) together with every pair of \( 1 \)'s on \( R_{i_{q-1}} \) that contains the \( 1 \in N_\ell \).

For each of these combinations, if \( \Gamma_h \) and \( \Gamma_k \) are the columns of \( M_G \) to which the \( 1 \notin N_\ell \) belong, let \( R_{i_1}, R_{i_q} \) be the rows of \( M_G \) on which the remaining entry \( 1 \) of \( \Gamma_h \) and \( \Gamma_k \) lies.

When \( R_{i_1}, R_{i_q} \) are different from each other and from \( R_{i_2}, \ldots, R_{i_{q-1}} \), they represent the ends of the \((q-1)\)-paths contained in \( G \).

Such \((q-1)\)-paths, for every choice of \( N_\ell \), have the form \( x_{i_1} \ldots x_{i_q} \), where \( x_{i_j} \) are related to \( R_{i_j} \), \( j = 1, \ldots, q \).

The last assertion derives from the definition of generalized graph ideal. \( \square \)

**Remark 3.1.** The question to calculate the number of paths of any length is fully solved for particular graphs such as cycles, complete and complete bipartite graphs.

**Example 3.1.** Determine the number of \((q-1)\)-paths, \( q \geq 2 \), of a complete bipartite graph \( K_{m,n} \), \((m+n) \geq 3\).

- \( K_{m,n} \) has \( m \) vertices of degree \( n \) and \( n \) vertices of degree \( m \);
- \( K_{m,n} \) has \( mn \) edges (or 1-paths);
- in \( K_{m,n} \) there are \( \frac{k!k!}{2k} \binom{m}{k} \binom{n}{k} \) cycle subgraphs \( C_{2k} \), \( k \geq 2 \);
- the incidence matrix of \( K_{m,n} \) is a \(((m+n) \times mn)\)-matrix.

According to Propositions 2.1, 2.2, 2.3, 2.4, and [4] Corollary 4, in the complete bipartite graph \( K_{m,n} \) there are:
\[ m \binom{n}{2} + n \binom{m}{2} \text{ 2-paths;} \]
\[ mn \left[ \binom{m}{n} \right] = m(m-1)n(n-1) = 2 \binom{m}{2} \binom{n}{2} \text{ 3-paths;} \]
\[ m \binom{2}{m} + n \binom{2}{n} \left[ \binom{m}{n} \right] - 4 \frac{2!2!}{4} \binom{m}{2} \binom{n}{2} = \]
\[ = 2 \binom{m}{2} \binom{n}{2} - \frac{m-1}{2} + 2 \binom{m}{2} \binom{n-1}{2} - 2 \binom{m}{2} \binom{n}{2} = \]
\[ = m(m-1)n(n-1) \frac{(m-2)+(n-2)}{2} = \]
\[ = 3! \binom{m}{3} \binom{n}{2} + \binom{m}{2} 3! \binom{n}{3} \text{ 4-paths;} \]
\[ mn \left[ \binom{m}{n} \right] = 4 \binom{m}{2} (2(m-2) + 2(n-2)) \left[ \binom{m}{n} \right] - \frac{2!2!}{4} \binom{m}{2} \binom{n}{2} = \]
\[ = 2 \binom{m}{2} \binom{n}{2} - (m-1)(n-1) - (1+m-2+n-2)) = \]
\[ = m(m-1)n(n-1) (mn - m - n + 1 - m - n + 3) = \]
\[ = m(m-1)(m-2)n(n-1)(n-2) = 3! \binom{m}{3} + 3! \binom{n}{3} \text{ 5-paths;} \]
\[ \frac{h!}{2} \binom{m}{h} \binom{n}{h+1} + (h+1)! \binom{m}{h+1} \frac{h!}{2} \binom{n}{h} \text{ (m+n-1)-paths} \]
\[ \text{if } m+n = 2h+1 \text{ is odd,} \]
\[ h! \binom{m}{h} h! \binom{n}{h} \text{ (m+n-1)-paths} \]
\[ \text{if } m+n = 2h \text{ is even.} \]

When \( q \geq 3 \), Theorem 3.3 gives the structure of \((q-1)\)-paths of \( K_{m,n} \) and the generators of the generalized graph ideals \( L_q(K_{m,n}) \).

### 4. An application

Let’s consider an interesting application of a transportation problem that can occur in tourist sphere.

**Problem 4.1.** An Italian travel agency proposes tourist packs, reserved to foreign visitors, that include sightseeing of famous Italian towns to be chosen from a group of 9 towns. These towns are linked up according to the following connected (bipartite not complete) graph \( G \):

[Diagram of the graph]

Determine how many and which distinct tourist packs including sightseeing of 3 towns the tour operator can propose. And of 4, 5, or 6 towns?

Are there distinct tourist packs including sightseeing of more than 6 towns?
It is clear that the difficulty of computation does not depend on the total number of available towns, but only on the number of towns that each pack includes together with the connections among these towns.

The algebraic interest of the argument is evident. The generalized graph ideals are still studied by several known algebraists (see [5]).

A solution to this problem is found by applying the expounded techniques.

Towns in $G$ are the generators of the generalized graph ideal:

$$I_1(G) = \langle x_1 = \text{Turin}, x_2 = \text{Rome}, x_3 = \text{Venice}, x_4 = \text{Bologna}, x_5 = \text{Genoa},$$

$$y_6 = \text{Verona}, y_7 = \text{Naples}, y_8 = \text{Milan}, y_9 = \text{Florence} \rangle.$$

Note that the maximal length of a path in $G$ is 8, so the tour operator can propose touristic packs that include sightseeing of all the towns in $G$.

Let’s first determine the number and the composition of distinct tourist packs that include sightseeing of 3 towns.

According to Proposition 2.1, the number of tourist packs is:

$$\binom{2}{2}+\binom{2}{2}+\binom{2}{2}+\binom{3}{2}+\binom{2}{2}+\binom{2}{2}+\binom{2}{2}+\binom{3}{2} = 18.$$

According to Theorem 3.1, their composition is:

$$x_1 y_7 x_2, x_1 y_8 x_3, x_1 y_8 x_4, x_1 y_8 x_5, x_2 y_9 x_4, x_2 y_9 x_5,$$

$$x_3 y_6 x_4, x_3 y_8 x_4, x_3 y_8 x_5, x_4 y_8 x_5, x_4 y_9 x_5, y_6 x_3 y_8,$$

$$y_6 x_4 y_8, y_6 x_4 y_9, y_7 x_1 y_8, y_7 x_2 y_9, y_8 x_4 y_9, y_8 x_5 y_9.$$

They are different from one another at least in a town, so

$$L_3(G) = \langle x_1 y_7 x_2, x_1 y_8 x_3, x_1 y_8 x_4, x_1 y_8 x_5, x_2 y_9 x_4, x_2 y_9 x_5,$$

$$x_3 y_6 x_4, x_3 y_8 x_4, x_3 y_8 x_5, x_4 y_8 x_5, x_4 y_9 x_5, y_6 x_3 y_8,$$

$$y_6 x_4 y_8, y_6 x_4 y_9, y_7 x_1 y_8, y_7 x_2 y_9, y_8 x_4 y_9, y_8 x_5 y_9 \rangle,$$

has 18 generators.

Let’s determine the number and the composition of distinct tourist packs that include sightseeing of 4 towns.

According to Proposition 2.2, the number of tourist packs is:

$$\binom{2}{2}+\binom{4}{2}+\binom{2}{2}+\binom{3}{2}+\binom{2}{2}+\binom{4}{2}+\binom{3}{2}+\binom{3}{2}+\binom{4}{2}+\binom{3}{2} + \binom{4}{2}+\binom{3}{2} =$$

$$= \binom{4}{3}+3 \binom{4}{2}+3 \binom{3}{2}+3 \binom{3}{2}+3 \binom{2}{2} = 6 + 9 + 4 + 6 + 3 = 28.$$
According to Theorem 3.2, their composition is:

\[
\begin{align*}
&x_1y_7x_2y_9, 
&x_1y_8x_3y_6, 
&x_1y_8x_4y_6, 
&x_1y_8x_4y_9, 
&x_1y_8x_5y_9, 
&x_2y_7x_1y_8, 
&x_2y_9x_4y_6, 
&x_2y_9x_5y_8, 
&x_2y_9x_5y_9, 
&x_3y_6x_4y_8, 
&x_3y_6x_4y_9, 
&x_3y_8x_1y_7, 
&x_4y_8x_3y_6, 
&x_4y_8x_5y_9, 
&x_4y_9x_2y_7, 
&x_4y_9x_5y_8, 
&x_5y_8x_1y_7, 
&x_5y_8x_3y_6, 
&x_5y_8x_4y_6, 
&x_5y_8x_4y_9, 
&x_5y_9x_2y_7, 
&x_5y_9x_4y_6, 
&x_5y_9x_4y_8.
\end{align*}
\]

Those different from one another at least in a town generate

\[
L_2(G) = (x_1y_7x_2y_9, 
 x_1y_8x_3y_6, 
 x_1y_8x_4y_6, 
 x_1y_8x_4y_9, 
 x_1y_8x_5y_9, 
 x_2y_7x_1y_8, 
 x_2y_9x_4y_6, 
 x_2y_9x_5y_8, 
 x_3y_6x_4y_8, 
 x_3y_6x_4y_9, 
 x_3y_8x_1y_7, 
 x_4y_8x_3y_6, 
 x_4y_8x_5y_9, 
 x_4y_9x_2y_7, 
 x_4y_9x_5y_8, 
 x_5y_8x_1y_7, 
 x_5y_8x_3y_6, 
 x_5y_8x_4y_6, 
 x_5y_8x_4y_9, 
 x_5y_9x_2y_7, 
 x_5y_9x_4y_6, 
 x_5y_9x_4y_8),
\]

and they are 22.

Let’s determine the number and the composition of distinct tourist packs that include sightseeing of 5 towns.

Observe that there are 18 \((9 \times 2)\)-submatrices of \(M_G\) having just one row of 1’s and two rows with a unique 1.

These three rows correspond respectively to the internal town and to the towns at the ends of each tourist pack that includes sightseeing of 3 towns.

By using software Mathematica, one can write out the above submatrices.

According to Proposition 2.3, the number of tourist packs is:

\[
\begin{align*}
&\left[\frac{4}{2}\right] + \left[\frac{2}{2}\right] + \left[\frac{2}{2}\right] + \left[\frac{3}{2}\right] + \left[\frac{2}{2}\right] + \left[\frac{3}{2}\right] + \left[\frac{2}{2}\right] + \left[\frac{4}{2}\right] + \left[\frac{3}{2}\right] + \\
&\quad + \left[\frac{3}{2}\right] + \left[\frac{2}{2}\right] + \left[\frac{2}{2}\right] + \left[\frac{3}{2}\right] + \left[\frac{3}{2}\right] + \left[\frac{3}{2}\right] + \left[\frac{3}{2}\right] - 4 \cdot 2 = \\
&= 2 \left[\frac{4}{3}\right] + 3 \left[\frac{4}{2}\right] + 8 \left[\frac{3}{2}\right] + 5 \left[\frac{2}{2}\right] - 8 = 12 + 9 + 16 + 5 - 8 = 34.
\end{align*}
\]

According to Theorem 3.3, their composition is:

\[
\begin{align*}
&x_1y_7x_2y_9x_4, 
&x_1y_7x_2y_9x_5, 
&x_1y_8x_3y_6x_4, 
&x_1y_8x_3y_6x_5, 
&x_1y_8x_4y_6x_3, 
&x_1y_8x_4y_6x_2, 
&x_1y_8x_4y_9x_5, 
&x_1y_8x_5y_9x_2, 
&x_1y_8x_5y_9x_4, 
&x_2y_7x_1y_8x_3, 
&x_2y_7x_1y_8x_4, 
&x_2y_9x_4y_6x_3, 
&x_2y_9x_4y_6x_5, 
&x_2y_9x_4y_8x_3, 
&x_2y_9x_4y_8x_5, 
&x_2y_9x_5y_8x_3, 
&x_2y_9x_5y_8x_4, 
&x_3y_6x_4y_8x_5, 
&x_3y_6x_4y_9x_5, 
&x_3y_8x_1y_7x_2, 
&x_3y_8x_1y_7x_4, 
&x_4y_6x_3y_5x_8, 
&x_4y_6x_3y_5x_9, 
&x_4y_6x_3y_8x_4, 
&x_4y_6x_3y_8x_5, 
&x_6y_3x_8y_4x_5, 
&x_6y_3x_8y_4x_9, 
&y_6y_3x_8y_4x_5, 
&y_6y_4x_8y_3x_5, 
&y_7x_1y_8y_4x_5, 
&y_7x_1y_8y_4x_9, 
&y_7x_2y_9x_5y_8, 
&y_7x_2y_9x_5y_9, 
&y_7y_2y_9y_4x_8, 
&y_7y_2y_9y_4x_9, 
&y_8x_1y_7y_2y_9, 
&y_8x_3y_6x_4y_9.
\end{align*}
\]

Those different from one another at least in a town generate

\[
L_5(G) = (x_1y_7x_2y_9x_4, 
 x_1y_7x_2y_9x_5, 
 x_1y_8x_3y_6x_4, 
 x_1y_8x_3y_6x_5, 
 x_1y_8x_4y_6x_3, 
 x_1y_8x_4y_6x_2, 
 x_1y_8x_4y_9x_5, 
 x_1y_8x_5y_9x_2, 
 x_1y_8x_5y_9x_4, 
 x_2y_7x_1y_8x_3, 
 x_2y_7x_1y_8x_4, 
 x_2y_9x_4y_6x_3, 
 x_2y_9x_4y_6x_5, 
 x_2y_9x_4y_8x_3, 
 x_2y_9x_4y_8x_5, 
 x_2y_9x_5y_8x_3, 
 x_2y_9x_5y_8x_4, 
 x_3y_6x_4y_8x_5, 
 x_3y_6x_4y_9x_5, 
 x_3y_8x_1y_7x_2, 
 x_3y_8x_1y_7x_4, 
 y_6y_3x_8y_4x_5, 
 y_6y_3x_8y_4x_9, 
 y_6y_3x_8y_5x_9, 
 y_6y_4x_8y_3x_5, 
 y_6y_4x_8y_3x_9, 
 y_7x_1y_8y_4x_5, 
 y_7x_1y_8y_4x_9, 
 y_7x_2y_9x_5y_8, 
 y_7x_2y_9x_5y_9, 
 y_7y_2y_9y_4x_8, 
 y_7y_2y_9y_4x_9),
\]

and they are 27.

Let’s determine the number and the composition of distinct tourist packs that include sightseeing of 6 towns.
Observe that there are 28 \((9 \times 3)\)-submatrices of \(M_G\) having just two rows with two 1’s and two rows with a unique 1.

These four rows correspond respectively to the internal towns and to the towns at the ends of each tourist pack that includes sightseeing of 4 towns.

By using software \textbf{Mathematica}, one can write out the above submatrices.

According to Proposition 2.4, the number of tourist packs is:

\[
\frac{4}{2} + \frac{2}{2} + \frac{3}{2} + \frac{2}{2} + \frac{3}{2} + \frac{3}{2} + \frac{2}{2} + \frac{2}{2} + \frac{3}{2} +
\]

\[
+ \frac{2}{2} + \frac{2}{2} + \frac{4}{2} + \frac{4}{2} + \frac{4}{2} + \frac{3}{2} + \frac{2}{2} + \frac{4}{2} + \frac{3}{2} + \frac{2}{2} +
\]

\[
+ \frac{3}{2} + \frac{2}{2} + \frac{2}{2} + \frac{3}{2} + \left(\frac{3}{2} + \frac{4}{3}\right) - (4+6) - (4+8) =
\]

\[
= 2 \left(\frac{4}{3}\right) + 5 \left(\frac{4}{2}\right) + \left(\frac{4}{3}\right) + 10 \left(\frac{1}{2}\right) - 10 - 12 =
\]

\[
= 12 + 15 + 4 + 20 + 10 - 22 = 39.
\]

According to Theorem 3.3, their composition is:

\[
x_1 y_7 x_2 y_3 x_4 y_6, \quad x_1 y_7 x_2 y_3 x_9 x_4 y_8, \quad x_1 y_7 x_2 y_9 x_4 y_5 x_8, \quad x_1 y_7 x_3 y_6 x_4 y_9, \quad x_1 x_8 x_4 y_9 x_2 y_7, \\
x_1 y_8 x_5 y_9 x_2 y_7, \quad x_1 y_8 x_5 y_9 x_4 y_6, \quad x_2 y_7 x_1 y_8 x_3 y_6, \quad x_2 y_7 x_1 y_8 x_4 y_6, \quad x_2 y_7 x_1 y_8 x_4 y_9, \\
x_2 y_7 x_1 y_8 x_5 y_9, \quad x_2 y_9 x_4 y_6 x_3 y_8, \quad x_2 y_9 x_4 y_8 x_1 y_7, \quad x_2 y_9 x_4 y_8 x_3 y_6, \quad x_2 y_9 x_5 y_8 x_1 y_7, \\
x_2 y_9 x_5 y_8 x_3 y_6, \quad x_2 y_9 x_5 y_8 x_4 y_6, \quad x_3 y_6 x_4 y_8 x_1 y_7, \quad x_3 y_6 x_4 y_8 x_5 y_9, \quad x_3 x_6 x_4 y_9 x_2 y_7, \\
x_3 y_6 x_4 y_9 x_5 y_8, \quad x_3 y_6 x_4 y_9 x_5 y_8, \quad x_3 y_8 x_1 y_7 x_2 y_9, \quad x_3 y_8 x_5 y_9 x_2 y_7, \quad x_3 y_8 x_5 y_9 x_4 y_6, \\
x_4 y_6 x_3 y_8 x_1 y_7, \quad x_4 y_6 x_3 y_8 x_5 y_9, \quad x_4 y_8 x_1 y_7 x_2 y_9, \quad x_4 y_8 x_5 y_9 x_2 y_7, \quad x_4 y_8 x_5 y_9 x_2 y_7, \\
x_4 y_9 x_5 y_8 x_1 y_7, \quad x_4 y_9 x_5 y_8 x_3 y_6, \quad x_5 y_8 x_1 y_7 x_2 y_9, \quad x_5 y_8 x_3 y_6 x_4 y_9, \quad x_5 y_8 x_4 y_9 x_2 y_7, \\
x_5 y_9 x_2 y_7 x_1 y_8, \quad x_5 y_9 x_4 y_6 x_3 y_8, \quad x_5 y_9 x_4 y_8 x_1 y_7, \quad x_5 y_9 x_4 y_8 x_3 y_6.
\]

Those different from one another at least in a town generate

\[
L_6(G) = \langle x_1 y_7 x_2 y_9 x_4 y_6, \quad x_1 y_7 x_2 y_9 x_4 y_8, \quad x_1 y_7 x_2 y_9 x_5 y_8, \quad x_1 y_8 x_3 y_6 x_4 y_9, \\
x_1 y_8 x_3 y_8 x_4 y_6, \quad x_2 y_7 x_1 y_8 x_3 y_6, \quad x_2 y_7 x_1 y_8 x_4 y_6, \quad x_2 y_9 x_4 y_6 x_3 y_8, \\
x_2 y_9 x_5 y_8 x_3 y_6, \quad x_2 y_9 x_5 y_8 x_4 y_6, \quad x_3 y_8 x_4 y_8 x_1 y_7, \quad x_3 y_8 x_4 y_8 x_5 y_9, \\
x_3 y_8 x_4 y_9 x_2 y_7, \quad x_3 y_8 x_4 y_9 x_2 y_7, \quad x_3 y_8 x_4 y_9 x_2 y_7, \quad x_3 y_8 x_5 y_9 x_2 y_7, \\
x_4 y_6 x_3 y_8 x_1 y_7, \quad x_4 y_6 x_3 y_8 x_5 y_9, \quad x_4 y_8 x_1 y_7 x_2 y_9, \quad x_4 y_8 x_5 y_9 x_2 y_7, \quad x_4 y_8 x_5 y_9 x_2 y_7, \\
x_4 y_9 x_5 y_8 x_1 y_7, \quad x_4 y_9 x_5 y_8 x_3 y_6, \quad x_5 y_8 x_1 y_7 x_2 y_9, \quad x_5 y_8 x_3 y_6 x_4 y_9, \quad x_5 y_8 x_4 y_9 x_2 y_7, \\
x_5 y_9 x_2 y_7 x_1 y_8, \quad x_5 y_9 x_4 y_6 x_3 y_8, \quad x_5 y_9 x_4 y_8 x_1 y_7, \quad x_5 y_9 x_4 y_8 x_3 y_6, \rangle,
\]

and they are 18.

Finally, by Theorem 3.3, we are able to know which distinct tourist packs including sightseeing of more than 6 towns the tour operator can propose.

To determine the composition of tourist packs that include sightseeing of 7 towns, let’s consider the 34 \((9 \times 4)\)-submatrices of \(M_G\) having just three rows with two 1’s and two rows with a unique 1, that correspond to the tourist packs including sightseeing of 5 towns.

So it is:

\[
x_1 y_7 x_2 y_9 x_4 y_6 x_3, \quad x_1 y_7 x_2 y_9 x_4 y_8 x_3, \quad x_1 y_7 x_2 y_9 x_4 y_8 x_5, \quad x_1 y_7 x_2 y_9 x_5 y_8 x_3, \\
x_1 y_7 x_2 y_9 x_5 y_8 x_4, \quad x_1 y_7 x_2 y_9 x_5 y_8 x_4 y_9 x_2, \quad x_1 y_8 x_3 y_6 x_4 y_9 x_5, \quad x_1 y_8 x_5 y_9 x_4 y_6 x_3, \\
x_2 y_7 x_1 y_8 x_3 y_6 x_4 y_6 x_3, \quad x_2 y_7 x_1 y_8 x_4 y_6 x_3, \quad x_2 y_7 x_1 y_8 x_4 y_6 x_5 y_9, \quad x_2 y_8 x_4 y_6 x_3 y_8, \\
x_2 y_9 x_5 y_8 x_4 y_6 x_3, \quad x_2 y_9 x_5 y_8 x_4 y_6 x_3, \quad x_3 y_8 x_1 y_7 x_2 y_9, \quad x_3 y_8 x_5 y_9 x_2 y_7, \\
x_3 y_8 x_5 y_8 x_2 y_7, \quad x_3 y_8 x_5 y_8 x_2 y_7, \quad x_3 y_8 x_5 y_8 x_2 y_7, \quad x_3 y_8 x_5 y_8 x_2 y_7, \\
x_4 y_6 x_3 y_8 x_1 y_7, \quad x_4 y_6 x_3 y_8 x_5 y_9, \quad x_4 y_8 x_1 y_7 x_2 y_9, \quad x_4 y_8 x_5 y_9 x_2 y_7, \quad x_4 y_8 x_5 y_9 x_2 y_7, \\
x_4 y_9 x_5 y_8 x_1 y_7, \quad x_4 y_9 x_5 y_8 x_3 y_6, \quad x_5 y_8 x_1 y_7 x_2 y_9, \quad x_5 y_8 x_3 y_6 x_4 y_9, \quad x_5 y_8 x_4 y_9 x_2 y_7, \\
x_5 y_9 x_2 y_7 x_1 y_8, \quad x_5 y_9 x_4 y_6 x_3 y_8, \quad x_5 y_9 x_4 y_8 x_1 y_7, \quad x_5 y_9 x_4 y_8 x_3 y_6, \rangle,
\]

and they are 28.
Those different from one another at least in a town generate

\[ L_7(G) = (x_1 y_7 x_2 y_9 x_4 y_6 x_3, x_1 y_7 x_2 y_9 x_4 y_8 x_3, x_1 y_7 x_2 y_9 x_4 y_8 x_5, x_1 y_7 x_2 y_9 x_5 y_8 x_3, x_1 y_8 x_3 y_6 x_4 y_9 x_2, x_1 y_8 x_3 y_6 x_4 y_9 x_5, x_2 y_7 x_1 y_8 x_3 y_6 x_4, x_2 y_9 x_5 y_8 x_3 y_6 x_4, y_6 x_3 y_8 x_1 y_7 x_2 y_9, y_6 x_3 y_8 x_2 y_9 x_1 y_7, y_6 x_3 y_8 x_5 y_9 x_2 y_7, y_6 x_3 y_8 x_5 y_9 x_2 y_7, y_6 x_4 y_8 x_1 y_7 x_2 y_9, y_6 x_4 y_8 x_5 y_9 x_2 y_7, y_6 x_4 y_9 x_5 y_8 x_1 y_7, y_7 x_1 y_8 x_3 y_6 x_4 y_9) , \]

and they are 15.

In a similar way, the composition of the remaining tourist packs is easily definable. In particular the tour operator can propose 21 tourist packs that include sightseeing of 8 towns, of which 5 generate \( L_8(G) \), and 8 tourist packs that include sightseeing of 9 towns.

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